## 1. Introduction

### 1.1 Links

- My homepage https://lpetrov.cc
- Course page https://publish.obsidian.md/particle-systems/
- [[../Syllabus | Syllabus]]


### 1.2 Some reminders

1. Lectures are recorded and available by request
2. It is helpful if you have camera on during lectures
3. If you're taking the class for grade, see [[../Syllabus\#Expectations and assessment | this section of the syllabus]]
4. The course is officially "hybrid", so there will be some in-person activities like outside walks and discussions, and maybe class meetings. Nothing is set and planned as of now. All in-person activities will either be optional, or with remote participation possible

### 1.3 Please introduce yourself

What is your background in...

- Probability theory, graduate and undergraduate?
- Symmetric functions?
- Real analysis (as in, measure theory)?


### 1.4 Preview

One rather famous particle system - the sticky tetris / ballistic deposition - models a lot of interesting behaviours. A readable account is here. Unfortunately, this system is not mathematically tractable (yet).
\{ \}


And here is an animation of an "integrable" particle system TASEP (totally asymmetric simple exclusion process), which demonstrates a lot of the same behavior (which is conjectural for the ballistic deposition). This system can be solved exactly - we know "everything" about its probability distribution and asymptotics.
\{ \}

1.5 Goal
\{ \}

$$
\begin{aligned}
& \text { Goal of lectures: present selt-contained, detailed } \\
& \text { description of souse results around random } \\
& \text { particle systems. } \\
& \text { Please ask many questions, and I'm happy to } \\
& \text { go slower rather two faster. }
\end{aligned}
$$

2 Background in measure theory and probability -
2.1 Measure spaces
\{ \}


Definition. Sigma-algebra $\}$

$$
\begin{aligned}
& \text { 3) if } A_{1}, A_{2}, A_{3}, \ldots \in F(\text { countably want }) \text {, } \\
& \text { then } \bigcup_{n=1}^{\infty} A_{n} \in F \\
& \frac{\text { Ex. L1-1 }}{A_{1}, A_{2}, \ldots \in \text {-alg. F, }} \\
& \text { then } \bigcap_{A_{n} \in \mathcal{F}}^{\infty} \\
& n=1
\end{aligned}
$$

Definition. Measure \{ \}

Def. Let $F$ be a $b$-algebra
$\mu: \underset{\sim}{f} \mathbb{R}$ is called a measure if

1) For all $A \in F, \quad \mu(A) \geqslant 0$
2) $\mu(\phi)=0 \quad$ (empty set)
3) Countably additive $\quad$ "b-additive"): if $A_{1}, A_{2}, \ldots \in J \quad$ and $\quad A_{i} \cap A_{j}=\phi, i \neq j$ them $\mu\left(\bigcup_{n=1}^{\infty} A_{n}\right)=\sum_{n=1}^{\infty} \mu\left(A_{n}\right)$.

If $\mu(\Omega)=1$, the measure is called probability measure and we denote it $P$

Example. Measure spaces $\}$

2) $([0,1], B, d x)$ - standard probability space
3) If $\mathcal{X}$-discrete finite or countable,
$\mathbb{Z}, \mu(n) \quad \mathcal{F}=2^{\notin}$ all subsets, $\mu^{\mu}$ is described by $\quad \mu(\{x\}), \forall x \in \notin$
4) $\mathbb{R}^{n}$, Morel $b$-alg $7 n\left(a_{i}, b_{i}\right)$, $d x_{1} \cdots d x_{n}$
5)

$$
\begin{aligned}
& \text { Spaces of functions like } C[0,1]) \\
& \text { there is a } b \text {-algebra of aylindric sets } \\
& \text { here (= same as Bored, ie., generated } \\
& \text { ley all open sets), but wont discuss... }
\end{aligned}
$$

We will need both probability measures (on them we define randomness), and measure spaces like $\mathbb{R}$ with Lebesgue measure. On $\mathbb{R}$ (or $\mathbb{Z}$ ) we will define random point configurations and particle systems in particular.

For the present lecture, we mostly stick to the probability context.
So we take the following running examples
of measure spaces:

- $\mathbb{R}$ with Lebesgue measure (length)

$$
\begin{aligned}
& \mu((a, b))=b-a \quad(a \leqslant b) \\
& \mu(\text { finite or countable })=0 \\
& \text { with counting measure }
\end{aligned} \begin{aligned}
& \text { Ex L1-2. } \\
& \mu(A)=\# \text { of points in } A, \quad A \leq \mathbb{Z}
\end{aligned}
$$

- An abstract probability space $(\Omega, J, P)$

$$
P(\Omega)=1
$$

### 2.2 Random variables

## Random variables

Let $(\Omega, \mathcal{F}, \mathrm{P})$ be a probability space, that is, $\mathrm{P}(\Omega)=1$.
Definition. Random variable $\}$

$$
\begin{aligned}
\text { Def. Random variable } \xi= & \text { function }^{\xi: \Omega \longrightarrow \mathbb{R}} \begin{aligned}
& \hat{i}_{\text {measurable: }} \\
& \xi^{-1}(B) \in \mathcal{F} \\
& \text { for all } B \in \mathcal{B}, \\
& \text { where } B-B_{\text {orel }} \text { ball }
\end{aligned} \\
& \\
&
\end{aligned}
$$

Example. Random variables taking values $\pm 1 \quad\{ \}$

$$
\begin{aligned}
& \text { Ex. } \Omega=[0,1] \text { / Lebesque } \\
& 13_{1} \quad P\left(s_{1}=1\right)=P\left(\omega \in \Omega \text { s.t. } s_{1}(\omega)=1\right) \\
& =P\left(\left[0, \frac{1}{2}\right]\right) \\
& =1 / 2 \\
& \text { Poth } \xi_{1}, \xi_{2} \text { take salues } \pm 1 \\
& P\left(\xi_{2}=1\right)=P\left(\left[0, \frac{L}{4}\right) \cup\left[\frac{1}{2}, \frac{3}{4}\right)\right]=\frac{1}{2} \\
& p\left(\xi_{2}=-1\right)=1 / 2 .
\end{aligned}
$$

Definition. Probability distribution \{ \}

Each randan variable $\xi$ induces probable mearive

$$
\begin{aligned}
& \operatorname{le}_{\xi} \text { on } \mathbb{R} \\
& \quad \mu_{\xi}(B)=P(\xi \in B)=P(w \in \Omega \text { sit. } \xi(\omega) \in B),
\end{aligned}
$$

Def $\int^{u g}$ is called the
(probability) distribution of $\}$

Ex. $\xi_{1}, \delta_{2}$ induce the same measure, but they are not the same riV.

In fact,

$$
\xrightarrow[\substack{\mu_{\xi_{1}}(B)=\\ \underset{-1}{1 / 2} \cdot \mathbb{1}_{-1 \in B}+\frac{1}{2} \cdot \mathbb{1}_{1 \in B}}]{\substack{1 / 2}} \rightarrow
$$

Definition. Equality in distribution $\}$
Def. If $\mu_{\xi_{1}}=\mu_{\xi_{2}}$, we say $\xi_{1} \stackrel{d}{=} \xi_{2}$, equal in distribution.

Definition. Discrete and absolutely continuous random variables \{\}

\{ \}
Discrete: Exists a discrete subset $X \subset \mathbb{R}$,

Absolutely Continuous; Exists density $f(x)$ sit.

$$
\begin{aligned}
& P(\xi \in B)=\int_{B} f(x) d x \\
& \Rightarrow \forall B \in B(\mathbb{R}) \\
& \int_{-\infty}^{+\infty} f(x) d x=1 \quad \text { (Note: for each such } f, \exists \xi \\
&\text { sit. } f \text { is a density of } \xi)
\end{aligned}
$$

Examples. Random variables $\}$

Ex. 1) Bernoulli(p) $\}=0$ or 1
2)

$$
\begin{aligned}
& \operatorname{Poisson}(\lambda) \quad \lambda \geqslant 0 \\
& \left\{\in \mathbb{Z}_{\geqslant 0}\right. \\
& P(\xi=k)=\frac{\lambda^{k}}{k!} e^{-\lambda}, \quad k=0,1,2, \ldots
\end{aligned}
$$

$$
(0 \leq p \leq 1)
$$

(Note: $\quad e^{\lambda}=\sum_{n=0}^{\infty} \lambda^{n} / n!$ )
3) Geometric $G e o(p), 0 \leqslant p \leqslant \frac{1}{1}$

$$
\begin{gathered}
\left\{\in \mathbb{Z}_{\geqslant 0},\right. \\
P(\xi=k)=(1-p) p^{k}, \quad k=0,1,2, \ldots \\
\left(\text { Note! } \sum_{k \geqslant 0} p^{k}=1_{1-p}\right)
\end{gathered}
$$

\{ \}
4) Exponential $\operatorname{Exp}(\lambda), \lambda>0$

$$
\begin{gathered}
f(x)= \begin{cases}\lambda e^{-\lambda x}, & x \geq 0 \\
0, & x<0\end{cases} \\
P(\zeta>t)=e^{-\lambda t}\left(=\int_{t}^{+\infty} \lambda e^{-\lambda x} d x\right)
\end{gathered}
$$

5) Standard normal (Stel Gaussian) $\left.\xi \sim N\left(O_{n}\right)\right)$

## Independence

\{ \}

Note: for a single r.s. \&, there is usually no difference between $\xi: \Omega \rightarrow \mathbb{R}$ and its distribution $\mu \xi$
For several riv. the way how $\mathcal{S}_{i}: \Omega \rightarrow \mathbb{R}$

Definition. Independence of two random variables \{\}

$$
\begin{aligned}
& \text { Def. Two random variables } \xi_{1}, \xi_{2} \text { are called } \\
& \text { independent if } \\
& P\left(\xi_{1} \in B_{1} \text { and } \xi_{2} \in B_{2}\right)=P\left(\xi_{1} \in B_{1}\right) P\left(\xi_{2} \in B_{2}\right) \\
& \text { for all } B_{1,}, B_{2} \in B(\mathbb{R})
\end{aligned}
$$

Definition. Independence of several random variables \{\}
 sumtuarly rudeperdent if
product ruk holds for all finite subcollections

$$
\begin{gathered}
P\left(\xi_{i 1} \in B_{1}, \ldots, \xi_{i_{k}} \in B_{k}\right)=\pi_{j_{1},}^{k} P\left(\xi_{i j} \in B_{j}\right) \\
\forall B_{j} \in B(\mathbb{R})
\end{gathered}
$$

Example. Independence \{ \}


- $\xi_{1}, \xi_{2}$ are independent
(and identically distributed $\rightarrow$ abbreviated cid)

Ex. 11-4. Use the previous example to explicitly construct countably many $\oint_{1}, \Im_{2}, \Im_{3},\left\{_{4}, \ldots\right.$ such that they are aid, $P\left(\xi_{j}=1\right)=P\left(\xi_{j}=-1\right)=1 / 2$.
2.3 Poisson random variable
\{ \}

$$
\begin{aligned}
& \text { Poisson }(\lambda) \quad \lambda \geqslant 0 \\
& \left\{\in \mathbb{Z}_{\geqslant 0}, P(\xi=k)=\frac{\lambda^{k}}{k!} e^{-\lambda}, \quad k=0,1,2, \ldots\right. \\
& \text { Note: } e^{\lambda}=\sum_{n=0}^{\infty} \lambda^{n} / n! \\
& \text { Note: } \lambda=0 \Rightarrow \begin{array}{l}
\}=0 \\
\text { with prob. } 1
\end{array}
\end{aligned}
$$

## Definition. Expectation \{ \}

De. $\xi \in \mathcal{H}$ discrete, Than $E g(\xi):=\sum_{x \in \mathcal{E}} g(x) P(\xi=x)$

Computation. Poisson random variable expectations Let us compute some expectations with respect to the Poisson random variable $\xi \sim \operatorname{Poisson}(\lambda)$.

First,

$$
\mathrm{E} \xi=\sum_{n=0}^{\infty} n \cdot \frac{\lambda^{n}}{n!} e^{-\lambda}=\sum_{n=1}^{\infty} \frac{\lambda^{n-1}}{(n-1)!} \lambda e^{-\lambda}=\lambda
$$

Another way to compute the same expectation is to apply $\frac{d}{d \lambda}$ to the Taylor expansion of $e^{\lambda}$. We have

$$
e^{\lambda}=\sum_{n=0}^{\infty} \frac{d}{d \lambda}\left(\frac{\lambda^{n}}{n!}\right)=\sum_{n=0}^{\infty} \frac{n \lambda^{n-1}}{n!}
$$

After multiplying by $\lambda e^{-\lambda}$ in both sides, we have

$$
\lambda=\sum_{n=0}^{\infty} n \cdot \frac{\lambda^{n}}{n!} e^{-\lambda}
$$

In fact, we can apply the operator $(d / d \lambda)$ several times (say, $k$ ), and then multiply by $\lambda^{k}$, to get

$$
\mathrm{E}(\xi(\xi-1) \ldots(\xi-k+1))=\lambda^{k}
$$

This is the so-called $k$-th factorial moment, where we denote $\xi^{\downarrow k}=\xi(\xi-1) \ldots(\xi-$ $k+1$ ) ( $k$ factors).

In particular, from this we also get the variance

$$
\operatorname{Var}(\xi):=\mathrm{E}(\xi-\mathrm{E} \xi)^{2}=\mathrm{E}\left(\xi^{2}\right)-(\mathrm{E} \xi)^{2}=\lambda
$$

## Exercise. Factorial moment $\}$

$$
\begin{aligned}
& \text { Ex. L1-S Use the idea of the previous } \\
& \text { computations to find the } k \sim t h \\
& \text { factorial moment of } \xi \sim \text { Poiss ( } \lambda \text { ) } \\
& \text { for all } k \geqslant 0 \text { : } \\
& E\left(\xi^{\downarrow k}\right):=E(\underbrace{s(\xi-1)(\xi-2) \ldots(\xi-k+1))}_{k \text { factors }}
\end{aligned}
$$

Exercise. Additivity of Poisson random variables $\}$

$$
\begin{aligned}
& \text { Ex. L1-6. (Additivity of Parson distribution) } \\
& \text { Let } s \sim \operatorname{Pas}(\lambda), \eta \sim P_{o i s}(\mu), \xi \text { and } \eta \\
& \text { are independent. } \\
& \text { Then } \xi+\eta \sim \operatorname{Pois}(\lambda+\mu) \\
& \text { (in fact, this works for } \\
& \text { countable additivity, too: } \\
& \left.\delta_{1}+\xi_{2}+\xi_{3}+\ldots \sim \operatorname{Poiss}\left(\sum_{i=1}^{\infty} \lambda_{i}\right)\right)
\end{aligned}
$$

## \# 3 Poisson Process

## 3 Poisson Process I

### 3.1 General definition

\{ \}

$$
\begin{aligned}
& \text { Let }(X, F, \mu) \text { be a measure space, } \\
& \text { with some additional alswnuptions } \\
& \text { - diagonal }\{x=y\} \text { is measurable in } X \times X \Rightarrow \\
& \text { all }\{x\}, x \in X \text {, below to } F \text {, ie., are measurable; } \\
& \text { and } M(\{x\})=0 \text { for all } x \in X \text { (non-atanic) } \\
& {\left[\text { Think } X=\mathbb{R}^{d} \text { for some } d \geqslant 1\right. \text {.] }}
\end{aligned}
$$

Definition. Poisson process $\}$
Def. Poisson process in $(X, F, \mu)$ is a random countable subset $\pi$ of $X$, suck that:

1) For any disjoint $A_{1}, A_{2}, \ldots, A_{n} \in \mathcal{F}$, the random varialks $N\left(A_{i}\right)=\#\left(\Pi \cap A_{i}\right)$ are independent
2) $N(A) \sim$ Poisson distribution with paracketer $\mu(A)$.

On this picture we have $N(A)=0, N(B)=3, N(C)=1$. These three random variables must be independent.
\{ \}


## Notes. Poisson process

1. There are no double points in a Poisson process, since $\mu(\{x\})=0$ for all points $x \in X$.
2. If $\mu(A)=\infty$, then there are infinitely many points of the Poisson process in $A$. An example, there are infinitely many points of the usual Poisson process in $\mathbb{R}$.
3. If $A \cap B=\varnothing$, then $N(A)$ and $N(B)$ are independent, and $N(A \cup B)=$ $N(A)+N(B)$. Note that here we use additivity of Poisson random variables, so that $N(A \cup B)$ is also Poisson, and definition is self-consistent.
4. Same additivity can be employed for countable disjoint unions. In this sense, the Poisson process can be viewed as a random atomic measure.
\{ \}

$$
\begin{aligned}
& \mu(A)=E N(A)=\text { mean } \# \text { of points of } M \text { in } A \\
& \text { so } \mu=\text { mean measure of } P . P \text {. } \Pi . \\
& {\left[\text { Most typical care, } X=\mathbb{R}^{d}, \quad \mu=\lambda d \vec{x}\right. \text { is }} \\
& \lambda>0 \text { times the Lebesgue measure on } \mathbb{R}^{d} \text {; } \\
& \text { so that } \lambda=\text { ruean } \# \text { of pts in the } \\
& \text { unit cube } \left.[0,1]^{d}\right] .
\end{aligned}
$$

3.2 Uniformity and independence in a Poisson process
\{ \}

$$
\begin{aligned}
& \text { Let } \Pi \text { be a Poisson Process on }(X, F, \mu u) \text {. } \\
& \text { Take } A \in \mathcal{F} \text {, with } \mu(A)<\infty \text {. }
\end{aligned}
$$

Theorem 3.2.1 \{ \}

$$
\begin{aligned}
& \text { Thur. Conditioned on } \mu(A)=K \text {, } \\
& \text { the distribution of the points inside } A \text { is } \\
& \text { that of inelep points, dropped into } A \\
& \text { according to pu(.)/ } / \mu(A)
\end{aligned}
$$

Proof 3.2.1 \{ \}

Proof. We show that for each $B \subseteq A$ (measurable),

$$
P(N(B)=m \mid N(A)=K)=\binom{k}{m} p^{m}(1-p)^{K-m},
$$

$$
\text { where } P=\operatorname{Ju}(B) / \mu(A) \text {. }
$$

Then points are independent, because each of them independently chooses whether to land in $B$ or $A \backslash B$, with probabilities $p$ and $1-p$, respectively.
Let us now show the binomial distribution. We have
$\mathrm{P}(N(B)=m \mid N(A)=K)=\frac{\mathrm{P}(N(B)=m, N(A)=K)}{\mathrm{P}(N(A)=K)}=\frac{\mathrm{P}(N(B)=m, N(A \backslash B)=K-m)}{\mathrm{P}(N(A)=K)}$,
and now we can use independence of the point counts corresponding to $B$ and $A \backslash B$ :

$$
=\frac{e^{-\mu(B)}(\mu(B))^{k} / k!\cdot e^{-\mu(A \backslash B)}(\mu(A \backslash B))^{K-m} /(K-m)!}{e^{-\mu(A)}(\mu(A))^{K} / K!}=\binom{K}{m} p^{m}(1-p)^{K-m},
$$

as desired.

## Converse statement: from uniform to Poisson \{ \}

$$
\begin{aligned}
& \text { Fact. (who proof) } \\
& \text { Let us tare } \Pi_{N}=2 N \text { unit. ranelom andes pts in } \\
& [-N, N] \text {. (mean density } 1) \text {. } \\
& \text { Then as } N \rightarrow \infty \text {, } \\
& \Pi_{N} \rightarrow \text { Poisson process on }\left(\mathbb{R}^{1}, \frac{d x}{n}\right. \text { Lebesgue }
\end{aligned}
$$



### 3.3 Using homogeneous Poisson process in id to model arrivals

The homogeneous Poisson process in $\mathbb{R}^{1}$ is a model for successive rings of an exponential clock, which is used in describing TASEP and other particle systems.
\{ \}


$$
\begin{aligned}
& \text { Here } \Pi \sim \text { Poiss. Process } \\
& \text { in }\left(\mathbb{R}^{1}, \lambda d x\right) \text {. ie. length of }(a, b) \text { is } \lambda(\ell-a) .
\end{aligned}
$$

Theorem 3.3.1 \{\}

$$
\text { The. } 1 \text { is the only model for arrival of calls which }
$$

ore independent, there ave no double arrivals

$$
(P(W(\{\times\})=2)=0) \text {, and where mean is } \lambda \cdot d x
$$

Proof 3.3.1 \{ \}
Proof (sketch).
Tare interval $0_{n}^{n} n^{1} 1 / n$
Break into $n$ pieces \& approximate by con flips:
In each piece, 1 arrival is with prob. $\frac{\lambda}{n}$

$$
0 \text { arrivals whip. } 1-\frac{\lambda}{n}
$$

$$
\begin{aligned}
& \text { (approx. wo double arrivals..) } \\
& \qquad \frac{(\lambda / n}{k \times 1}=2 \text { arr. }
\end{aligned} \rightarrow \frac{)^{2}}{2!} e^{-\lambda / n}=\partial\left(\frac{d}{n}\right.
$$

\{ \}

$$
\begin{aligned}
& \& \quad P\left(\xi_{n}=k\right)=\binom{n}{k}\left(\frac{\lambda}{n}\right)^{k}\left(1-\frac{\lambda}{n}\right)^{n-k} \\
& \text { As } n \rightarrow \infty \text {, } \\
& \mathbb{P}\left(\xi_{n}=k\right) \rightarrow e^{-\lambda} \frac{\lambda^{k}}{k!}, k=0,1,2 \ldots \\
& \text { \{ \} }
\end{aligned}
$$

## 4 Notes and references

1. On measure theory, most graduate books work well. At UVA we typically use Gerald B. Folland. Real Analysis: Modern Techniques and Their Applications. 2nd Edition.
2. Basic graduate probability theory is the subject of numerous textbooks. I was learning from Albert N. Shiryaev. Probability, there is a new English edition: Part I, II.
3. A great short book on Poisson processes which does not require too much background: J. F. C. Kingman. Poisson Processes. Clarendon Press, 1993.

## Problems

[[_Lecture 1, 2-1| Lecture 1]]
Due date February 15, solutions posted around that day.

## 1

If $\mathcal{F}$ is a $\sigma$-algebra and $A_{1}, A_{2}, \ldots \in \mathcal{F}$, show that $\bigcap_{n=1}^{\infty} A_{n} \in \mathcal{F}$. $\# 2$

Does there exist an uncountable set $A \subseteq \mathbb{R}$ such that its length (i.e., its Lebesgue measure) is zero? \# 3 Show that

$$
\int_{-\infty}^{+\infty} e^{-x^{2} / 2} d x=\sqrt{2 \pi}
$$

## 4

Using the example from section 2.3 of the lecture, explicitly construct countably many random variables $\xi_{1}, \xi_{2}, \xi_{3}, \ldots$ which are iid (independent and identically distributed) and have $\mathrm{P}\left(\xi_{j}=+1\right)=\mathrm{P}\left(\xi_{j}=-1\right)=\frac{1}{2}$.

## 5

Let $\xi$ be the Poisson random variable with parameter $\lambda \geq 0$. For each integer $k \geq 0$, find the $k$-th factorial moment $\mathrm{E}\left(\xi^{\downarrow k}\right):=\mathrm{E}(\xi(\xi-1) \ldots(\xi-k+1))$.

## 6

Let $\xi$ be a Poisson random variable with parameter $\lambda$, and $\eta$ be a Poisson random variable with parameter $\mu$. Let $\xi, \eta$ be independent. Then $\xi+\eta$ is a Poisson random variable with parameter $\lambda+\mu$.
[[Solutions, 2-1]]

## Solutions

[[../../Lecture 1, 2-1/Problems, 2-1|Problems 1]]
[[../../Lecture 1, 2-1/_Lecture 1, 2-1|Lecture 1]]
1
(1) $\bigcap_{n=1}^{\infty} A_{n}=\left(\bigcup_{n=1}^{\infty} A_{n}^{c}\right)^{c}$, so by closedkes, vader complements \& countable unions, we have $\bigcap_{n=1}^{\infty} A_{n} \in F$

$$
\begin{gathered}
\text { (used } A^{c} \cup B^{C}=(A \cap B)^{C} \\
\text { for constable } \\
\text { unisu/intersection) }
\end{gathered}
$$

2
\{ \}
(2) Yes, Cantor set $A=\bigcap_{n=1}^{\infty} A_{n}$

$$
\text { where } A_{n}=[0,1] \text {, removed }
$$

$$
\text { middle third } n \text { times }
$$

- A uncountabk because
it has all numbers on base 3 which avoid"
- A has $\mu(A)=0$ because $j^{\mu}\left(A_{n}\right) \rightarrow 0$
and they are decreasing, $A_{1}>A_{2} \supset A_{3} \supset \ldots$

See also https://en.wikipedia.org/wiki/Cantor_set

## 3

\{ \}
$(3)$

$$
\begin{aligned}
\int_{-\infty}^{+\infty} e^{-x^{2} / 2} d x & =2 \int_{0}^{\infty} e^{-x^{2} / 2} d x \\
& =2 \sqrt{\int_{0}^{\infty} e^{-\frac{x^{2}+y^{2}}{2}} d x e l y} \\
\begin{aligned}
& \text { polar word, }_{\infty}^{\infty}=2 \sqrt{\int_{0}^{\pi / 2} d \theta \int_{0}^{\infty} e^{-r^{2} / 2} r d r} \\
& \text { integrate over } \\
& \text { quadrant } \mathbb{R}_{7}^{2} \\
& 70
\end{aligned} & =2 \sqrt{\pi / 2} \sqrt{\int_{0}^{\infty} e^{-r^{2} / 2} d\left(r^{2} / 2\right)} \\
& =\sqrt{2 \pi} \cdot \sqrt{1} D^{\infty}
\end{aligned}
$$

(4) Let us divide $[0,1]$ vito $2^{n}$ segments of length $\frac{1}{2^{n}}$.

$$
\begin{aligned}
\delta_{n} & =1 \quad \text { on even segments, } \\
& -1 \text { on odd segments. }
\end{aligned}
$$


etc.

$$
\mathbb{P}\left(\xi_{i_{1}}=1, \ldots, \xi_{i_{k}}=1\right)=1 / 2^{k}
$$

(because of dyadic representation of numbers from $[0,1]$, for example)
So $\xi_{j}$ are iud

5
\{ \}
(B)

$$
\begin{aligned}
\mathbb{E} \xi^{V^{k}} & =e^{-\lambda} \sum_{n=0}^{\infty} \lambda^{n} / n!n^{j k} \\
& =e^{-\lambda} \lambda^{k} \sum_{n=0}^{\infty}\left(\frac{d}{d \lambda}\right)^{k}\left(\frac{\lambda^{n}}{n!}\right) \\
& =e^{-\lambda} \lambda^{k}\left(\frac{d}{d \lambda}\right)^{k} e^{\lambda}=\lambda^{k}
\end{aligned}
$$

6
\{\}
(6)

$$
\begin{aligned}
& P(\xi+\eta=n)=\sum_{k=0}^{n} P(\xi=k) P(\eta=n-k) \\
& =e^{-(\lambda+\mu)} \sum_{k=0}^{n} \frac{\lambda^{k} \mu^{n-k}}{k!(n-k)!}=\frac{e^{-(\lambda+\nu)}}{n!}(\lambda+\mu)^{n} \\
& \begin{array}{c}
\text { use bivarial } \\
\text { uncorem }
\end{array}
\end{aligned}
$$

[[Problems, 2-3| $3+$ ? problems]], due $2 / 17$

## 1 Poisson process

We continue from [[../Lecture 1, 2-1/3 Poisson Process I|the end of L1]], and finish the discussion of Poisson process in
-

## 1 Poisson Process II

### 1.1 Recall

## Definition of Poisson process $\}$

$$
\begin{aligned}
& \text { Def. Poisson process in }(X, F, j u) \text { is a randoun } \\
& \text { countable subset } T \text { of } X \text {, such that: } \\
& \text { 1) For any disjoint } A_{1}, A_{2}, \ldots, A_{n} \in J \text {, the randoun } \\
& \text { Vorralks } N\left(A_{i}\right)=\#\left(\Pi \cap A_{i}\right) \text { are independcut } \\
& \text { 2) } N(A) \sim \text { Poisson distribution with parameter } \mu(A) .
\end{aligned}
$$

## Example of Poisson vs determinantal process $\}$


1.2 Poisson process in Id
\{ \}
Tare $\Pi$ - Poos Prove in $(\mathbb{R}, \lambda d x), \lambda>0$ ( $\lambda>0$ is called rate).


$$
\ldots<X_{-3}<X_{-2}<X_{-1}<0<X_{1}<X_{2}<X_{3}<\ldots
$$

Lemma 1.2.1 \{ \}

Leva 1.2.1. Distribution of $X_{1}$ is $E_{x p}(\lambda)$.

$$
\text { Proof } P\left(X_{1}>x\right)=P(N([0, x))=0)=e^{-\lambda x}
$$

Theorem 1.2.2 \{ \}
Times between arrivals are indef. \& have Exp $(\lambda)$ distribution.

Proof 1.2.2 Step 1
\{ \}
we umber points in $(0, \infty)$ by

$$
0<x_{1}<x_{2}<x_{3}<\ldots\binom{(\text { case of }(-\infty, 0)}{\text { is symmetric }}
$$

1) $X_{1} \sim E_{x p}(\lambda)$ by Lemme. 1.2.1

Step 2
\{ \}
2)

$$
\text { Let } \Pi^{\prime}=\left(x_{2}-x_{1}, x_{3}-x_{1}, \ldots\right)
$$

We show that $\Pi^{\prime}$ is ingle of $X_{1}$ and has the same distr. as
$\Pi=\left(x_{1} ; x_{2}, \ldots\right)$. This shows by ind. what we want

Step 3
\{ \}
3) Let $f(x)=0, x<0$; $f$ cont. \& bod support

Let $\Sigma=\sum_{1}^{\infty} f\left(x_{n}\right), \quad \Sigma^{\prime}=\sum_{2}^{\infty} f\left(x_{n}-x_{1}\right)$
we wows to show $\Sigma^{\frac{d}{\Sigma}} \Sigma^{\prime} \& \Sigma^{\prime}$ ind of $X_{1}$.

Step 4
\{ \}
4) Approx. $X_{1}$ by dyadic numbers from above:

define $\Sigma^{k}=\sum_{n=2}^{\infty} f\left(X_{n}-\xi_{k}\right)$
Then $\Sigma^{k} \rightarrow \Sigma^{\prime}, k \rightarrow \infty$
\{\}
Wherever, $P\left(\Sigma^{k} \leqslant z, X_{1} \leqslant x\right)=$

$$
\begin{aligned}
& =\sum_{l=1}^{\infty} P(\Sigma^{k} \leq z, \quad X_{1} \leq x, \underbrace{}_{\substack{\left.\xi_{\text {all chores }}^{k}=l / 2^{k}\right) \\
\text { for values }}} \begin{array}{l}
\text { of } \xi_{k}
\end{array} \\
& l /{ }_{2}>X_{1}
\end{aligned}
$$

If $\oint_{k}=\ell_{2^{k}}>X_{1}$,
then $\left(x_{2}, x_{3}, \ldots\right)$ are Foist. proc.
on $(l / 2 k, \infty)$, indep of $x_{1}$
(ky def. of P.proc.). So, out this set,
\{\}

$$
\sum^{k}=\sum_{n=2}^{\infty} f\left(x_{n}-l 2^{-k}\right)
$$

has the same distr. as $\sum$.

$$
\begin{aligned}
& \Rightarrow P\left(\sum^{k} \leq z, \quad X_{1} \leq x, \quad \quad_{k}=l / 2^{k}\right) \\
& \quad=P\left(\sum \leq z\right) P\left(X_{1} \leq x, \quad \xi_{k}=l / 2^{k}\right)
\end{aligned}
$$

Step 5
\{ \}
5)

Now suing over $l$, we have

$$
\begin{aligned}
& P\left(\Sigma^{k} \leq z, \quad X_{1} \leq x\right) \\
& \\
& =\sum_{l=1}^{\infty} P(\Sigma \leq z) P\left(X_{1} \leq x, \quad \xi_{k}=l / 2^{k}\right) \\
& \\
& =P(\Sigma \leq z) P\left(X_{2} \leq x\right) .
\end{aligned}
$$

\{ \}

So as $x \rightarrow \infty$,

$$
P\left(\Sigma^{\prime} \leqslant z, \quad X_{1} \leqslant x\right)=P(\Sigma \leqslant z) P\left(X_{1} \leqslant x\right)
$$

We see $\Sigma^{-1} \stackrel{d}{=}$ (boy taking $x \rightarrow+\infty$ ) and rudependerice

Remark 1.2.3. Waiting time paradox \{ \}

$$
\begin{aligned}
\pi= & \{x_{-2}, \underbrace{X_{-1}, X_{1}}_{0 \text { is }}, X_{2}, \ldots\} \circ \sim \mathbb{R} \\
& X_{1}, X_{2}-X_{1}, \ldots \sim E_{x p}(\lambda) \\
& X_{-1}, X_{-1}, X_{-2}, \ldots \sim E_{x p}(\lambda)
\end{aligned}
$$

but $X_{1}-X_{-1}$ is mot $\operatorname{Exp}(\lambda) 6$
this is not the inser-arrisal time bus rather the length of an interval $\rightarrow O$, this has a olffferent distribution $(\operatorname{Same}(2, \lambda))$
\{ \}

$$
\begin{aligned}
& \text { ("waiting time paradox": } \\
& \text { (auger inter - arrival interval } \\
& \text { has a better chance to } \\
& \text { eoutaín } \quad \text { ). }
\end{aligned}
$$

Theorem 1.2.4 \{ \}


Proof 1.2.4 \{ \}

$$
\begin{aligned}
P\left(T_{n} \in[x, x+d x]\right) & =P(N([0, x])=n-1, N((x, x+d x])=1) \\
& =\frac{(\lambda x)^{n-1} e^{-\lambda x}}{(n-1)!} \cdot \lambda d x e^{-\lambda d x} \\
& \simeq \frac{\lambda^{n} x^{n-1} e^{-\lambda x}}{(n-1)!} d x .
\end{aligned}
$$

Here is an associated problem: [[Problems, 2-3\#1]]

## 2 TASEP. Definition and existence

- 


### 2.1 Definition of TASEP

\{ \}
Let $G=(V, E)$ be a directed graph (finite or $\begin{gathered}\text { infinite, } \\ \text { bounded } \\ \text { degree) }\end{gathered}$

Definition. TASEP $\}$

Def. A TASEP on $G$ is a stochastic particle sys with configurations $\{0,1\}^{V}$ \& transitions associated w. expon clocks of rate 1 on edges.
$\forall$ Edge carries a piss -prices.


$$
\Rightarrow 0<0
$$

blecring of jo $\Rightarrow$

(we get a continuous-time

$$
\text { precess } \left.\left(X_{t}\right)_{t \in \mathbb{R} \geqslant 0} \text { on }\{0,1\}^{G}\right)
$$

Examples. TASEP \{ \}

Ex.

$$
\begin{aligned}
& G=\text { ring } \\
& G=\mathbb{Z}
\end{aligned}
$$

$$
\begin{aligned}
& \mathbb{Z}_{+}+ \\
& \text {+ some boundary } \\
& \text { interactions }
\end{aligned}
$$

Definition. Height function \{ \}


### 2.2 Existence of TASEP

Theorem 2.2.1 \{\}

$$
\begin{aligned}
& \text { For any initial } \\
& \text { configuration } X_{0} \in\{0,1\}^{2} \text {, } \\
& \text { there exists TASEP started from } X_{0} \\
& \text { process w. this initial } \\
& \text { contignation } X_{0}
\end{aligned}
$$

Remark 2.2.2 The existence of TASEP is not too obvious, as the process makes infinitely many jumps in finite time. Sure, the interaction is local, but there is no deterministic bound on the lenght of the interaction for any finite time. Therefore, one cannot say that the system evolves independently in some finite-size blocks.
\{ \}

Why not too obvious? many jugs...


Proof 2.2.1 \{ \}

$$
\begin{aligned}
& \text { We realize TASEP as a } \\
& \text { deterministic function of a Poisson } \\
& \text { process on }{\underset{i}{i} \in \mathbb{Z}}(\mathbb{R} \geqslant 0, d x)
\end{aligned}
$$

\{ \}

$\square$

After the proof of Theorem 2.2.1 see [[Problems, 2-3\#2]].

Remark 2.2.3. Basic coupling From the proof of Theorem 2.2.1, we see that it is possible to put onto the same probability space all TASERs started from all possible initial configurations. Indeed, for this we only need to use the same Poisson process, sampled once. This probability space is the space for the Poisson process on $\bigsqcup_{i \in \mathbb{Z}} \mathbb{R}_{\geq 0}$.
In particle systems literature, this construction of all TASERs on the same probability space is called basic coupling.

Remark 2.2.4 There is an explicit procedure for sampling the Poisson process. It is not required for our process on $\bigsqcup_{i \in \mathbb{Z}} \mathbb{R}_{\geq 0}$, because here it suffices just to take countably many independent exponential random variables $\operatorname{Exp}(\lambda)$, and they would be the inter-arrival times.

However, for Poisson processes on general spaces (even consider the Poisson process on the plane $\mathbb{R}^{2}$ - there are no exponential inter-arrival times), there is a nice explicit procedure. Suppose that our space $X=\bigsqcup_{n} X_{n}$ is a countable union of spaces with $\mu\left(X_{n}\right)<\infty$ (such a space $X$ is called $\sigma$-finite). Then, for each $X_{n}$, sample a Poisson random variable $N_{n}$ with mean $\mu\left(X_{n}\right)$. After that, sample $N_{n}$ independent points in $X_{n}$ with distribution $\mu(\cdot) / \mu\left(X_{n}\right)$. The collection of all these points is the Poisson process.

See [[Problems, 2-3\#3]], where you need to prove that this sampling indeed produces the desired Poisson process.

### 2.3 Markov property

Here we discuss the Markov property of TASEP.
Recall. Markov property $\}$

$$
\begin{aligned}
& \text { Recall: a precess is Markov if } \forall n \\
& \begin{aligned}
& P\left(X_{t}\right.\left.=y \mid X_{s}=x, X_{s_{1}}=x_{1}, \ldots X_{s_{n}}=x_{n}\right) \\
&=P\left(X_{t}=y \mid X_{s}=x\right), \\
& t>s>s_{1}>\ldots>s_{n} \\
& \text { future }
\end{aligned}
\end{aligned}
$$

Equivalently, the future and the past are independent conditioned on the state of the process at present.

Theorem 2.3.1 TASEP on $\mathbb{Z}$ started from any initial configuration is a Markov process on $\{0,1\}^{\mathbb{Z}}$.

Proof 2.3.1 This is quite obvious, especially if we adopt the definition through independence of past and future. Indeed, if we fix the present, then the past and the future are determined by the states of the Poisson process on $\bigsqcup_{i \in \mathbb{Z}} \mathbb{R}$ corresponding to two disjoint sets. Therefore, we get independence directly from the properties of the Poisson process.

Remark 2.3.2 Is there another clock distribution, besides the exponential, such that TASEP constructed with these clocks is Markov? For example, what if the waiting time before the jump has uniform distribution on $[0,1]$ ?

It turns out that no. It is not hard to see that the Markov property of TASEP is equivalent to the memoryless property of the waiting time $\xi$ :

$$
\mathrm{P}(\xi>a+b \mid \xi>a)=\mathrm{P}(\xi>b)
$$

This property, in its turn, is equivalent to the fact that the function $f(a)=\mathrm{P}(\xi>$ $a)$ is multiplicative in $a$. Given the appropriate boundary conditions $f(0)=1$, $f(+\infty)=0$ coming from probability, we see that it must be $f(a)=e^{-\lambda a}$, which singles out the exponential distribution.
-

### 2.4 Some other particle systems

## ASEP

\{ \}

$$
\begin{aligned}
& \text { SEP. } \\
& 0 \leqslant t \leqslant 1 \\
& (t \rightarrow 1 \text {, place tr. }) \begin{array}{l}
t=0 \quad \text { Ti SEP } \\
t=1 \quad \text { StEP }
\end{array} \\
& \text { t<1: }\left(t_{\text {time }}\right)^{1 / 3} \text { fluct. } \\
& t=1 \text { : (time) fluct. }
\end{aligned}
$$

Here parameter $t$ is a number between 0 and 1 , and not a time parameter. While this notation might be unfortunate, it has reasons within integrability.

## Coloured TASEP

## 2 Colours

\{ \}


The configuration $X^{(2)}$ should be "bigger" than $X^{(1)}$, notation $X^{(2)} \succ X^{(1)}$. This means that if a location $i \in \mathbb{Z}$ is occupied under $X^{(1)}$, then it must be occupied under $X^{(2)}$.

Possible nontrivial transitions (each at rate 1) in this TASEP are


Therefore, we call $(1,1)$ a first-class particle, and $(0,1)$ a second-class particle. That is, first-order particles treat second-class particles as empty space.
See the "question to think", [[Problems, 2-3\#?]].

Many colours
\{ \}
color = class, reverse ordered

\{ \}


## PushASEP

Under PushASEP, each particle jumps to the right according to the TASEP rules (in particular, the jump is blocked if the destination is occupied), at rate $R$. And in parallel, each particle jumps to the left at rate $L$ to the left, but any particle can jump, and the jumping particle lands at the nearest empty space to the left. Therefore, there is no blocking mechanism in left jumps, but rather one can say that there is a "pushing" mechanism.


## q-TASEP

Here $0 \leq q<1$ is a parameter. Each particle jumps to the right by one at rate $1-q^{\text {gap }}$, where gap is the distance to the nearest particle to the right. If gap $=+\infty$ (i.e., the configuration has the rightmost particle), then the jump is with rate 1.
\{ \}


When $q=0, q$-TASEP turns into the usual TASEP, as

$$
1-0^{n}=1, n \geq 1, \quad 1-0^{0}=0
$$

- ASEP - coloured TASEP - PushASEP - $q$-TASEP


## Notes and references

1. A great short book on Poisson processes which does not require too much background: J. F. C. Kingman. Poisson Processes. Clarendon Press, 1993.
2. Picture of Poisson vs determinantal point processes is taken from J. Ben Hough, Manjunath Krishnapur, Yuval Peres, and Bálint Virág. Determinantal Processes and Independence, https://projecteuclid.org/euclid.ps/1146832696
3. TASEP was introduced in probability theory and biology almost simultaneously, around 50 years ago:

- C. MacDonald, J. Gibbs, and A. Pipkin. "Kinetics of biopolymerization on nucleic acid templates". In: Biopolymers 6.1 (1968), pp. 1-25.
- F. Spitzer. "Interaction of Markov processes". In: Adv. Math. 5.2 (1970), pp. 246-290.

4. The graphical construction (in the proof of the existence of TASEP) is due to Harris, T. E. Additive set-valued Markov processes and graphical methods. Ann. Probability 6 (1978), no. 3, 355-378.
5. The "basic coupling" leading to the coloured TASEP dates back to at least T. M. LIGGETT, Interacting Particle Systems, Springer, Berlin, 1985.
6. A survey on point processes (including the notion of convergence of probability measures on point configurations) is Soshnikov's survey.

## Problems

[[_Lecture 2, 2-3| Lecture 2]]
Due date February 17, solutions posted around that day.

## 1

Let $\Pi$ be a Poisson process on $(X, \mathcal{F}, \mu)$. For each point in $\Pi$, independently for all points, erase this point with probability $p \in(0,1)$. Denote the resulting configuration by $\Pi^{\prime}$. Show that $\Pi^{\prime}$ is again a Poisson process, and find the corresponding mean measure $\mu^{\prime}$ for it.

## 2

Adapt the proof of the existence of TASEP to the case of a generic countable bounded degree directed graph $G=(V, E)$. The bounded degree condition means that there exists $d \geq 1$ such that all degrees (incoming and outgoing, if needed) do not exceed $d$.

## 3

Show that the sampling procedure of a Poisson process on a general $\sigma$-finite space $X$ described in Remark 2.2.4 indeed produces the point configuration distributed as the Poisson process.

This is a question to think for a week, the answer will be given in the next lecture on Feb 7.
\{ \}


$$
\frac{L(t)}{t} \rightarrow ?
$$

[[../Lecture 3, 2-8/1.1 TASEP second-class particle asymptotics|Answer]]
[[Solutions, 2-3]]

## Solutions

[[Problems, 2-3|Problems 2]]
[[_Lecture 2, 2-3|Lecture 2]]

1
\{ \}
(1) Check def. of Poisson proc.

1) Indep of $N\left(A_{i}\right)$ for disjoint sets is obvious
2) distrib of single $N(A)$

- was a Poiss. riv with $\lambda=\mu(A)$
-Now, we erase points $w$ probab.p
So, $\quad N \sim \operatorname{Poss}(\lambda)$
New $\#$ of points is $M$ (left points)
$\operatorname{Law}(M(N) \sim$ Binomial $(N, 1-p)$
So, uncondét. law of $M$ is

$$
P(M=m)=\sum_{n=0}^{\infty} \frac{b^{n}}{\not n!} e^{-\lambda} \frac{n_{1}^{\prime}}{m!(n-m)!} p^{n-m}(r p)^{m}
$$

$$
\begin{aligned}
& =\frac{\lambda^{m}(1-p)^{m}}{m!} e^{-\lambda} \sum_{h \geqslant m} \frac{(\lambda p)^{n-m}}{(n-m)!} \\
& =\frac{(\lambda(1-p))^{m}}{m!} e^{-\lambda(1-p)} \\
& \Rightarrow \pi^{\prime} \text { is a piss. proc. on }(X,(1-p) \mu) .
\end{aligned}
$$

2
\{ \}
(2) Put the Poisson processeg on edges of $G=(V, E)$.

\& $X_{t}=$ deterministic function of $\lambda_{0}$ and of the realization of this Poisson process.

3
\{ \}
(3) Let us show that for $1 x_{1}$ with $\mu\left(x_{1}\right)<\infty$, the procedure gives the Poise process. then the union is again a Poisson process, this is quite clear.

Let $\mu\left(X_{1}\right)=J, \quad N \sim \operatorname{poiss}(\lambda)$
Enough to show that ultimo ar sal distribution with $N$ objects,
where $N$ is random Poisson, produces inelepaudert Poisson r.v.'S.
This is a simple computation Mich I oust.

## 1 Leftovers from the previous lecture

- 1.1 [[../Lecture 2, 2-3/2.4 Some other particle systems\#Coloured TASEP|Coloured TASEP]] and answer [[../Lecture 2, 2-3/Problems, $2-3 \#$ ?|to the question about the limiting behaviour of the second class particle]], without proof. Here is the [[1.1 TASEP second-class particle asymptotics|answer]].
- 1.2 [[../Lecture 2, 2-3/2.4 Some other particle systems\#q-TASEP $\mid$ qTASEP]] and [[Problems, 2-8\#1|problem of its existence]]


## 2 TASEP and Last Passage Percolation

- 


### 2.1 Height function

Recall the TASEP's height function:
\{ \}


0

Now let us rotate it, and we obtain a model of a down-right interface in $\mathbb{Z}^{2}$ which grows in continuous time ("invades in the up-right direction") by adding boxes.


### 2.2 Random interface growth

The model of random interface growth is a more efficient way of running TASEP, as there are no "lost" events in the governing Poisson process, like it was in the [[../Lecture 2, 2-3/2.2 Existence of TASEP|Harris graphic construction]].
In the growth model, each cell $(i, j)$ of $\mathbb{Z}^{2}$ is equipped with an independent exponential random variable $\alpha_{i, j} \sim \operatorname{Exp}(1)$ (with mean 1 ). When the interface reaches $(i, j)$ and $(i, j)$ becomes the corner of the unoccupied zone, then the interface waits time $\alpha_{i, j}$ and after that covers the cell $(i, j)$.
\{ \}

2.3 Percolation times
\{ \}

Exp r.v. is nemoryless, so we cau"turn on" the wait time when interface reaches the cell


Let $L_{i, j} \in \mathbb{R}$ be a random variable representing the time $t$ at which the growing interface $h_{t}$ covers $(i, j)$. Clearly, $L_{i, j}$ depend on the initial condition, the initial interface $h_{0}$.
We have $L_{i, j} \geq L_{i-1, j}, L_{i, j} \geq L_{i, j-1}$.
\{ \}




$$
L_{i j} \geqslant L_{i-1, j}
$$

$$
L_{\hat{j}} \geqslant L_{i, j-1}
$$

Proposition 2.3.1 \{ \}

$$
L_{i j}=\max \left(L_{i, j},-1, L_{i-1, j}\right)+\alpha_{i, j}
$$

This immediately follows from definitions.

Corollary 2.3.2 The space covered by $h_{t}$ at time $t$ is $\left\{(i, j): L_{i, j} \leq t\right\}$.
Corollary 2.3.3 \{ \}


This interpretation explains the name oriented (= directed) last-passage percolation.

Proof 2.3.3 Show that the right-hand side (maximum of sums) satisfies the same recursion as the $L_{i, j}$ 's in [[2.3 Percolation times\#Proposition 23 1 |proposition]].

Remark 2.3.4 In [[2.3 Percolation times\#Corollary $233 \mid$ Corollary 2.3.3]], the paths $\pi$ start at the initial interface $h_{0}$ and ends at $(i, j)$. Note that when $h_{0}$ has a straight part, the path must start at the corner of the initial interface we cannot start the growth from the middle of a straight part.
\{ \}


Remark 2.3.5 There is also first-passage percolation (FPP), in which max is replaces by min. Moreover, by first-passage percolation people usually mean the undirected process, i.e., we do not restrict minimization to directed paths but rather look at paths of certain length $\leq n$. Although the FPP model is related, it is usually not integrable.
-
2.4 Point-to-point directed LPP
\{ \}

$$
L(x, y)=\max _{x+\pi} \sum_{p \in \pi}^{+} \alpha_{p \in \mathbb{Z}^{2}}
$$



$$
\Leftrightarrow \text { growth starting from the comer }
$$

$$
(=\text { "corner growth") }
$$

\{ \}


In fact, the step initial configuration in TASEP is the simplest one to study, and this connection to LPP also motivates to study the step initial configuration.
2.5 Other environment weights
\{ \}
$\alpha_{i j} \in \mathbb{R}$, lid ranelom variables
$\Rightarrow$ LPP / grourtu model is vell-defined
(and has a Limit shape
which we will prove today)

Definition 2.5.1 $\}$

Def. discrete integrable LPP

- with id geometric weights

$$
\text { Geo (q): } \quad \begin{gathered}
P\left(\alpha_{i j}=k\right)=q^{k}(1-g), \quad k=0,1,2, \ldots \\
(0<q<1)
\end{gathered}
$$

Remark 2.5.2 \{ \}

Rework
Geo $(q)$ is the vague family of discrete memorylen r. variables $\in \mathbb{Z}_{\geqslant 0}$.

$$
\begin{aligned}
& P\left(\alpha_{i j} \geqslant k\right)=\sum_{n=k}^{\infty} q^{n}(1-q)=q^{k} \\
& P\left(\alpha_{i j} \geqslant a+b \mid \alpha_{i j} \geqslant a\right)=P \mid \alpha_{i j} \geqslant b
\end{aligned}
$$

Summary of section 2 \{ \}
Conclusion. For any distributions, there is a LPP model.
out of these, there are two "integrable" exacuples:
(1) Exponential weights $\longleftrightarrow$ TASEP in continuous time
(2) Geometric weights $\leftrightarrows$ ?

See [[Problems, 2-8\#2| problem]] about a TASEP with geometric weights. Here is a slight hint:
\{ \}


## 3 Subadditive ergodic theory and limit shape

- 


### 3.1 Subadditivity in LPP

We take weights $\alpha_{i, j} \geq 0$ in the environment, where the weights are:

1. Independent identically distributed
2. Nonnegative
3. Have enough moments, i.e., $\mathbb{E}\left|\alpha_{i, j}\right|^{r}<\infty$ for large enough $r$.

Recall the definition of the point-to-point last-passage time $L(x, y), x, y \in \mathbb{Z}^{2}$.

Remark 3.1.1 Unless $x=\left(x_{1}, x_{2}\right), y=\left(y_{1}, y_{2}\right)$ are in the "up-right position" $\left(x_{1} \leq y_{1}\right.$ and $\left.x_{2} \leq y_{2}\right)$, then clearly $L(x, y)=0$.

Definition 3.1.2. Percolation time along a vector Pick a unit direction $\vec{v} \in \mathbb{R}_{\geq 0}^{2},|\vec{v}|=1$. Define $L(0, n \vec{v})$ to be the last-passage percolation time from 0 to $n \vec{v}$.

Remark 3.1.3 Instead of $n \vec{v}$, we should take the integer part in both coordinates. This is always assumed throughout the lecture. Moreover, some proofs rely on $\vec{v}$ being a rational vector, and more care is needed when one wants to extend the limits to irrational $\vec{v}$ s. Especially this is important when considering invariance under translations by $\vec{v}$. I am not touching these issues for simplicity of the discussion, and refer to Timo Seppalainen's notes (see [[_Lecture 3, $2-8 \#$ Notes and references|ref 3]]) for a more careful analysis.

Proposition 3.1.4 As a function of $n$, the quantities $L(0, n \vec{v})$ satisfy superadditivity:

$$
L(0, n \vec{v}) \geq L(0, m \vec{v})+L(m \vec{v}, n \vec{v}), \quad m \leq n
$$

Proof 3.1.4 The left-hand side maximizes over a larger set of up-right paths. \{ \}


Definition 3.1.5 Now take the averages, and define the following nonrandom quantities: $T(n)=\mathbb{E} L(0, n \vec{v})$. Then $T(n) \geq T(m)+T(n-m)$. Indeed, because the iid environment $\left(\alpha_{i, j}\right)$ is translation-invariant, the expectation of $L(m \vec{v}, n \vec{v})$ is the same as $T(n-m)$.

Thus, the nonrandom sequence $T(n)$ satisfies superadditivity.
Definition 3.1.6 A sequence $\left(a_{n}\right)$ is called superadditive, if $a_{m+n} \geq a_{m}+a_{n}$ for all $m, n$.

A sequence $\left(a_{n}\right)$ is called subadditive, if $a_{m+n} \leq a_{m}+a_{n}$ for all $m, n$.
Remark 3.1.7 Subadditivity and superadditivity are essentially equivalent (take negation of all the elements of the sequence). In last-passage percolation we need superadditivity, but the convergence result we need is traditionally called subadditive ergodic theorem.
-

### 3.2 Limit in expectation

Theorem 3.2.1 The limit $T(n) / n$ exists.

Definition 3.2.2 Assuming the previous theorem, define

$$
\ell(\vec{v})=\lim _{n \rightarrow+\infty} \frac{\mathbb{E} L(0, n \vec{v})}{n}, \quad|\vec{v}|=1 .
$$

Now, extend this to all vectors $\vec{v}=(a, b) \in \mathbb{R}_{>0}^{2}$. Clearly, the resulting function $\ell(\vec{v})=\ell(a, b)$ is homogeneous:

$$
\ell(\gamma a, \gamma b)=\gamma \ell(a, b), \quad \gamma>0
$$

Proof 3.2.1 Theorem follows from the next classical result:
Lemma 3.2.3. Fekete's lemma for superadditive sequences $\}$

$$
\begin{aligned}
& \text { if } a_{n+m} \geqslant a_{n}+a_{m} \text { for all } n, m \\
& \text { Wen exists } \\
& \qquad \operatorname{Lim}_{n \rightarrow \infty} \frac{a_{n}}{n}=\sup _{n} \frac{a n}{n}
\end{aligned}
$$

The proof is left as an [[Problems, 2-8\#3| exercise]].

Remark 3.2.4 The limit in the above lemma can be $+\infty$, consider $a_{n}=\log (n!)$.

Definition 3.2.5. Limiting percolation cluster $\}$

Def.

$$
G:=\left\{(a, b) \in \mathbb{R}^{2} \geqslant 0 \text { s.t. } l(a, b) \leq 1\right\}
$$


3.3 Subadditive ergodic theorem

Definition 3.3.1. Measure preserving transformations \{ \}

$$
\text { Lat }(\Omega, \mp, P) \text { be a paravility pace }
$$

Dat. $T: \Omega \rightarrow \Omega$ meagrable, is measure preserving $3,3.1$

$$
\text { if } P\left(T^{-1} A\right)=P(A) \text { for all } A \in \mathcal{F}
$$

Example 3.3.2. Bernoulli shift $\}$

$$
\begin{aligned}
& \text { Example 3,3,2 }\{0,1\}^{N /} \text { \& bacmord shift } \\
& \text { (Bernoulli shift) } \\
& \left(x_{1}, x_{2}, x_{3}, \ldots\right) \stackrel{\top}{\mapsto}\left(x_{2}, x_{3}, \ldots\right) \\
& \begin{array}{r}
P=\text { Bernoulli project measure } \\
(1 / 2,1 / 2) \text { prob. of } 0 / 1 .
\end{array}
\end{aligned}
$$

Let us discuss this example in more detail. Here $P$, the probability measure, which corresponds to tossing a fair coin infinitely many times.

The preimage is
\{ \}

$$
T^{-1} A=\{(0, a): a \in A\} \cup\{(1, a): a \in A\}
$$

Note that for this map, it is not true that $\mathrm{P}(T A)=\mathrm{P}(A)$ for all $A$. Indeed, if $A=\left\{x_{1}=0\right\}$, then $T A$ is the whole space $\Omega$.

In the Venn diagram form, the map $T$ looks as
\{ \}

[[Problems, 2-8|Problems 4-6]] discuss measure-preserving maps and ergodicity a little further.

Theorem 3.3.3. Subadditive ergodic theorem (in superadditive form) We are giving this theorem in a superadditive form so that it's easier to apply later to last-passage percolation.

Let $g_{n}(x), x \in \Omega$, be a sequence of integrable functions which satisfy superadditivity:

$$
g_{n+m}(x) \geq g_{n}(x)+g_{m}\left(T^{n} x\right), \quad \forall x \in \Omega
$$

Then 1. Almost surely (i.e., for P-almost every $x$ ) there exists a limit

$$
g(x)=\lim _{n \rightarrow+\infty} \frac{g_{n}(x)}{n}
$$

which is $\leq+\infty$. 2. The limiting function is invariant, i.e., $g(T x)=g(x)$ for P-almost every $x \in \Omega$.

Proof 3.3.3 We leave the proof of this result out of this course. However, you are welcome to give a 10 -minute talk on this proof. See [[Problems, 28\#T1|Assignment T1]] and let me know if you're interested.

We also discussed applications of the subadditive ergodic theorem to the lastpassage percolation limit shape, but I am going to repeat this in the next lecture.

## Notes and references

1. The uniform limit law of the second class particle extends to arbitrary product measures with densities $\rho$ to the left of the origin and $\lambda$ to the right of the origin, where $\rho>\lambda$ (then the segment on which the uniform distribution lives should be suitably modified). There are many papers discussing second class particles, and several beautiful exact couplings available, including:
2. P. A. Ferrari. Shock fluctuations in asymmetric simple exclusion. Probab. Theory Related Fields, 91(1), 1992.
3. P. A. Ferrari and C. Kipnis. Second class particles in the rarefaction fan. Ann. Inst. H. Poincare Probab. Statist., 31(1), 1995.
4. P. A. Ferrari and L. P. R. Pimentel. Competition interfaces and second class particles. Ann. Probab., 33(4), 2005.
5. O. Angel, A. Holroyd, and D. Romik. The oriented swap process. Annals of Probability, 37(5):1970-1998, 2009.
6. Subadditive ergodic theorem is originally due to Kingman
7. J. F. C. KINGMAN, The Ergodic Theory of Subadditive Stochastic Processes, J. Roy. Statist. Soc. Ser., Vol. B30, 1968, pp. 499-510.
8. J. F. C. KINGMAN, Subadditive Ergodic Theory, Ann. Probability, Vol. 1, 1973, pp. 883-909.
9. J. F. C. KINGMAN, Subadditive Processes, Lecture Notes in Math., Vol. 539, 1976, pp. 167-223.
10. See also J. MICHAEL STEELE. Kingman's subadditive ergodic theorem Annales de l'I. H. P., section B, tome 25 , no 1 (1989), p. 93-98 (link) for a proof.
11. The proofs of the limit shape results were done without a careful exploration of integer parts. In particular, note that the subadditive ergodic theorem holds for sequences. Therefore, one has to extend the function $\ell$ from integer/rational points to all points. This exercise is done with great care in Seppalainen's lecture notes, Theorem 2.1.
12. First-passage (undirected) percolation is a subject of intense study, too. See, for example, the survey Antonio Auffinger, Michael Damron, Jack Hanson. 50 years of first passage percolation.

## Problems

[[_Lecture 3, 2-8|Lecture 3]]
Due date February 22, solutions posted around that day.

## 1

Show that the $q$-TASEP defined in [[../Lecture 2, 2-3/2.4 Some other particle systems\#q-TASEP|L2]] exists, using a graphical construction similar to [[../Lecture 2, 2-3/2.2 Existence of TASEP\#Proof 221 |the one for TASEP]].

## 2

Describe the Markov chain with discrete time on $\{0,1\}^{\mathbb{Z}}$ which corresponds to the Last Passage Percolation with independent $\operatorname{Geo}(q)$ weights. That is, "discretize" the time in TASEP.

Hint: consider the case of one particle. Which random jumping mechanism corresponds to the waiting time till jump distributed as $\operatorname{Geo}(q)$ ?

## 3

Show that if for a sequence $\left(a_{n}\right)_{n}$ we have $a_{n+m} \geq a_{n}+a_{m}$ for all $n, m$, then there exists

$$
\lim _{n \rightarrow+\infty} \frac{a_{n}}{n}=\sup _{n} \frac{a_{n}}{n}
$$

(this limit may be equal to $+\infty$ ).

## 4

In the setup of [[3.3 Subadditive ergodic theorem|section 3.3]], use the subadditive ergodic theorem to establish the classical Birkhoff theorem. That is, for $T$ an ergodic map, and $f(x)$ an integrable function on $\Omega$ define

$$
S_{n}(x)=\frac{1}{n} \sum_{k=0}^{n-1} f\left(T^{k} x\right), \quad x \in \Omega
$$

So $S_{n}$ is the "time", or "orbital" average. As $n \rightarrow+\infty, S_{n}$ converges to the space ("probabilistic") average,

$$
\lim _{n \rightarrow+\infty} S_{n}=\int_{\Omega} f(x) \mathrm{P}(d x)
$$

Show that

$$
\lim _{n \rightarrow+\infty} \frac{1}{n} \sum_{k=0}^{n-1}\left\{2^{k} x\right\}=\frac{1}{2}
$$

for (Lebesgue-)almost every $x \in[0,1]$, where $\{a\}$ is the fractional part of $a$.

## 6

A number $x \in[0,1$ ) is called normal (in base 2 ), if the number of 1 -s in its base 2 expansion $x=0 . a_{1} a_{2} a_{3} \ldots$ among the first $n$ digits $a_{1}, \ldots, a_{n}$ goes to $\frac{1}{2}$ as $n$ grows. Show that (Lebesgue-)almost all numbers in $[0,1$ ) are normal.

## T1

By Ti I will suggest topics for 10-minute talks, where $i$ is a counter throughout all lectures. This first talk idea is to present the proof of the subadditive ergodic theorem (Theorem 3.3.3 [[3.3 Subadditive ergodic theorem|here]]).
[[Solutions, 2-8]]

## Solutions

[[../../Lecture 3, 2-8/Problems, 2-8|Problems 3]]
[[../../Lecture 3, 2-8/_Lecture 3, 2-8|Lecture 3]]
1
\{ \}
(1) Put countably many Poisson processes at each $x \in \mathbb{Z}$ :
with rates $1,1-q, 1-q, 1-q^{3}, \ldots$
Then when gap $=n$, the particle $x_{i}$ waits for a point in the process with rate $1-q^{n}$ (rate 1 is $1-q^{\infty}$,
 weeded for the right most particle if there is one)
rate $=1-q^{n}$
A subtle point is if the particle $x_{i-1}$ juts first. Then use the strong Markov property to coulude twat the distribution of the waiting tare is wot affected., as $x_{i}$ jugs at an independent time

2
\{ \}
(2)

$$
\begin{aligned}
& \text { Geo }(q) \text { waiting time } \\
& =\text { random umber of } \\
& \text { coin flips }(P(\text { heads })=1-q) \\
& \text { til we dee a heads }
\end{aligned}
$$

$\Rightarrow$ in discrete time TASEP, in
parallel each pastie guys to the right (it it cam) with probab. $1-q$, and stays with proser. q

(3) (Terete's lemma) $a_{m+n} \geqslant a_{n}+a_{n}$

$$
\Rightarrow \lim _{u \rightarrow \infty} \frac{a_{u}}{u} \leq+\infty \text { exists }
$$

Proof Let $M=\sup _{n} \frac{a_{u}}{n}$, choose $n$ sit.

$$
\begin{aligned}
& a_{n}>h(M-\varepsilon) \\
& \text { let } b=\min _{1 \leq i \leq n} a_{i} \\
& \text { For } m \geqslant n: \quad m=q n+r, \text { so } \\
& a_{m} \geqslant q a_{n}+a_{r} \geqslant 4 a_{n}+b \text {, so } \\
& \frac{a_{m}}{m} \geqslant \frac{4 a_{n}}{m}+\frac{b}{m}>\frac{4 n(M-\varepsilon)}{m}+\frac{b}{m}
\end{aligned}
$$

For $m \rightarrow \infty$, this is close to $M-\varepsilon$.

4
\{ \}
(4) Clear, because for

$$
g_{n}(x)=\sum_{0}^{n-1} f\left(T^{k} x\right)
$$

we have

$$
g_{n}(x)+g_{m}\left(T^{n} x\right)=g_{n+m}(T x)
$$

Which is sub / super additive,
So

$$
\frac{g_{n}(x)}{\eta} \rightarrow g(x) \text {, exists }
$$

(aud is an invar. function)
if $T$ is ergodic, $g=$ count.
so, $g(x)=\int_{\Omega} \frac{g_{n}(x)}{n} P(d x)=\int_{\Omega} f(x) P(d x)_{D}$

5
\{ \}
(5) We world like to show that

$$
\left\{2^{k} x\right\}=f\left(T^{k} x\right) \text { for }
$$

some esfedic transformation $T$.
Let $\Omega=[0,1]$ with lebesgue measure
(a probability space), and

$$
T \text { - Bernoulli shift } \quad x \longmapsto\{2 x\}
$$

which is the same as

$$
O_{1} a_{1} a_{2} a_{3} \longmapsto 0, a_{2} a_{3} \ldots
$$

in binary fractions $\quad(a, \in\{0,1\})$
Note: $\quad T^{k} x=\{2\{2 \ldots\{x x\} \ldots\}\}=\left\{2^{k} x\right\}$
$f(x)=x \quad$ (so, $f$ is the uniform on $[0,1]$ random variable)

So, by Birkhoff or swbadd ergodic then,

$$
\frac{1}{n} \sum_{0}^{n-1}\left\{2^{k} x\right\} \rightarrow \int_{0}^{1} f(x) d x=\frac{1}{2}
$$

6
\{ \}

Use notation from Prool. 5
(6) $\#$ of 1 's among $a_{1}, \ldots, a_{n}$

$$
\text { in } 0 \cdot a_{1} \ldots a_{n} \ldots . \text { is }
$$

$$
\begin{array}{r}
\sum_{0}^{n-1} f_{1}\left(T_{x}^{k}\right), \text { where } f_{1}(x)=\text { forst } \\
\text { digitt of } x
\end{array}
$$

So, $\frac{1}{n} \sum_{0}^{n-1} f_{1}\left(T^{k} x\right) \underset{\text { a,s. }}{\longrightarrow} \int_{0}^{1} f_{1}(x) d x$

$$
=\frac{1}{2} \cdot 0+\frac{1}{2} \cdot 1=\frac{1}{2}
$$

$\Rightarrow$ aluost all urexbers
are norwal).
[[Problems, 2-10|2 problems]], due 2/24

## 1 Remark / correction

- 


## 1 Remark correction in LPP

\{ \}


$$
\begin{aligned}
L(a, b) & =\max _{\pi ; h_{0}-(a, b)} \sum_{(i j) \in \pi} \omega_{i j} \\
& \omega_{i}
\end{aligned}
$$

Let me clarify that the last passage percolation paths need to end at corners of the initial interface $h_{0}$ (orange in the picture), and not at arbitrary points of $h_{0}$. This is consistent with how the random growth proceeds.

## 2 Subadditive ergodic theorem

Recall the material from [[../Lecture 3, 2-8/ _Lecture 3, 2-8|L3]], and add more details. I am going to report the notes from L 4 below, but there is significant overlap with the L3 material.
2.1 Measure preserving transformations Definition 2.1.1 \{ \}

Def.

$$
\begin{array}{cc}
(\Omega, F, P) & P(\Omega)=1 \\
T: \Omega \rightarrow \Omega \quad \begin{array}{c}
\text { meas warble (lot meyer })
\end{array} \\
\text { wees pres if } & P\left(T^{-1} A\right)=P(A) \\
\forall A \in F
\end{array}
$$

Example 2.1.2 \{ \}

$$
\begin{aligned}
& \text { Ex, } \frac{\text { Rotation }}{T_{\alpha} x=S^{1}, \quad P=\text { woruatized }} \begin{array}{c}
T_{\alpha} x \\
T_{\alpha}^{\alpha} x
\end{array} \\
& \text { Six } \alpha \in[0,2 \pi) \\
& P(A)=P(T A) \\
& =P\left(T_{\alpha}^{-1} A\right)
\end{aligned}
$$

Example 2.1.3 \{ \}


This is called the (one-sided) Bernoulli shift. There are several important aspects of this construction. First, this is a way to look at an independent sequence of coin flips from the measure-theoretic perspective, by introducing the measure space $\{0,1\}^{\mathbb{Z}_{\geq 1}}$ with product measure. Then the sequence of independent coin flips becomes the sequence of functions $\left(x_{1}, x_{2}, \ldots\right) \mapsto x_{i}$.
Then, the Bernoulli shift $T$ is an example of a not one-to-one measure preserving transformation.

Finally, the Bernoulli shift is a model of the shift by a vector that is used in the LPP limit shape result.

Definition 2.1.4. Ergodicity of a measure preserving map $\}$
Def. Ergodic $\frac{\text { transformations }}{\text { if all immanent sets }} \quad A \quad(T A=A)$ have measme, 0 or 1.
Equiv., $\forall f$ on $\Omega, f(T x)=f(x) \forall x \Rightarrow \begin{aligned} & f=\text { cost. } \\ & p-a . e .\end{aligned}$

See also [[Problems, 2-10\#2|Problem 2]] related to ergodicity of the rotation of the circle.

Example 2.1.5 \{ \}

$$
\begin{aligned}
& \text { bernoulli: shift }
\end{aligned}
$$

$$
\begin{aligned}
& \left.\begin{array}{l}
\text { enough to check for } \\
\text { ns simple funct". for first (dep. on. Fin. .nay } \\
\text { coordinates }
\end{array}\right)
\end{aligned}
$$

Here we use the structure of the product $\sigma$-algebra on the countable product space. Namely, any function measurable with respect to this $\sigma$-algebra (in plain terms, any function depending on the id coin flips $x_{i}$ ) is a limit of functions which depend on finitely many coordinates. Therefore, it essentially is enough to check the condition for invariant functions of finitely many coordinates.

Proposition 2.1.6. Kolmogorov 0-1 law This is [[Problems, 2-10\#1| Problem 1 from this lecture]].
\{ \}

Prop.(Ex. L4-1) kolmogoros 0-1 law
Let $X_{1}, X_{2}, \ldots$ be independent random variables, and $\mathcal{T}$ be the tail $\sigma$-algebra:

$$
\mathcal{T}=\bigcap_{n=1}^{\infty} \sigma\left(X_{n}, X_{n+1}, X_{n+2}, \ldots\right) . \quad \text { tail b-algebra }
$$

In words, $\mathcal{T}$ consists of events which are independent of any finite subcollection of the $X_{j}$ 's. Then, if $A \in \mathcal{T}$, then $\mathrm{P}(A)=0$ or 1 .

$$
P(A)^{2}=P(A)
$$

ex. $\left\{\exists \underset{i \rightarrow \infty}{\lim } X_{i}\right\} \in[$

Proof 2.1.6 (sketch) \{ \}

$$
\begin{aligned}
& \text { wts } P(A \cap A)=P(A) \\
& \rightarrow b\left(x_{1} \ldots x_{n}\right), \sigma\left(x_{n+1}, \ldots\right) \text { indep. } \\
& \rightarrow b\left(x_{1}, \ldots, x_{n}\right), \quad \tau \quad \text { indep. } \\
& \rightarrow b\left(U_{n} b\left(x_{1} \ldots x_{n}\right)\right), \quad \tau \text { indep. }
\end{aligned}
$$

$$
A \in I \text { but also } A \in E\left(\bigcup_{n} \sigma\left(x_{1}, \ldots, x_{n}\right)\right)
$$

$$
\text { because } T \text { is meas wrt }\left\{x_{i}\right\}
$$

2.2 Ergodic theorem

Let
\{ \}
$g_{n}$ :- insegrable functions on $\Omega$

$$
\begin{aligned}
& \text { satiffying superqelditivity } \\
& g_{h+m}(x) \geqslant g_{n}(x)+g_{m}\left(T_{x}^{n}\right) \quad x \in \Omega
\end{aligned}
$$

Theorem 2.2.1. Superadditive ergodic theorem \{ \}

$$
\begin{aligned}
& \text { Them Almost surely (i.e with P-a.e } x \text { ) } \\
& \text { there exists a limit }
\end{aligned}
$$

$$
\begin{aligned}
& \lim _{n \rightarrow \infty} \frac{g_{n}(x)}{n}=g(x) \quad(\leqslant+\infty) \\
& \text { and } g(x) \text { is an insaviaut function } \\
& \text { hie. } g(T x)=g(x) \quad \text { (almost surely) }
\end{aligned}
$$

As discussed before, the proof of this result is not given in the course, and is left as an additional readong.

Corollary 2.2.2. The classical Birkhoff ergodic theorem Let $f$ be an integrable function on $\Omega$, and define

$$
g_{n}(x)=\sum_{k=0}^{n-1} f\left(T^{k} x\right)
$$

Then clearly

$$
g_{n}(x)+g_{m}\left(T^{n} x\right)=g_{m+n}(x)
$$

which means that $g$ satisfies subadditivity / superadditivity. Then

$$
\lim _{n \rightarrow+\infty} \frac{1}{n} g_{n}(x) \rightarrow g(x)
$$

Moreover, because $T$ is measure preserving, in case of ergodic $T$ we have (by taking averages with respect to P ):

$$
\frac{1}{n} \int g_{n}(x) \mathrm{P}(d x)=\int f(x) \mathrm{P}(d x) \quad \Rightarrow \quad \frac{1}{n} g_{n}(x) \rightarrow \int f(x) \mathrm{P}(d x)
$$

Corollary 2.2.3. Law of large numbers For the Bernoulli shift, we can define $f\left(x_{1}, x_{2}, \ldots\right)=x_{1}$. Then $\$ \mathrm{~g} \_\mathrm{n}(\mathrm{x})=\mathrm{x} \_1+\ldots+\mathrm{x} \_\mathrm{n} \$$, and the result says

$$
\frac{x_{1}+\ldots+x_{n}}{n} \rightarrow \int x_{1} \mathrm{P}(d x)=\frac{1}{2} .
$$

In fact, the same results holds for any id sequence of random variables $X_{n}$, $n \geq 1$, where we define the shift $T$ in the same way. This result

$$
\frac{X_{1}+\ldots+X_{n}}{n} \rightarrow \mathbb{E}\left(X_{1}\right) \quad \text { almost surely }
$$

is called the (strong) law of large numbers.

## 3 Application to limit shapes

- 


### 3.1 Setup

\{ \}

$$
\begin{aligned}
& \text { sid rv } \\
& S=\left\{\alpha_{i j}^{\mathbb{Z}} ; \quad i, j \in \mathbb{Z}_{\geqslant 1}\right\} \\
& P \text { - proket measure (so that } \alpha_{i j} \text { iss) } \\
& \begin{array}{l}
\sqrt{\left.|E| \alpha_{i j}\right|^{P}<\infty} \text { large enough } P \\
\text { not a necessary covelition } \\
\text { but wto it } l \text { may be intimate }
\end{array}
\end{aligned}
$$

### 3.2 Limit shape

Definition 3.2.1 \{\}

Theorem 3.2.2 \{ \}

$$
\begin{aligned}
& \text { Then } \quad \forall(a, b) \in \mathbb{R}_{\geq 0}^{2} \\
& \text { 1) } \ell(a, b) \text { exists }(\leq+\infty)
\end{aligned}
$$

2) is deterministic
if $l(a, b)<\infty$ :
3) coincides with

$$
\operatorname{Lim}_{n \rightarrow \infty}{\underset{n}{n}}^{\mathbb{E} L(0,(n a), \operatorname{Ln} b]))}
$$

Proof 3.2.2 We assume that $\vec{v}=(a, b)$ is an integer vector. Using homogeneity, this allows to extend the result to all rational points. Extending to irrational points is a standard limiting argument.
Let $T$ be the shift by $\vec{v}$. This is like the Bernoulli shift, but along a direction in $\mathbb{Z}^{2}$.

Recall that $L(0, n \vec{v})$ is the last-passage time from 0 to $n \vec{v}$.
Then $L$ satisfies superadditivity:
\{ \}


So:
\{ \}
$\begin{aligned} \Rightarrow & \text { Erg, tu. shows } \quad l(a, b)=\frac{1}{n} \lim _{u} L(0,(n a, n b)) \\ & l \text {-invar. vader } T \quad \text { exists }\end{aligned}$

Finally, $\ell$ is nonrandom because of the ergodicity of $T$, which follows from Kolmogorov 0-1 law.

Definition 3.2.3 Recall the notion of the scaled percolation cluster $G$ :
\{ \}


### 3.3 Properties of the limit shape function

Theorem 3.3.1 The last passage limit shape function $\ell(a, b)$ satisfies the following properties: 1. $\ell$ is homogeneous, symmetric in $a, b$, and nondecreasing in both arguments 2. Either $\ell=+\infty$ or $\ell<\infty$ for all points 3 . $\ell$ is superadditive: $\ell(p+q) \geq \ell(p)+\ell(q)$ for all $p, q \in \mathbb{R}_{\geq 0}^{2} 4$. $\ell$ is concave: $\gamma \ell(p)+(1-\gamma) \ell(q) \leq$ $\ell(\gamma p+(1-\gamma) q), \gamma \in[0,1] 5$. $\ell$ is continuous

## Proof 3.3.1

1. Is straightforward
2. If $\ell=+\infty$ at some point, then it is infinite in the whole quadrant. Moreover, the other points can be shifted (using homogeneity) into this quadrant.
\{ \}

3. Superadditivity follows from the same prelimit statement:
\{ \}
$n^{-1} L(0, n \vec{v})+n^{-1} L(n \vec{V}, n(\vec{v}+\vec{w})) \leq n^{-1} L(0, n(\vec{v}+\vec{w}))$

4. Superadditivity + homogeneity implies concavity:

$$
\gamma \ell(p)+(1-\gamma) \ell(q)=\ell(\gamma p)+\ell((1-\gamma) p) \leq \ell(\gamma p+(1-\gamma) q)
$$

5. Finite concave functions are continuous.

Corollary 3.3.2 Therefore, the boundary of the limit shape cluster looks as follows:
\{ \}


It is concave, continuous, and may contain straight or curved pieces.

### 3.4 Explicit limit shapes

Theorem 3.4.1 For exponental weights with mean 1, we have:
\{ \}


This is the piece of the parabola which is tangent to both axes at unit points.
We will prove this theorem eventually in the course.

Theorem 3.4.2 For geometric weights $\in \mathbb{Z}_{\geq 0}$ with parameter $q \in(0,1)$, we have:

$$
\ell(a, b)=b\left(\frac{\left(\sqrt{\frac{a q}{b}}+1\right)^{2}}{1-q}-1\right)
$$

\{ \}

$$
G=\{(a, b): l(a, b) \leq 1\} \rightarrow \text { spewed parched } a
$$


(note that this function is symmetric in $a, b$ )
We will prove this theorem eventually in the course.
Remark 3.4.3 \{ \}
A more symmetric answer is

$$
\begin{aligned}
& \text { for geo }(q)+1, \\
& \text { distr. } q^{k-1}(1-q), k \in \mathbb{Z}_{\geqslant 1} \\
& \text { then } l^{*}(a, b)=l(a, b)+a+b
\end{aligned}
$$



Open problem 3.4.4
The case of any other iid weights is wide open. Namely, we don't know any other explicit limit shapes in the last passage percolation model.

### 3.5 From LPP to TASEP limiting density

If we believe in the parabola limit shape ([[3.4 Explicit limit shapes\#Theorem 3 $41]]$ ), we can derive the limit shape of the TASEP density.
Moreover, there is a general relation between the height and the density.
Let $H(t, x), t \in \mathbb{R}_{\geq 0}, x \in \mathbb{Z}_{\geq 0}$, be the finite-time height function of TASEP. Define

$$
h_{t}(x)=\lim _{R \rightarrow+\infty} H(R t,\lfloor R x\rfloor),
$$

and similarly define the limiting density $\rho(t, x)$.

Lemma 3.5.1 \{ \}


Proof 3.5.1 Straightforward from the picture
\{ \}


Corollary 3.5.2 Modulo [[3.4 Explicit limit shapes\#Theorem 341$]$ ], the limiting density of TASEP started from the step initial configuration is

$$
\rho(t, x)=\frac{1}{2}-\frac{x}{2 t} .
$$

Proof 3.5.2 This is obtained by scaling the parabola by $t$, and differentiating as in [[\#Lemma 3 5 1 1]].

Here is an illustration of the density's evolution:
\{ \}


4 Heuristic hydrodynamics of TASEP
-
4 Heuristic hydrodynamics of TASEP
Let us begin by formulating principles on which the hydrodynamics approach is based.
\{ \}
(1) Local distrib. should be invar.
under finite time TA SEP
evolution
(2) distrib. is transl invariant
(under shifts at 2)
\{ \}


Definition 4.0.1 \{ \}


Propopsition 4.0.2 \{ \}

$$
\text { Prop TASEP preserves Ber ( } \rho \text { ) }
$$

Proof 4.0.2 (sketch) \{ \}


Proposition 4.0.3 \{ \} ~


Proof 4.0.3 Straightforward.

Continuity equation for the TASEP limiting density $\}$


Proposition 4.0.4 The density $\rho(t, x)=\frac{1}{2}-\frac{x}{2 t}$ solves the Burgers equation

$$
\frac{\partial}{\partial t} \rho(t, x)+\frac{\partial}{\partial x}(\rho(t, x)(1-\rho(t, x)))=0
$$

Proof 4.0.4 Straightforward check.

## Notes and references

1. Limit shape theorem for TASEP using hydrodynamics is established in Rost, H. Non-equilibrium behaviour of a many particle process: Density profile and local equilibria. Z. Wahrscheinlichkeitstheorie verw Gebiete 58, 41-53 (1981), https://doi.org/10.1007/BF00536194. This paper also essentially shows that the limit shape of the exponential last passage percolation is a parabola.
2. Explicit limit shape for last passage percolation with geometric weights can be found in K. Johansson. Shape fluctuations and random matrices. Commun. Math. Phys., 209(2):437-476, 2000, arXiv:math/9903134 [math.CO]. However, this is not the first place where this limit shape was obtained. The original references are:
3. H. Cohn, N. Elkies, and J. Propp. Local statistics for random domino tilings of the Aztec diamond. Duke Math. J., 85(1):117-166, 1996. arXiv:math/0008243 [math.CO].
4. W. Jockusch, J. Propp, and P. Shor. Random domino tilings and the arctic circle theorem. arXiv preprint, 1998. arXiv:math/9801068 [math.CO].
5. T. Seppäläinen. Hydrodynamic scaling, convex duality and asymptotic shapes of growth models. Markov Process. Related Fields, 4(1):1-26, 1998.
6. A more careful exposition of the last passage percolation limit shape result is found in Seppalainen's lecture notes, Theorem 2.1.

## Problems

[[_Lecture 4, 2-10|Lecture 4]]

## 1

Prove or find in the literature the proof of the Kolmogorov 0-1 law. That is, let $X_{1}, X_{2}, \ldots$ be independent random variables, and $\mathcal{T}$ be the tail $\sigma$-algebra:

$$
\mathcal{T}=\bigcap_{n=1}^{\infty} \sigma\left(X_{n}, X_{n+1}, X_{n+2}, \ldots\right)
$$

In words, $\mathcal{T}$ consists of events which are independent of any finite subcollection of the $X_{j}$ 's. Then, if $A \in \mathcal{T}$, then $\mathrm{P}(A)=0$ or 1 .

## 2

Let $(\Omega, \mathcal{F}, \mathrm{P})$ be the unit circle with the normalized length measure. Let $T_{\alpha}$ be the counterclockwise rotation by the angle $\alpha$. 1 . Show that $T_{\alpha}$ is measurepreserving 2. Show that for $\alpha$ irrational multiples of $2 \pi, T_{\alpha}$ is ergodic 3 . Is $T_{\alpha}$ ergodic for $\alpha$ rational multiples of $2 \pi$ ?
\{ \}

[[Solutions, 2-10|Solutions]]

## Solutions

[[../../Lecture 4, 2-10/Problems, 2-10|Problems 4]]
[[../../Lecture 4, 2-10/_Lecture 4, 2-10|Lecture 4]]

1
\{ \}
(1) Let $A \in \tau \rightarrow$

$$
\begin{aligned}
A & =\sigma\left(x_{1}, x_{2}, \ldots\right)= \\
& =b(\sum_{n=1}^{\infty} \underbrace{\infty}_{\text {the term } b\left(x_{1}, \ldots\right.}, x_{n}))
\end{aligned}
$$

$$
\mathbb{H}
$$

$$
\exists A_{n} \in \zeta\left(x_{1}, \ldots, x_{n}\right)
$$

approx. A inture seuse

Is the $b$-ars. gever.
by these r.v.'s
(ie. the suallest $z$-als with respect to which twese r.r are
neasurable)
\{ \}

$$
\begin{aligned}
& P\left(A \Delta A_{n}\right) \rightarrow 0, n \rightarrow \infty \\
& \Rightarrow P\left(A_{n}\right) \rightarrow P(A), \quad P\left(A_{n} \cap A\right) \rightarrow P(A)
\end{aligned}
$$

But $A \in \tau \Rightarrow A$ is indies of $A_{m}$, so

$$
P\left(A_{n} \cap A\right)=P\left(A_{n} \mid P(A) \rightarrow P(A)\right.
$$

So $P(A)=P(A)^{2}$, as desired
(Note: Kolmgorow 0-1 law is for inter. rv.
tube is a generalization for aid riv, where we con have there general exchangeable events $\hat{j}$ their prob. is also Dor 1)

2
Parts 1 and 3
\{ \}
(2) 1) $T_{\alpha}$ is a motion of
space $\Rightarrow$ preserves lengths and hence the Nearbure
(Dote, Th a is one to one)
3) No, for example fer $\alpha=\pi$,


Hence, any centrally symmetric set is invariant under $T_{\alpha}$, ad it is not necessarily of measure $O$ or 1 .

Part 2
\{ \}
2) Step 1. $\{n+\}_{n \geqslant 1}$ is cause in $S^{1}$

Step 2. Any invariant $L^{1}$ function $f$ is $\varepsilon$-close in worm to a cont. $f_{\varepsilon}$,

$$
\left\|f_{2}-f\right\|_{L_{1}}<\varepsilon
$$

Step 3. If $f$ is invariant,

$$
\begin{gathered}
f\left(T_{\alpha} x\right)=f(x) \quad \text { a.e. } x \\
\Downarrow \\
\left\|T_{\alpha} f_{\varepsilon}-f_{\varepsilon}\right\|_{L^{\prime}}<2 \varepsilon
\end{gathered}
$$

similarly

$$
\left\|T_{\alpha}^{n} f_{2}-f_{\varepsilon}\right\|_{L} 1<2 \varepsilon
$$

\{ \}

$$
\Rightarrow \| f_{\varepsilon}(\underbrace{}_{\text {addition wat }}(+t)-f^{1}(\cdot) \|_{2^{1}}<2 \varepsilon \quad \forall t \in S^{1}
$$

become \{nd\} ~ i s ~ d u s e ~

$$
\begin{aligned}
& \text { Step } 4 . \\
& \left\|f_{\varepsilon}-\int_{0}^{2 \pi} f_{\varepsilon}(t) \cdot d t\right\|_{L^{1}}= \\
& \Rightarrow \int\left|\int\left(f_{\varepsilon}(x)-f_{\varepsilon}(x+t)\right) d x\right| d t \leqslant \\
& \leq \iint\left|f(x)-f_{\varepsilon}(x+t)\right| d x d t \leq 2 \varepsilon-
\end{aligned}
$$

\{\}
$\Rightarrow f_{\varepsilon}$ is $\varepsilon$-constant in $L^{1}$ norm
$\Rightarrow f$ is constant in $L^{1}$ norm $(\varepsilon \rightarrow 0)$
so $f=$ court. a.e. :

$$
\begin{aligned}
\|f-c\|_{L^{\prime}}<\varepsilon \quad \forall \varepsilon & \Rightarrow \int|f-c|=0 \\
& \Rightarrow f=c \text { a.e. }
\end{aligned}
$$

[[Problems, 2-15|3 problems $+\mathrm{T} 2+\mathrm{T} 3]$ ], due $3 / 1$
1 Hydrodynamics of TASEP
1.1 Burgers equation

Definition 1.1.1 \{ \}
Def. Liuctivy density $\rho(x, t) \quad x \in \mathbb{R}$ $h_{t}(x)_{x \in \mathbb{Z}}$ finite height function $H_{t}(x)=\lim _{L \rightarrow \infty} \frac{1}{L} h_{L t}(L L x J)$
exists by subadditivity Lipschitz

$$
\rho(t, x)=\frac{1}{2}\left(1-\frac{\partial}{\partial x} \mathcal{H}_{t}(x)\right) \quad \text { density } \quad \text { (if exists) }
$$

Definition 1.1.2 Alternatively, $\rho(t, x)$ can be defined as \{ \}
(if the limit exists)

Claim 1.1.3 \{ \}
$\rho$ satisfies Burgers equation

$$
\frac{\partial}{\partial t} \rho(t, x)+\frac{\partial}{\partial x}(\rho(t, x)[1-\rho(t, x)])=0
$$

Heuristics supporting Claim 1.1.3 \{\}
(1)? Locally around $x$, distribution is $\operatorname{Ber}(\rho(t, x))=$ Berwoulli prodect wearure
(2) (porve belou) Ber $(\rho)$ is stationary under TASEP (s trauslation ) invariant)
\{ \}
(3) Speed of $\operatorname{Ber}(\rho)$ is $j(\rho)=\rho(1-\rho)$ avg $\#$ of particles crossing a given eelge
(4) Contimuity equation is $\frac{\partial}{\partial t} \rho+\frac{\partial}{\partial x} j(\rho)=0$

$$
\frac{\partial}{\partial x} j(\rho) \leftrightarrow j(\rho(t, x+d x))-j(\rho(t, x))
$$

\{ \}


Application to step initial condition $\}$


At $t=0$, the density is not differentiable at 0 . However, $\rho(t, x)=\frac{1}{2}-\frac{x}{2 t}$ has pointwise limit to the step function $\rho(0, x)$ as $t \rightarrow 0$ (for all $x \neq 0$ ).
The Burgers equation with this initial condition does not have a unique solutimon. To select this particular $\rho(t, x)$, additional conditions are needed, such as
conservation of particles, or "entropy" solution requirement.
1.2 Stationarity of Bernoulli product measures

Definition 1.2.1 Let $\operatorname{Ber}(\rho)$ denote the Bernoulli product measure on $\{0,1\}^{\mathbb{Z}}$.
Proposition 1.2.2 For each $\rho \in[0,1]$, the measure $\operatorname{Ber}(\rho)$ is stationary under the TASEP evolution.

Lemma 1.2.3. Single site. \{ \}


$$
\operatorname{prob}\left(\frac{1}{x}\right)=\rho
$$

'Burke property
\& incoming parties at this location $\sim$ Poisson process $(\beta)$
indef of $x$
\{ \}
then at any time $t>0, \operatorname{prob}\binom{x}{x}=\rho$ stays \& the outgoing process is Poisson $(\beta)$, indep. of port at $x$.

Proof 1.2.3 \{ \}

Proof, Let the coming proc. be Poiss ( $\alpha$ ), outgoing Poiss $(\beta$ [more general]

In time $d t$, $\operatorname{prob}(0 .$.$) is$

$$
\frac{0}{0}=(1-\rho)[1-\alpha d t]
$$

\{ \}
similarly:

$$
\begin{aligned}
& \xrightarrow{0} \xrightarrow{0} \quad \rho \cdot \beta d t \\
& \cdots \cdots \rightarrow_{0}^{0} \cdots \\
& \rho(1-\beta d t)+(1-\rho) \alpha d t \\
& \rightarrow \quad 0 \quad O\left(d t^{2}\right)
\end{aligned}
$$

\{ \}

-...3) $\rho+d t[\beta-\rho \beta-\beta]=[\rho \underbrace{[1-\beta d t]}$
This implies ivelepandence for all $\rho, \beta$.

## Lemma 1.2.4. Process on the half-line \{ \}


ie: $\forall \mathrm{Hm}$, Doiss $(p)$ of imps

$$
\text { \& cudep. of sites from } n \text { to } m
$$

Remark 1.2.5 In fact, [[\#Lemma 123 Single site|Lemma 1.2.3]] is not necessary, and we discussed it just for an illustration.

Proof 1.2.4 We prove this by induction on $n$. That is, it suffices to look at a single site.
\{ \}

$\Rightarrow$ get independence \& $\operatorname{Ber}(\rho)$ on $\mathbb{Z}_{\geqslant n}$ preserved.

Proof 1.2.2 \{ \}

Proof of Prop. (Berg $\rho$ ) on all $Z$ is stat. for TASPD) Take measure $j_{n} \leftrightarrow$ TASEP evol. on $\operatorname{Ber}(\rho)$ an $\mathbb{Z}_{\geqslant-n}$
Lem shows tres these meas are compatible for different $n$

$\Rightarrow \exists \lim$ as $n \rightarrow+\infty$
\{ \}

```
Under this limit, meas. is \(\operatorname{Ber}(\rho)\) on \(\mathbb{Z}\)
    \& evolution is TASEP.
```

Remark 1.2.6 This technique of constructing stationary processes works for most integrable systems, including stochastic six vertex model and random polymers.
-

### 1.3 Stationarity of geometric LPP

In the last-passage percolation model with geometric weights, its stationary version (meaning in a sense the full-plane LPP) may be constructed by adding certain carefully selected boundary weights.
\{ \}


$$
\begin{aligned}
& P\left(y_{i, 0}=k\right)=\frac{p-r}{1-r}\left(\frac{1-p}{1-r}\right)^{k} \quad k \geqslant 0 \quad \text { (bor) } \\
& P\left(y_{0, j}=k\right)=r(1-r)^{k} \quad k \geqslant 0 \quad \text { (vert) }
\end{aligned}
$$

The stationarity of this model is checked in [[Problems, 2-15\#1|Problem 1]] and [[Problems, 2-15\#2| Problem 2]].

2 Liggett's characterization of invariant measures
2.1 Coupled process

Definition 2.1.1. Basic coupling Having 2 TASERs $\eta_{t}$ and $\zeta_{t}$ on $\mathbb{Z}$, we couple them into a single process $\left(\eta_{t}, \zeta_{t}\right)$, such that the processes use the same exponential clocks attached to vertices of $\mathbb{Z}$. In particular, once two particles glue:
\{ \}

then they stay glued together forever
Definition 2.1.2. Ordering of states $\}$

$$
\begin{gathered}
3, \eta \in\{0,1\}^{\mathbb{Z}} \quad 3 \geqslant \eta \text { if } x \in \eta \Rightarrow x \in \zeta \\
\text { fer all } x \in \mathbb{Z} \\
\qquad 0-3
\end{gathered}
$$

Definition 2.1.3. Ordering of measures We say that $\mu_{1} \leq \mu_{2}$ if there exists a probability measure on $\{0,1\}^{\mathbb{Z}} \times\{0,1\}^{\mathbb{Z}}$ with marginals $\mu_{1}, \mu_{2}$, under which $\eta \leq \zeta$ with probability one (where $(\eta, \zeta)$ is the configuration on $\mathbb{Z} \sqcup \mathbb{Z}$ ). We refer to a measure with prescribed marginals as a "coupling".

Lemma 2.1.4 \{ \}
Lemma. If $z_{0} \geqslant \eta_{0} \quad 3_{0}=\eta_{0} \quad \zeta_{0} \leqslant \eta_{0}$
at time 0
$\Rightarrow$ these hold for all $t$

$$
\left(3_{t} \geqslant \eta_{t}, \quad 3_{t} \geqslant \eta_{t}, \quad 3_{t} \leq \eta_{t}, \text { app. }\right)
$$

The proof is straightforward.
2.2 Properties of the coupled process

Definition 2.2.1. Stationary, translation invariant, extreme $\}$

$$
\begin{gathered}
\frac{\text { Stationary probab. seas on }\{0,1\}^{\mathbb{Z}}\left(\times\{0,1\}^{Z}\right)}{\text { T }} \text { ( meas dols not change } \\
\text { under time evolution } \\
\text { of TA SEP }
\end{gathered}
$$

Transl. inv probab. seas on $\{0,1\}^{\mathbb{Z}}(\times\{0,1\})$

$$
A \subseteq \mathbb{Z} \cup \mathbb{Z}
$$

$$
\begin{aligned}
& A \subseteq \mathbb{Z} \\
& \mu(A)=p^{n}(A+1)=\mu^{u}(A-1)
\end{aligned}
$$

\{ \}


Example 2.2.2. Some convex sets $\}$


Proposition 2.2.3 \{ \}

$$
\begin{aligned}
& \text { Prop If } v \text { is extreme, stationery for coupled proc } \\
& \text { Then } \nu(J=\eta), \nu(\eta \geqslant 3), \nu(\eta \leq 3) \\
& \text { are either } 0 \text { or } 1 \text {. }
\end{aligned}
$$

Remark 2.2.4 Moreover, under an extreme stationary measure, every invariant set has probability 0 or 1 .

Proof 2.2.3 \{ \}

$$
\begin{aligned}
& \text { Proof. Let } \quad 0<\widehat{\nu(B)<1,} \quad B=\{3=\eta\} . \\
& \Rightarrow \nu(0)=p \nu(0 \mid B)+(1-p) \nu\left(0 \mid B^{c}\right) \\
& \text { Now } \nu(0 \mid B) \text { is stat. wears because } B \text { is an } \\
& \text { meas. for } \nu . \\
& \text { Also } \nu\left(0 \mid B^{c}\right) \text { - stat meas. } \\
& \Rightarrow \quad p=0 \text { or } 1 \text { because } \nu \quad \text { is extreme. }
\end{aligned}
$$

### 2.3 Ordering of measures

Proposition 2.3.1. Main proposition $\}$

$$
\begin{aligned}
& \text { Prop. Let jus, jus be } 2 \text { extreme, stationcoy, } \\
& \text { transl. - moniont. } \\
& \text { Then either } \mid u_{1} \leq \int u_{2} \text { or } j x_{1} \geqslant j u_{2}
\end{aligned}
$$

Proof 2.3.1

Step 0 \{ $\}$

$$
\begin{aligned}
& \text { Choose au } v 0,1\}^{2} x\{0,1\}^{\mathbb{2}} \\
& \text { with /marginals } \mu_{1}, j_{2} \text {. How? } \\
& \text { stat., trausl-inv, exticure }
\end{aligned}
$$

Step 1 \{
Take $\nu_{1}=\mu_{1} \times \mu_{2}$, rum TASEP from $\nu_{1}$

$$
\Rightarrow v_{1}^{(t)} \text { have marg. } j^{u_{1}}, j^{u_{2}} \not \forall t
$$

Space $\{0,1\}^{2} x\{0,1\}^{2}$ is complect $\Rightarrow \exists t_{n} \rightarrow \infty$ s.t. $\frac{1}{t_{n}} \int_{0}^{t_{n}} v_{1}^{(t)} d t \rightarrow v, n \rightarrow \infty$

So $r$ is stationary (and clearly tr. musari)

Step $2\}$
(2) Let $f=\{V$ - stat. Ir.inver. with wrgivinals $\left.j_{1}, \mu_{2}\right\}$
At - compact, convex, wompty lay (1)

$$
\Rightarrow A_{\text {ext }} \neq \varnothing
$$

Let $v \in A_{\text {ext }}$
Kelt - Millman them
A compact convex subset of a Hausdorff locally convex topological vector space is equal to the closed convex hull of its extreme points.
We wand to show that $V$ is also extreanal for $\left\{\begin{array}{c}\text { all measures., stat. } \\ \text { under the }\end{array}\right.$ Under the
coupled TASEP $\}$
\{ \}

Let

$$
\nu=\gamma \alpha+(1-\gamma) \beta \quad 0<\gamma<\perp
$$

$\alpha_{1, \beta-s t a t . ~ t r . ~ t u v e r i a n t ~ f e e f ~ c o u p l e d ~ T A S E P ~}^{\text {and }}$ because $j_{1}, \mu_{2}$ cire extrenal, $\alpha_{1} \beta \in A$
$\Rightarrow V=\alpha=\beta$ because $\nu$ was in fextr.

Step 3 \{ \}

$$
\begin{aligned}
& \text { (3) So now } V \sim \text { extreme, stat, tr. invar. } \\
& \text { for the coupled process, } \\
& \text { with marginals } \mu_{1}, \mu_{2} .
\end{aligned}
$$

$$
\begin{aligned}
& \text { We whom } \\
& \qquad(\zeta \leq \eta \text { or } z \geqslant \eta)=0 \text { or } 1 . \\
& 3, \eta \in\{0,1\}^{\mathbb{Z}} \quad(\text { wort } 1)
\end{aligned}
$$

It suffices to check that:
\{ \}
$\forall x$ :


This is where I am omitting some smaller details, and refer to Lemmas 2.4, 2.5, and 3.1 in Liggett's 1976 paper (see [[_Lecture 5, 2-15\#Notes and references|references]]). In broad terms, that these configurations have zero probability makes sense because $\nu$ is stationary. Each of these configurations has a nonzero chance of gluing the particles together (to a state $(0,1)$ ), and because $\nu$ is stationary, this gluing should have already happened.

Step 4 So, for a pair of extreme, translation invariant, stationary measures $\mu_{1}, \mu_{2}$ we have constructed a coupling $\nu$ which is itself extreme, translation invariant, stationary for the coupled process. Moreover, we have shown that

$$
\nu(\zeta \leq \eta \text { or } \zeta \geq \eta)=1
$$

which means that $\mu_{1} \leq \mu_{2}$ or $\mu_{1} \geq \mu_{2}$. This proves the proposition.

Remark 2.3.2 The above proposition means that the extreme, translation invariant, stationary measures form a totally ordered set, which suggests that they are indeed indexed by a single parameter.
2.4 Liggett's theorem

Theorem 2.4.1 \{ \}

Thu. Let $\mu$ be transl. invar., stationary, extreme for TASEP en $\mathbb{Z}$ then there exists $\mathrm{O} \in[0,1]$

$$
\text { Sit } y=\operatorname{Ber}(\rho) \text { prochut measure }
$$

Proof 2.4.1 \{ \}

$$
\text { Per }(\rho) \text { is transl. invar. \& stationary }
$$

Also it is extreme

$$
(E x, L 5-3)
$$

\{ \}

So for all product $B \in(\rho), \forall \rho$ we have $\mu \leqslant \operatorname{Ber}(\rho)$ or $\mu \geqslant \operatorname{Ber}(\rho)$ Let $\rho_{0}=\sup \left\{\rho: j^{u} \geqslant \operatorname{Ber}(\rho)\right\}$.
\{ \}

To show $\mu=\operatorname{Ber}\left(\rho_{0}\right)$ :

Let $f$ be a monotone cont. function on $\{0,1\}^{\mathbb{Z}}$. (These funct. determine the marne)
Then

$$
\rho \longmapsto \int_{\{0,1\}^{\mathbb{Z}}} f d(\mathbb{B e r}(\rho))
$$

is an incr. f. of $\rho$.
\{ \}

$$
\begin{aligned}
\Rightarrow \text { we have } & \int f d \mu=\int f \operatorname{Ber}\left(\rho_{0}\right) \\
& \Rightarrow \quad \mu=\operatorname{Ber}\left(\rho_{0}\right) \quad
\end{aligned}
$$

## Notes and references

1. The proof that all extreme stationary translation invariant measures for the TASEP are the Bernoulli product measures is given in Liggett, Thomas M. Coupling the Simple Exclusion Process. Ann. Probab. 4 (1976), no. 3, 339--356. doi:10.1214/aop/1176996084. We follow this paper in our proof.
2. Construction of the stationary stochastic six vertex model is described in Amol Aggarwal. Current fluctuations of the stationary ASEP and six-vertex model. Duke Math. J., Volume 167, Number 2 (2018), 269-384, https://arxiv.org/abs/1608.04726, Lemma A.2.

## Problems

[[_Lecture 5, 2-15|Lecture 5]]

## 1

Notation. In this and the next problem we denote by $a^{+}=\max (a, 0)$ the positive part of $a$.
Let $0<r<p<1$. Let $I, J$ and $Y$ be independent geometric random variables with distributions

$$
P[I=k]=\frac{p-r}{1-r}\left(\frac{1-p}{1-r}\right)^{k}, P[J=k]=r(1-r)^{k}, P[Y=k]=p(1-p)^{k}
$$

for $k \in \mathbb{Z}_{\geq 0}$. Let $I_{1}=(I-J)^{+}+Y, J_{1}=(J-I)^{+}+Y$ and $X=\min (I, J)$. Then the triple $\left(I_{1}, J_{1}, X\right)$ has the same distribution as $(I, J, Y)$.
Hint: Use the joint moment generating function $\mathbb{E}\left[a^{I_{1}} b^{J_{1}} c^{X}\right]$ and show that it's the same as for $(I, J, Y)$.

## 2

Consider the last-passage model with boundaries:


$$
\begin{array}{ll}
P\left(y_{i, 0}=k\right)=\frac{p-r}{1-r}\left(\frac{1-p}{1-r}\right)^{k} & k \geqslant 0 \quad \text { (hor) } \\
P\left(y_{0, j}=k\right)=r(1-r)^{k} & k \geqslant 0 \quad \text { (vert) }
\end{array}
$$

Recall the notation $L(m, n)$ for the last-passage times from $(0,0)$ to $(m, n)$.
In the last-passage model with boundaries we define some new random variables. Horizontal and vertical increments are given by

$$
\begin{array}{ll}
I_{i, j}=L(i, j)-L(i-1, j) & \text { for } i \geq 1, j \geq 0 \\
\text { and } \quad J_{i, j}=L(i, j)-L(i, j-1) & \text { for } i \geq 0, j \geq 1
\end{array}
$$

An alternative formula for $I_{i, j}$ develops as follows, if $i, j \geq 1$ :

$$
\begin{aligned}
I_{i, j} & =L(i, j)-L(i-1, j) \\
& =\max (L(i-1, j), L(i, j-1))+Y_{i, j}-L(i-1, j-1) \\
& -[L(i-1, j)-L(i-1, j-1)] \\
& =\max \left(J_{i-1, j}, I_{i, j-1}\right)+Y_{i, j}-J_{i-1, j} \\
& =\left(I_{i, j-1}-J_{i-1, j}\right)^{+}+Y_{i, j}
\end{aligned}
$$

Similar formula works for $J_{i, j}$ by symmetry, so we have

$$
\begin{gathered}
I_{i, j}=\left(I_{i, j-1}-J_{i-1, j}\right)^{+}+Y_{i, j} \\
J_{i, j}=\left(J_{i-1, j}-I_{i, j-1}\right)^{+}+Y_{i, j} \quad \text { for }(i, j) \in \mathbb{Z}_{\geq 1}^{2}
\end{gathered}
$$

Define

$$
X_{i, j}=\min \left(I_{i+1, j}, J_{i, j+1}\right) \quad \text { for }(i, j) \in \mathbb{Z}_{\geq 0}^{2}
$$

Take a down-right path $\sigma$ and let

$$
Z_{\ell}(\sigma)= \begin{cases}L(\sigma(\ell+1))-L(\sigma(\ell))=I_{\sigma(\ell+1)} & \text { if } \sigma(\ell+1)-\sigma(\ell)=(1,0) \\ L(\sigma(\ell))-L(\sigma(\ell+1))=J_{\sigma(\ell)} & \text { if } \sigma(\ell+1)-\sigma(\ell)=(0,-1)\end{cases}
$$

Prove the following:

Theorem The random variables $X_{i, j}$ below $\sigma$, and $Z_{\ell}(\sigma)$ for all $\ell$ are ingependent and geometrically distributed. Moreover, $X_{i, j}$ are distributed as "bulk" LPP weights; along $\sigma$ horizontal increments have the "horizontal" distribution; and along $\sigma$ the vertical increments have the "vertical" distribution.
\{ \}

satisfy:

$\% \% \# 3$
Show that the Bernoulli product measure $\mu_{\rho}$ on $\{0,1\}^{\mathbb{Z}}$ is extreme within the class of translation invariant measures.

Notation. $\mu_{\rho}$ is the measure corresponding to the sequence of id Bernoulli random variables (with $\rho$ the probability of 1 ) at each component of $\mathbb{Z}$. A translation invariant probability measure is called extreme if the equality $\mu=$ $\gamma \mu_{1}+(1-\gamma) \mu_{2}$, where $\gamma \in(0,1)$ and $\mu_{1}, \mu_{2}$ are translation invariant probability measures, implies $\mu_{1}=\mu_{2}=\mu$.
Hint. We have $\mu_{1} \leq 1 / \gamma \mu$, so $\mu_{1}$ is absolutely continuous with respect to $\mu$. Therefore by Radon-Nikodym, there exists a function $f$ for which $\mu_{1}=f d \mu$. Show that $f$ must be a translation invariant function on $\{0,1\}^{\mathbb{Z}}$. Show that this leads to a contradiction with $\gamma \in(0,1)$.

## 4

Extreme translation invariant probability measures on $\{0,1\}^{\mathbb{Z}}$ form a wider family than just the Bernoulli product measures. Produce an example of an extreme translation invariant measure which is not $\operatorname{Ber}(\rho) . \% \%$

## 3

Take the action of another group of transformations on $\{0,1\}^{\mathbb{Z}}$, namely, finitary permutations. These are permutations $\sigma$ which fix all but finitely many points (but which points are fixed depends on $\sigma$ ); they form a group which is usually called the infinite symmetric group. A measure invariant under this group is usually called exchangeable.

De Finetti's theorem states that all extreme exchangeable probability measures on $\{0,1\}^{\mathbb{Z}}$ are Bernoulli product measures.

Prove this theorem, or find its proof in the literature and understand the argument.

## T2

By $\mathrm{T} i \mathrm{I}$ will suggest topics for 10 -minute talks, where $i$ is a counter throughout all lectures. This second talk idea is to continue the discussion of the stationary LPP field using Seppalainen's lecture notes, and explain how to get the explicit limit shape via a certain variational principle.

## T3

(By Ti I suggest topics for 10-minute talks, where $i$ is a counter throughout all lectures.)

Explain a proof of de Finetti's theorem in a 10 -minute talk.
[[Solutions, 2-15|Solutions]]

## Solutions

[[../../Lecture 5, 2-15/Problems, 2-15|Problems 5]]
[[../../Lecture 5, 2-15/_Lecture 5, 2-15|Lecture 5]]
1
\{ \}
(1)

$$
\begin{aligned}
& \mathbb{E}\left(a^{I_{1}} b^{I_{2}} c^{X}\right)= \\
= & \mathbb{E}\left(a^{(I-J)^{+}} b^{\left.(J-I)^{+} c^{\min (I, J)}(a b)^{Y}\right]}\right. \\
= & \mathbb{E}\left(a^{(I-J)^{+}} b^{(J-I)^{+}} c^{\min (I, J)}\right]_{(\text {indepandence })} \mathbb{E}\left[(a b)^{Y}\right] \\
= & \left\{\sum_{0 \leq j \leq i} a^{i-j} c^{j} \frac{p-p}{1-r}\left(\frac{1-p}{1-r}\right)^{i} r(1-r)^{j}\right. \\
& \left.\sum_{0 \leq i<j} b^{j-i} c^{i} \frac{p-r}{1-r}\left(\frac{1-p}{1-r}\right)^{i} r(1-r)^{j}\right] \\
& x \sum_{k \geqslant 0}^{p(r-p)^{k}(a b)^{k}} \\
= & \frac{p-r}{1-r-(1-p) a} a_{0}^{1-(1-r) b} \frac{r}{1-(1-p) e} \\
= & \mathbb{E}(a) \mathbb{E}\left(b^{J}\right) \mathbb{E}\left(a^{Y}\right)
\end{aligned}
$$

\{ \}

this is a local more which
veeps the distributions \& ivelependence, by problem 1


3
We refer to the book of Borodin-Olshanski, see section 5 there with two proofs, explained in great detail.

## 1 Push-block process

- 


### 1.1 Interlacing arrays

## Definition 1.1.1. \{ \}



GT stands for "Gelfand-Tsetlin". Elements of $G T_{N}$ are usually called signatures.
Connection to representation theory: $G T_{N}$ encodes irreducible representatons of the unitary group $U(N)$. This is the group of complex $N \times N$ matrices $A$ such that $A^{*} A=I d$, where "*" is conjugate transpose. (This is the complex analogue of orthogonal matrices.)

Definition 1.1.2. Interlacing $\}$

$$
\begin{aligned}
& \text { Def. } \lambda \in G T_{N}, \quad \mu \in G T_{N-1} \\
& \text { interlace if } \\
& \lambda_{1} \geqslant \mu_{1} \geqslant \lambda_{2} \geqslant \ldots \geqslant \lambda_{N-1} \geqslant \mu_{N-1} \geqslant \lambda_{N}
\end{aligned}
$$

Definition 1.1.3. Interlacing arrays Let $\mathcal{C} \subset G T_{1} \times G T_{2} \times G T_{3} \times$ be the set of interlacing arrays, ie., of sequences $\left\{\lambda^{(k)} \in G T_{k}\right\}$ with the property that $\lambda^{(1)} \prec \lambda^{(2)} \prec \lambda^{(3)} \prec \ldots$. For example:
\{ \}


Note. Interlacing naturally appears in several contexts: - roots of a polynomial and of its derivative; - matrix spectra (this context is closer to us here).

See [[Problems, 2-22\#1|Problem 1]], for example.

Example 1.1.4. Lozenge tiling interpretation There is a bijection of the set of interlacing arrays of a fixed depth $N$ to lozenge tilings of a strip of width $N$ with $N$ "defects" on top. Therefore, infinite interlacing arrays are in bijection with tilings of the upper half plane. Lozenges are three types of rhombs on the regular triangular grid.

For step 1, shift the coordinates so that they are distinct on each level:
\{ \}


Then draw vertical lozenges at these coordinates, in a coordinate system with $120^{\circ}$ angle between the axes:
\{ \}

1.2 Push-block process

Definition 1.2.1 On $\mathcal{C}$, we define a continuous time Markov process as follows:
\{ \}
Def. Push-blocn Markov chain
on interacting aras of depth $N($ or $\infty)$, in continuous time

- each $\lambda_{j}^{(k)}$ has exp. clock of rate 1

$$
\begin{gathered}
\lambda_{j}^{(k)} \rightarrow \lambda_{j}^{(k)}+1 \\
\text { generically }
\end{gathered}
$$

- blocking: if $\lambda_{j}^{(k)}=\lambda_{j-1}^{(k-1)}$

$$
\lambda_{j}^{(k)} \underset{\lambda_{j-1}^{(k-1)}}{ }
$$

then-no jump.
\{ \}


You may think that this dynamics preserves interlacing, and also under it the lower particles are heavier than the higher particles.
See also [[Problems, 2-22\#2| Problem 2]].

Example 1.2.2 Here are a few transitions of the push-block process \{ \}

Ex.


$$
{ }_{1}^{0} 2_{1}^{1} 2_{2}^{2} 2^{2} \longrightarrow 0_{0}^{0} 1_{1}^{1} 2^{2} 3^{3} \text { blowing, wo jump. }^{3}
$$

Corollary 1.2.3 \{ \}

$$
\begin{array}{r}
\lambda_{1}^{(1)} \text { performers a simple Poisson } \\
\text { waite. } \\
(\text { when clock sings, it jugs) }
\end{array}
$$

So, the asymptotic behaviour of the first particle is as follows, by the CLT: \{ \}

$$
\begin{aligned}
& \lambda_{1}^{(1)}(t)=t+\sqrt{t} \cdot \int \mathcal{N}(0,1) \\
& \lambda_{1}^{(1)}(0)=0 \quad \text { (ct) } t \rightarrow \infty
\end{aligned}
$$

## 2 TASEP and push-block

### 2.1 TASEP as a marginal

It is known that a function of a Markov process is not necessarily a Markov process itself. Indeed, if a function is not one-to-one, it could "forget" some information.

However, we can see that TASEP is represented as a function of the push-block dynamics, and moreover it is marginally Markovian.
Theorem 2.1.1 Let $x_{k}=\lambda_{k}^{(k)}-k$, where $\left\{\lambda_{j}^{(k)}\right\}$ is an interlacing array evolving as a push-block process. Then $\left\{x_{k}\right\}$ evolves as TASEP.

Proof 2.1.1 This follows in a straightforward way from the definition of the push-block process. Here is an example of the mapping:
\{ \}

$$
0_{1}^{0} 2_{1}^{1} 2^{2} 2^{2}
$$


2.2 PushTASEP

There is one more Markovian marginal in the push-block process - the PushTASEP, or long-range TASEP.
\{ \}


$$
\begin{aligned}
& \begin{array}{l}
\text { each part. has Exp clock } \\
\text { when it rings, a tors. aus }
\end{array} \\
& \text { by } 1 \text { to the right, \& } \\
& \text { purines cargboody to the right }
\end{aligned}
$$



See [[Problems, 2-22\#3|Problem 3]] on existence of this process.

## 3 Gibbs measures

### 3.1 Definition of Gibbs measures

Definition 3.1.1 Let $N$ be fixed. An extreme Gibbs measure on interlacing arrays $\lambda^{(1)} \prec \ldots \prec \lambda^{(N)}$ of depth $N$ is a probability measure on these arrays, under which $\lambda^{(N)}$ is fixed, and the joint distribution of

$$
\lambda^{(1)} \prec \ldots \prec \lambda^{(N-1)}
$$

is uniform on the set of interlacing arrays of depth $N$ with fixed top row $\lambda^{(N)}$. In other words, extreme Gibbs measures of finite depth are just uniform measures.

Example 3.1.2 If $N=3$ and $\lambda^{(3)}=(4,1,1)$, then the corresponding extreme Gibbs measure places probability weight $\frac{1}{10}$ onto each of the 10 arrays:
\{ \}


$$
1 \leq y \leq x \leq 4
$$

Definition 3.1.3 A Gibbs measure on interlacing arrays $\lambda^{(1)} \prec \ldots \prec \lambda^{(N)}$ of depth $N$ is a probability measure on these arrays which is a mixture ( $=\sim$ convex combination of extreme ones).

In other words, conditioned on $\lambda^{(N)}=\lambda$, the conditional distribution of all the lower rows $\lambda^{(1)}, \ldots, \lambda^{(N-1)}$ is uniform among all interlacing arrays with top row $\lambda^{(N)}$.

Definition 3.1.4 A measure $M$ on interlacing arrays of infinite depth is called Gibbs, if for every $N$, conditioned on $\lambda^{(N)}=\lambda$, the conditional distribution of the lower rows $\lambda^{(1)}, \ldots, \lambda^{(N-1)}$ does not depend on the higher rows $\lambda$, and is uniform on the set of interlacing arrays with top row $\lambda$. That is,
$M\left(\lambda^{(1)}, \ldots, \lambda^{(N-1)} \mid \lambda^{(N)}=\lambda\right)=\frac{1}{\#\{\text { interlacing arrays of depth } N \text { with top row } \lambda\}}$

Example 3.1.5 The delta measure with $\lambda_{j}^{(k)}=0$ for all $1 \leq j \leq k<+\infty$ is Gibbs for trivial reasons.

Example 3.1.6 Take $\beta \in(0,1)$ and a random walk. Place 0 everywhere to the left of the random walk, and 1 everywhere to the right of it. Thus, we get a random interlacing array. In fact, this measure is Gibbs.
\{ \}


## See [[Problems, 2-22\#4|Problem 4]].

### 3.2 Harmonic functions

There is an equivalent description of the Gibbs property using so-called harmonic functions.

Definition 3.2.1 Let $\lambda \in G T_{N}$. By $\operatorname{Dim}_{N} \lambda$ denote the number of interlacing arrays of depth $N$ with top row $\lambda$.

For example, $\operatorname{Dim}_{3}(4,1,1)=10$.
See [[Problems, 2-22\#5| Problem 5]].
In fact, $\operatorname{Dim}_{N} \lambda$ is the dimension of the irreducible representation of $U(N)$ corresponding to $\lambda$.

Definition 3.2.2 Let $M$ be a Gibbs measure on infinite interlacing arrays. Associate to it a family of functions on $G T_{N}$ for each $N$, by

$$
\varphi_{N}(\lambda)=\frac{M\left(\lambda^{(N)}=\lambda\right)}{\operatorname{Dim}_{N} \lambda}
$$

For a Gibbs measure this is the same as $M\left(\lambda^{(1)}=\mu, \ldots, \lambda^{(N-1)}=\nu, \lambda^{(N)}=\lambda\right)$ for any fixed interlacing array $(\mu, \ldots, \nu, \lambda)$ of depth $N$ with top row $\lambda$. In this equivalent description we use the Gibbs property in an essential way, as an independence from the lower rows $\mu, \ldots, \nu$.

Proposition 3.2.3 The space of Gibbs measures on infinite interlacing arrays is in a one-to-one correspondence with the space of harmonic functions $\left\{\varphi_{N}\right\}$, that is, which satisfy

$$
\varphi_{N}(\lambda)=\sum_{\nu \in G T_{N+1}} \varphi_{N+1}(\nu)
$$

for all $N$ and all $\lambda \in G T_{N}$.
Example of the harmonicity:
\{ \}


Proof 3.2.3 For any measure $M$ on infinite interlacing arrays we have \{ \}

$$
\begin{aligned}
& \sum_{v \in G T_{N+1}} M\left(\lambda^{(1)}=\mu, \ldots, \lambda^{(N-1)}=x, \lambda^{(N)}=\lambda, \lambda^{(N+1)}=v\right) \\
& =M\left(\lambda^{(1)}=J^{n}, \ldots, \lambda^{(N-1)}=x, \lambda^{(N)}=\lambda\right)
\end{aligned}
$$

If the measure is Gibbs, we immediately get the harmonicity of $\varphi_{N}$ defined by $\varphi_{N}(\lambda)=M\left(\lambda^{(N)}=\lambda\right) / \operatorname{Dim}_{N} \lambda$.

On the other hand, if the functions $\varphi_{N}(\lambda)$ are harmonic, let us define a measure $M$ using $\varphi_{N}$ 's and the Gibbs property:

$$
M\left(\lambda^{(1)}=\mu, \ldots, \lambda^{(N-1)}=\varkappa, \lambda^{(N)}=\lambda\right):=\varphi_{N}(\lambda)
$$

independently of $\mu, \ldots, \varkappa$. This measure is automatically Gibbs, and the measure $M$ is well-defined thanks to the harmonicity condition.
-

### 3.3 Classification (answer)

Recall that a Gibbs measure (on infinite interlacing arrays) is called extreme if it cannot be represented as a nontrivial convex combination of other such Gibbs measures.

The problem of classification of extreme Gibbs measures is solved, but its solution is very interesting and has lead to many developments. There are also several equivalent ways to formulate this problem, including a representation-theoretic one (classify all indecomposable normalized characters of the infinite-dimensional unitary group $U(\infty)$ ).
Let us mention the answer. The space of parameters of extreme measures is a certain subset $\Omega \subset \mathbb{R}^{4 \infty+2}$, and the mapping at the level of harmonic functions is as follows:
\{ \}

$$
\begin{aligned}
& \sum_{n \in \mathbb{Z}} \varphi_{1}(n) u^{n}=e^{\gamma^{+}(u-1)+\gamma^{-}\left(u^{-1}-1\right)} x \\
& x \prod_{k} \frac{\left(1+\beta_{k}^{+}(u-1)\right)\left(1+\beta_{k}^{-}\left(u^{-1}-1\right)\right)}{\left(1-\alpha_{k}^{+}(u-1)\right)\left(1-\alpha_{k}^{-}\left(u^{-1}-1\right)\right)} \\
& \varphi_{N}(\lambda)=d \ln \left[\varphi_{1}\left(\lambda_{i}+j^{0}-1\right)\right]_{i_{j} j^{0}=1}^{N}
\end{aligned}
$$

## Notes and references

1. Push-block process was introduced in A. Borodin and P. Ferrari. Anisotropic growth of random surfaces in $2+1$ dimensions. Commun. Math. Phys., 325:603-684, 2014. arXiv:0804.3035 [mathph ].
2. A simulation of the push-block process due to P.Ferrari can be found here.
3. The problem of classification of extreme Gibbs measures on tilings of the upper half plane has a long history. For example, see A. Borodin and G. Olshanski. The boundary of the Gelfand-Tsetlin graph: A new approach. Adv. Math., $230: 1738-1779,2012$. arXiv:1109.1412 [math.CO].

## Problems

[[_Lecture 6, 2-22|Lecture 6]]

## 1

Let $A$ be an $N \times N$ complex hermitian or real symmetric matrix. Let $\lambda_{1} \geq \ldots \geq$ $\lambda_{N}, \lambda_{i} \in \mathbb{R}$, be eigenvalues of $A$. Let $B$ be the $(N-1) \times(N-1)$ matrix which is obtained by deleting, say, the $N$-th row and the $N$-th column of $A$. Show that the eigenvalues $\mu_{1} \geq \ldots \geq \mu_{N-1}$ of $B$ interlace with those of $A$ :

$$
\lambda_{1} \geq \mu_{1} \geq \ldots \geq \lambda_{N-1} \geq \mu_{N-1} \geq \lambda_{N}
$$

Hint. Use the "variational" characterization of the eigenvalues: $\lambda_{1}$ is the coefficient of maximal dilation of the norm of a vector by $A$, and so on:

$$
\begin{gathered}
\lambda_{k}=\min _{U}\left\{\left.\max _{x}\left\{\left.\frac{(A x, x)}{(x, x)} \right\rvert\, x \in U \text { and } x \neq 0\right\} \right\rvert\, \operatorname{dim}(U)=k\right\} \\
\lambda_{k}=\max _{U}\left\{\left.\min _{x}\left\{\left.\frac{(A x, x)}{(x, x)} \right\rvert\, x \in U \text { and } x \neq 0\right\} \right\rvert\, \operatorname{dim}(U)=n-k+1\right\}
\end{gathered}
$$

## 2

Show that the push-block process on infinite two-dimensional interlacing arrays (starting from an arbitrary initial configuration) exists.

Hint. Use compatibility of the dynamics restricted to the first $N$ levels, for all $N$.

## 3

Show that the PushTASEP (started from an arbitrary initial particle configuradion on $\mathbb{Z}$ ) exists.
Hint. Use a suitable version of the graphical construction, and maybe it is useful to replace particles by holes, and vice versa.

## 4

Show that the measure on interlacing arrays coming from a random walk with a fixed $\beta \in(0,1)$ satisfies the Gibbs property.
\{ \}


5
Prove the formula for $\operatorname{Dim}_{N} \lambda$, the number of interlacing arrays of depth $N$ and with top row $\lambda$ :

$$
\operatorname{Dim}_{N} \lambda=\prod_{1 \leq i<j \leq N} \frac{\lambda_{i}-\lambda_{j}+j-i}{j-i}, \quad \lambda \in G T_{N}
$$

Hint. You may use induction: $\operatorname{Dim}_{N} \lambda$ is the sum of $\operatorname{Dim}_{N-1} \mu$ over all $\mu \in$ $G T_{N-1}$ which interlace with $\lambda$.
[[Solutions, 2-22|Solutions]]

## Solutions

[[../../Lecture 6, 2-22/Problems, 2-22|Problems]]
[[../../Lecture 6, 2-22/_Lecture 6, 2-22|Lecture]]
1
\{ \}
(1) $A$


$$
B=P A D
$$

where $P$ is the projection onto the subspace

$$
\mathbb{C}^{N-1} \subset \mathbb{C}^{N}
$$

So,

$$
j_{k}=\min _{U}\left(\max _{x} \frac{(B x, x)}{(x, x)}\right) \text { or } \max _{U}^{\text {spanned }}\left(\min _{x} \frac{(B x, x)}{(x, x)}\right)
$$

Nate $(B x, x)=(P A P x, x)$

$$
=\left(A\left(P_{x}\right), P_{x}\right)
$$

because $P$ is selt-adjoint

Let $S_{j}=\operatorname{spam}\left(b_{1} \ldots b_{j}\right)$, where $b_{j}$ are eigenvectors of $B: \quad B b_{j}=\mu_{j} b_{j}$

Then

$$
\mu_{j}=\min _{x \in S_{j}}\left(B_{x}, x\right)=\min _{x \in \|_{j}}\left(A\left(P_{x}\right), P_{x}\right) \leqslant \lambda_{j}
$$

\{ \}

$$
\begin{aligned}
& \text { because } P x \text { also belongs to at } \\
& \text { vest } k \text { dime subspaces } \\
& \text { so } \lambda g \text { just tres max. } \\
& \text { In the other direction, }
\end{aligned}
$$

$$
\begin{aligned}
& \tilde{S}_{j}=\operatorname{bjan}\left(b_{j}, b_{j+1}, \ldots, b_{N-1}\right) \\
& \text { So, } \quad \int_{j} \geqslant \lambda_{j}+1 \\
& \text { ロ. }
\end{aligned}
$$

## 2

The push-block process exists by Kolmogorov extension theorem. Namely, for each finite $N$ the evolution of $\lambda^{(1)}, \ldots, \lambda^{(N)}$ exists (starting from an arbitrary initial configuration) because it is a finite-dimensional Markov jump process (like a Poisson process, but possibly more complicated).

Then, for different $N$ 's, the stochastic processes are compatible with each other. That is, given $N$ and the evolution of $\lambda^{(1)}, \ldots, \lambda^{(N)}$, the marginal evolution of $\lambda^{(1)}, \ldots, \lambda^{(k)}, k<N$, is independent of levels $k+1, \ldots, N$, and coincides with the push-block process on the first $k$ levels. Therefore, by Kolmogorov extension, we can define the process on infinitely many levels.

## 3

Let us assume that the initial configuration is not densely packed to the left.
Model the PushTASEP process by putting independent Poisson clocks at each site. Then make the graphical construction as follows. When a Poisson clock
rings at a site, if there is a hole then nothing happens. If there is a particle, then the particle jumps to the next available hole on the right.
\{ \}


There could be an issue with having a densely packed configuration of particles to the right, so that there is no available hole. In this case, employ Kolmogorov extension theorem and construct the process on $\mathbb{Z}_{\leq N}$ for some $N$, with modified jumping rule: if there is no available hole before $N$, then the particle which wants to jump past $N$ must disappear. These processes are compatible with each other, and so a global process on $\mathbb{Z}$ exists. If initially there is a densely packed configuration to the right, then the process makes infinitely many jumps in finite time.

4
\{ \}
(4) Gibbs property is equivalent to the fact the at in a random walk $S_{n}, n=0,1, \ldots$,
given $S_{n}=k$, all trajectories
leaching to $(n, k)$ are equally lively
(In fact, they have probability $\beta^{k}(1-\beta)^{n-k}$ )


5
\{ \}
(5) Want

$$
\operatorname{Dim}_{\operatorname{mont}} \lambda=\sum_{j<\lambda}^{\operatorname{Dim}_{N-1} M_{j}\left[\begin{array}{l}
\text { where } D_{i u_{N}} \lambda= \\
=\prod_{i \leq i<j \leq N} \frac{\lambda_{i}-\lambda_{j}+j-1}{j-i}
\end{array}\right]}
$$



Let $l_{j}=\lambda_{i}-i, \quad m_{i}=u_{i}-i$

$$
\begin{aligned}
\lambda_{i+1} \leq \mu_{i} \leq \lambda_{i} \quad & \Leftrightarrow l_{i+1}+i+1 \leq m_{i}+i \leq l_{i}+i \\
& \Leftrightarrow l_{i+1}<m_{i}^{c} \leq l_{i}
\end{aligned}
$$

We have $\prod_{i<j \leqslant N-1}\left(m_{i}-m_{j}\right)=\operatorname{det}\left[m_{i}^{N-1-j}\right)_{j}^{0-1}$

Sum over $l_{i} \geqslant m_{i}>l_{i+1}$
invelves only the $i-t h$ now of the determinant, and hence is linear
\{ \}

$$
\begin{aligned}
& \Rightarrow \sum_{l_{i} \geq m i>l_{i+1}} m_{i}^{N-j-1}= \text { polynomial in } l_{i}, l_{i+1} \\
& \text { of degree } N-j \\
&= p_{N-j}\left(l_{j}\right)-p_{N-j}\left(l_{i+1}\right) \\
& \quad\left(\operatorname{dg} p_{N-j}=N-j\right)
\end{aligned}
$$

$\Rightarrow$ we get det of $p_{v j}\left(l_{j}\right)-p_{N-j}\left(l_{i+1}\right)$ :
$\operatorname{det}\left[\begin{array}{c}p_{N-1}\left(l_{i}\right)-p_{N-1}\left(l_{i+1}\right) \\ \vdots \\ p_{2}\left(l_{i}\right)-p_{2}\left(l_{i+1}\right) \\ l_{i+1}-l_{i}\end{array}\right]_{(N-1) \times(N-1)}=$
(dearty we canpars back bey row/wi aperations)

$$
\operatorname{det}\left[\begin{array}{cccc}
p_{N-1}\left(l_{1}\right) & p_{N-1}\left(l_{2}\right) & \cdots & p_{N-1}\left(l_{N}\right) \\
p_{N-l}\left(l_{1}\right) & \cdots & \cdots & p_{N-2}\left(l_{N}\right) \\
& \vdots & & \\
l_{1} & \cdots & \cdots & l_{N} \\
1 & \cdots & \cdots & 1
\end{array}\right]_{N \times N}
$$

\{\}

The latter is count. $V\left(l_{1} \ldots l_{N}\right)$.

To get constant, note $D_{\text {imp }}^{k}(0, \ldots, 0)=1$ for all $k$
$\Rightarrow$ answer is proven

## 1 Gibbs measures on interlacing arrays

- 


## 1 Gibbs measures on interlacing arrays

Recall the definition of Gibbs measures:
Definition 1.0.1 \{ \}

Def. Meas. $M$ on interlacing arrays

$$
\begin{aligned}
& \lambda^{(1)}<\lambda^{(2)} \alpha \lambda^{(3)}<\ldots \quad(\infty \operatorname{dept} A) \\
& \text { is called Globs if } \forall N, \forall \lambda \in G T_{N} \\
& M\left(\lambda^{(1)}, \ldots, \lambda^{(N-1)} \mid \lambda^{(N)}=\lambda, \lambda^{(N+1)}=v, \ldots\right) \\
& \text { dep-overy on } \lambda \text { (and not floors } \geqslant N+1 \text { ) } \\
& \text { and is uniform among int. rang } \\
& w \text {, top row } \lambda \text {. }
\end{aligned}
$$

Definition 1.0.2 Gibbs measures form a convex set. That is, for $\mu_{1}, \mu_{2}$ Gibbs, their convex combinations $\gamma \mu_{1}+(1-\gamma) \mu_{2}$ are also Gibbs - as convex combinations of uniform conditional distributions are uniform, too.

Recall that a Gibbs measure $\mu$ is called extreme if from $\mu=\gamma \mu_{1}+(1-\gamma) \mu_{2}$, $0<\gamma<1, \mu_{1}, \mu_{2}$ Gibbs, it follows that $\mu_{1}=\mu_{2}=\mu$.

Example 1.0.3 The "Bernoulli random walk" measure is an example of a nontrivial Gibbs measure. In fact, it is extreme.
\{ \}


Now, let us formulate several equivalent problems of classifying extreme Gibbs measures.

Problem 1.0.4 Classify extreme Gibbs measures on infinite interlacing arrays.

## Problem 1.0.5 \{ \}

(2) Class extreme woruabized harmonic functions


$$
\forall_{N}, \quad \varphi_{N}(\lambda)=\sum_{\nu \in G T_{N+1}} \varphi_{N+1}(\nu)
$$

The connection between Gibbs measures and harmonic functions is $\varphi_{N}(\lambda)=$ $M\left(\lambda^{(N)}=\lambda\right) / \operatorname{Dim}_{N} \lambda, \lambda \in G T_{N}$.

Problem 1.0.6 Classify irreducible normalized characters of the infinitedimensional unitary group $U(\infty)$.

Problem 1.0.7 \{ \}
(4) class. totally nonbeg sequences

$$
\begin{aligned}
& \text { (Edrei 1953) } \\
& a_{n} \in \mathbb{C}, \quad n \in \mathbb{Z} \text { sit. all minors } \\
& \text { of }\left[a_{j-i}\right]_{i, j \in \mathbb{Z}} \text { are } \geqslant 0
\end{aligned}
$$

Example of a minor:
\{ \}


Theorem 1.0.8 All the 4 problems formulated above are equivalent to each other. Details are outside of our scope for now; see the many papers in [[_Lecture 7, 2-24\# Notes and references|Notes and references]].

Answer to all these problems $\}$

$$
\begin{aligned}
& \text { Extreme meas } \longleftrightarrow \Omega \subset \mathbb{R}^{4 \infty 0+2} \\
& \Omega=\left\{\quad \alpha_{1}^{ \pm} \geqslant \alpha_{2}^{ \pm} \geq \ldots \geqslant 0,\right. \\
& \beta_{1}^{ \pm} \geqslant \beta_{2}^{ \pm} \geqslant \ldots \geqslant 0, \quad \beta_{1}^{+}+\beta_{1}^{-} \leqslant 1, \\
& \gamma^{ \pm} \geqslant 0 \text {, } \\
& \sum_{i}\left(\alpha_{i}^{ \pm}+\beta_{i}^{ \pm}\right)<\infty \quad \begin{array}{l} 
\\
6 T_{2}=\left\{\lambda_{2} \leqslant \lambda_{1}\right\} \\
6 T_{\perp}=\mathbb{Z}
\end{array} \\
& \omega \in \Omega \quad \sum_{n \in D} \varphi_{1}(n) u^{n}=\quad|u|=1 \\
& =e^{\gamma^{+}(u-1)+\gamma^{-}\left(u^{-1}-1\right)} \prod_{k \geq 1} \frac{\left(1+\beta_{k}^{+}(u-1)\right)\left(1+\beta_{k}^{-}\left(u^{-1}-1\right)\right)}{\left(1-\alpha_{k}^{+}(u-1)\right)\left(1-\alpha_{k}^{-}\left(u^{-1}-1\right)\right)} \\
& \varphi_{N}^{w}(J)=\operatorname{det}\left[\psi_{1}^{w}\left(\lambda_{i}+j-i\right)\right]_{i j}^{N}=1
\end{aligned}
$$

For a not necessarily extreme normalized harmonic function $\left\{\varphi_{N}\right\}_{N}$, there exists a unique probability measure $\mu$ on $\Omega$ such that

$$
\varphi_{N}(\lambda)=\int_{\Omega} \varphi_{N}^{\omega}(\lambda) \mu(d \omega),
$$

for all $N$ and all $\lambda \in G T_{N}$. Here $\varphi_{N}^{\omega}(\lambda)$ are the extreme harmonic functions (which have determinantal form).

Example 1.0.9 (Bernoulli random walk) Below are some rough notes on the Bernoulli random walk extreme Gibbs measure.
\{ \}
w

$$
\gamma^{ \pm}=0, \quad \alpha_{i}^{*}=0, \quad \beta_{i}^{ \pm}=0
$$


\{\}

$$
\begin{gather*}
\sum \varphi_{1}(n) u^{n}=1+\beta(u-1)=(1-\beta) \cdot u^{0}+\beta u^{1} \\
\varphi_{1}(0)=1-\beta, \quad \varphi_{1}(1)=\beta \\
11 \\
\operatorname{Prob}\left(\lambda_{1}^{\prime}=0\right) \quad \text { Prob }\left(\lambda_{1}^{\prime}=1\right)  \tag{0}\\
0 \rho_{1-\beta} \quad \prod_{0}^{1}
\end{gather*}
$$

\{\}


## 2 Translation invariant Gibbs measures

- 


## 2 Translation invariant Gibbs measures

Here I am discussing the analogue of the Liggett's classification leading to Bernoulli product measures, but in two dimensions. For two dimensions, the translation invariant Gibbs measures on $\mathbb{Z}^{2}$ are much more complicated than the Bernoulli measures.
\{ \}

Gibbs property for lozenge tilings


Measure is invowiont under uniform resampling in any finite subset

Problem 2.0.1 \{ \}
Probkm classify extreme transl. ins
Gibbs measures on

$$
\text { tilings of } \mathbb{Z}^{2}
$$

Solution 2.0.2 \{ \}


Deplumed on 2
parameters - slope $(s, t)$


3 Hydrodynamics of the push-block process
-
3.1 Push-block and Gibbs measures

Recall the push-block process:
\{ \}

$$
{ }_{0}^{0} 1_{0}^{1} 1_{0}^{2} 2_{2}^{2}
$$

Theorem 3.1.1 The push-block process preserves the class of Gibbs measures on interlacing arrays. Moreover, it preserves the class of extreme Gibbs measures. The action on the extreme Gibbs measures' parameters is as follows:

$$
\gamma^{+} \mapsto \gamma^{+}+\text {time elapsed under the push-block process }
$$

We do not prove this theorem now, as it needs some Schur polynomials machinery. It might be proven using some intertwining of noncolliding Poisson random walks, but this intertwining also follows from the Schur polynomials machinery.

Example 3.1.2 Let us focus on one example which confirms the previous theorem in a particular case.
\{ \}

\{ \}

$$
\begin{aligned}
& \text { - If we start from a Gibbs, measure } \\
& \qquad P\binom{04}{*}=\alpha, P\binom{14}{*}=\beta, P\binom{13}{*}=\gamma \\
& \text { - all arrows have probability, } d t \\
& \text { - } P\left(\begin{array}{c}
04 \\
2
\end{array} \begin{array}{c}
14 \\
2
\end{array}\right)=\frac{\alpha}{5} d t, \text { etc } \\
& \text { - cher that we arrive at a } \\
& \text { Gibbs measure }
\end{aligned}
$$

In general, to show the preservation of the class of Gibbs measures on interlacing arrays, we need:
\{ \}

$$
\begin{aligned}
& \text { N, } J \in G T N \text { fixed } \\
& \Rightarrow \text { for any array } \lambda^{(1)} \alpha \ldots<\lambda^{(N-1)} 2 \lambda
\end{aligned}
$$

compute rates from all pasts

$$
\left[\begin{array}{c}
\mu \\
\mu^{(N-1)} \\
\vdots \\
\mu^{(1)}
\end{array}\right] \longrightarrow\left[\begin{array}{c}
\lambda \\
\lambda^{(N-1)} \\
\vdots \\
\lambda^{(1)}
\end{array}\right]
$$

\& assuming that an's wee Gills $_{(j)}$,
show hat $\lambda^{(\rho)}$ are coulis uniform.
3.2 Hydrodynamics

Recall the 1d situation with TASEP:
\{ \}


Theorem 3.2.1 $\}$

Thu. For $2 d, V_{s, t}$ on $\mathbb{Z}^{2}$ is invariant under push-block process (No proof given here)

Discussion 3.2.2. Hydrodynamics for the ed process $\}$

\{ \}

$$
\begin{aligned}
& \alpha(t, x, y) \\
\Rightarrow & \text { satisfies }
\end{aligned}
$$

$$
\frac{\partial h}{\partial t}=\forall /\binom{\Delta \Delta}{s, t}
$$

$$
V=\text { speed }
$$

of charge

$$
\text { of } n \text { st }
$$

$$
=T \Gamma\left(\frac{\partial h}{\partial x}, \frac{\partial h}{\partial y}\right)
$$

\{ \}


$$
V(s, t)=\frac{1}{\pi} \frac{\sin (\pi s) \cdot \sin (\pi t)}{\sin (\pi(s+t))}
$$

$$
\Rightarrow \frac{\partial h}{\partial t}=\frac{1}{\lambda} \frac{\sin \left(\pi \frac{\partial h}{\partial x}\right) \sin \left(\pi \frac{\partial h}{\partial y}\right)}{\sin \left(\pi\left(\frac{\partial h}{\partial x}+\frac{\partial h}{\partial y}\right)\right)}
$$



This final equation is what replaces the Burgers equation.
4 Schur polynomials
4.1 Two definitions of Schur polynomials Definition 4.1.1. Schur polynomial as a determinant $\}$

Def 1.

$$
\begin{aligned}
\lambda_{1} \geqslant \ldots \geqslant \lambda_{N} & \geqslant 0 \\
S_{\lambda}\left(x_{1} \ldots x_{N}\right) & =\frac{\operatorname{det}\left[x_{i}^{\lambda_{j}+N-j}\right]_{1}^{N}}{\operatorname{det}\left[x_{i}^{N-j}\right]_{1}^{N}}
\end{aligned}
$$

This expression is clearly symmetric in the $x_{i}$ 's. Moreover, it is a polynomial because the numerator is divisible by $x_{i}-x_{j}$, and the denominator is the Vandermonde determinant (see [[Problems, 2-24\#1 Vandermonde determinant|Problem 1]]).

Example 4.1.2 \{ \}

$$
\text { Ex. } \begin{aligned}
& \quad S_{(a, b)}(x, y)= \\
&=\frac{1}{x-y} d e t\left[\begin{array}{cc}
x^{a+1} & y^{a+1} \\
x^{b} & y^{b}
\end{array}\right] \\
&=\frac{(x y)^{b}\left\{x^{a-b+1}-y^{a-b+1}\right\}}{x-y} \\
&=(x y)^{b}\{\underbrace{x^{a-b}+y x^{a-b+1}+\ldots+y^{a-b}}_{a-b+1 \text { terms }}\}
\end{aligned}
$$

Definition 4.1.3. Schur polynomial as a sum $\}$


This represents the Schur polynomial as a sum over interlacing arrays of depth $N$ with fixed top row $\lambda^{(N)}=\lambda$.

For example, for $N=2$ and $\lambda=(a, b)$, there are $a-b+1$ such arrays, and the formula is identical to the one in [[\#Example 412 ]] above.

Theorem 4.1.4 Two definitions of Schur polynomials - [[[\#Definition 411 Schur polynomial as a determinant |1]] and [[\#Definition 413 Schur polynomial as a sum $\mid 2]]$ - are equivalent.

Corollary 4.1.5 The sum over interlacing arrays in [[\#Definition 413 Schur polynomial as a sum|Definition 4.1.3]] is symmetric in the $x_{i}$ 's.

Proof 4.1.5 It is informative to prove [[\#Corollary 415$]$ ] independently of [[\#Theorem 4114$]]$. Graphically, let us swap $x_{i}$ and $x_{i+1}$.
\{ \}


\{ \}


In combinatorics, this operation of flipping is known as the Bender-Knuth
involution.
Proof 4.1.4 \{ \}
Proof of then. Show the recursion for dot: (branching rule)

$$
\begin{aligned}
& \text { (*) } S_{\lambda}\left(x_{1} \ldots x_{N}\right)=\sum_{\mu} S_{\mu}\left(x_{1} \ldots x_{N-1}\right) x_{N}^{|\lambda|-|\mu|} \\
& (\text { comb formula clearly satisfies }) \\
& \sum_{x}, \underline{L 7-2} \text {; SNow that it suffices } \\
& \text { to prove }(x) \text { for } x_{N}=1
\end{aligned}
$$

\{ \}

Let us take $N=4$ for simplicity. Set
$\ell_{i}=\lambda_{i}+N-i=\lambda_{i}+4-i$, and
$m_{i}=\mu_{i}+N-1-i=\mu_{i}+3-i$.
We will perform the following operations with the
determinant in the numerator:

- Subtract row $j$ from row $j-1$ for all $j=2, \ldots, N$.
- The resulting determinant's last column contains only one 1 and all other elements are zero, so we can reduce order of the determinant by 1 .
- The $i$-th column then is divisible by $x_{i}-1$, this is how the $=\sum_{\mu}\left(x_{1}, x_{2}, x_{3}\right)$ Vandermonde drops in order, too.
- After the division, use the multilinearity of the determinant to get the desired recurrence.


### 4.2 Eigenoperators

\{ \}

$$
\begin{gathered}
S_{\lambda}\left(x_{1}, \ldots, x_{N}\right) \text { are eigenfunctions } \\
\text { of some operators acting in } x_{i} \\
(\text { Utlyo, they form basis in } \\
\mathbb{C}\left[x_{1}, \ldots, x_{N}\right]
\end{gathered}
$$

The basis claim will be proven in the next lecture.
Example of an eigenoperator is the following analouge of the Laplacian:

Example 4.2.1 $\}$

$$
\text { Example. } \begin{aligned}
D_{2} & =V^{-1} \circ \sum_{i}\left(\frac{\partial}{\partial x_{i}}\right)^{2} \circ V \\
V & =\text { ult. by } V(\vec{x})=\prod_{i<j}\left(x_{i}-x_{j}\right)
\end{aligned}
$$

See [[Problems, 2-24\#3|Problem 3]].
Definition 4.2.2 Let $q \in \mathbb{C}$ be fixed, and define the following $q$-difference operator:
\{ \}

$$
\begin{gathered}
\text { Def, } \int_{q}=V_{i=1}^{-1} \circ \sum_{i=}^{N} T_{q, x_{i}} \circ V \\
T_{q, x} f(x)=f(q x)
\end{gathered}
$$

Proposition 4.2.3 \{ \}

$$
\text { Prop, } D_{g}=\sum_{i=1}^{N}\left(\prod_{j \neq i} \frac{4 x_{i}-x_{j}}{x_{i}-x_{j}}\right) T_{q_{1} x_{i}}
$$

See [[Problems, 2-24\#4|Problem 4]].
Proposition 4.2.4 \{ \}

$$
\text { Prop } D_{q} S_{\lambda}=\left(\sum_{i=1}^{N} 9^{\lambda_{i}+N-i}\right) S_{\lambda} . \quad \forall q \in \mathbb{C}
$$

Proof 4.2.4 \{ \}
Prod $D_{q} s_{\lambda}=V^{-1} \circ \sum T_{q, x_{i}} \cdot \operatorname{det}\left[x_{i}^{\lambda_{j+N}-j}\right]$

$$
\begin{aligned}
& T_{p, x} x^{n}=q^{n} x^{n} . \\
& \Theta V^{-1} \circ \sum_{i} T_{q, x_{i}}\left(\sum_{b}(-1)^{6} T x_{j}^{\lambda_{j}\left(j_{j}^{+N-6 t i t}\right)}\right) \\
& =V^{-1} 0\left(\sum_{i} q^{\lambda_{i}+N_{-1}}\right) \cdot \operatorname{det}\left[x_{i}^{\lambda_{j}+N_{-}-j}\right]
\end{aligned}
$$

## Notes and references

1. Push-block process was introduced in A. Borodin and P. Ferrari. Anisotropic growth of random surfaces in $2+1$ dimensions. Common. Math. Phys., 325:603-684, 2014. arXiv:0804.3035 [math-ph].
2. Works on Gibbs measures on interlacing arrays:

- In the context of totally nonnegative sequences:

1. M. Aissen, A. Edrei, I. J. Schoenberg, and A. Whitney, On the generating functions of totally positive sequences, Proc. Nat. Acad. Sci. U. S. A. 37 (1951), 303-307.
2. M. Aissen, I. J. Schoenberg, and A. Whitney, On the generating functions of totally positive sequences $I$, J. Analyse Math. 2 (1952), 93-103.
3. A. Edrei, On the generating functions of totally
positive sequences. II, J. Analyse Math. 2 (1952), 104-109.
4. A. Edrei, On the generating function of a doubly infinite, totally positive sequence, Trans. Amer. Math. Soc. 74 (1953), 367-383.

- In the context of representation theory:

1. D. Voiculescu, Representations factorielles de type II1 de U(infinity), J. Math. Pures Appl. 55 (1976), 1-20.
2. R. Boyer, Infinite traces of AF-algebras and characters of U(infinity), J. Operator Theory 9 (1983), 205-236.

- In symmetric functions / combinatorics context:

1. A. Vershik and S. Kerov, Characters and factor-representations of the infinite unitary group, Dokl. Akad. Nauk SSSR 267 (1982), no. 2, 272-276.
2. A. Okounkov and G. Olshanski, Asymptotics of Jack polynomials as the number of variables goes to infinity, Int. Math. Res. Notices 1998 (1998), no. 13, 641-682, arXiv:q- alg/9709011.
3. A. Borodin and G. Olshanski, The boundary of the Gelfand-Tsetlin graph: A new approach, Adv. Math. 230 (2012), 1738-1779, arXiv:1109.1412 [math.CO].
4. L. Petrov. The Boundary of the Gelfand-Tsetlin Graph: New Proof of Borodin-Olshanski's Formula, and its q-analogue (2012) • Moscow Mathematical Journal, 14 (2014) no. 1, 121-160 • arXiv:1208.3443 [math.CO]
5. Sheffield's paper on translation invariant extreme Gibbs measures: Sheffield, S.: Random surfaces, Asterisque 304 (2005). arXiv:math/0304049 $$
math.PR
$$

## Problems

[[_Lecture 7, 2-24|Lecture 7]]

## 1. Vandermonde determinant

Show that

$$
\operatorname{det}\left[x_{i}^{N-j}\right]_{i, j=1}^{N}=\prod_{1 \leq i<j \leq N}\left(x_{i}-x_{j}\right)
$$

2
Show that to prove the branching rule for the Schur polynomials

$$
s_{\lambda}\left(x_{1}, \ldots, x_{N}\right)=\sum_{\mu: \mu \prec \lambda} s_{\mu}\left(x_{1}, \ldots, x_{N-1}\right) x_{N}^{|\lambda|-\mu}
$$

it suffices to show this for $x_{N}=1$.

3
\{ \}

$$
\begin{aligned}
& D_{2}=V \sum_{\substack{-1}}\left(\frac{\partial}{\partial x_{i}^{c}}\right)^{2} 0 \\
& V=\text { melt, by } V(\vec{x})=\prod_{0}^{0}\left(x_{i}-x_{j}\right)
\end{aligned}
$$

Write $D_{2}$ as an explicit differential operator acting in the $x_{i}$ 's, and compute its eigenvalues when acting on the Schur polynomials.

4
Show that $D_{q}$
\{ \}

$$
\begin{aligned}
& D_{q}=V^{-1} \circ \sum_{i=1}^{N} T_{q_{i} x_{i}} \circ V \\
& T_{q, x} f(x)=f(q x)
\end{aligned}
$$

has the following form:
\{ \}

$$
D_{g}=\sum_{i=1}^{N}\left(\prod_{j \neq i} \frac{4 x_{i}-x_{j}}{x_{i}-x_{j}}\right) T_{q_{1} x_{i}}
$$

## Solutions

[[Problems, 2-24|Problems]]
[[_Lecture 7, 2-24|Lecture]]

1
\{ \}
(1) $\underbrace{\operatorname{det}\left[x_{i}^{N-j}\right]}=\prod_{q<j}\left(x_{i}-x_{j}\right)$
polynomial in $x_{2}$,
vanishes when $x_{i}=x_{j} \quad \forall i \neq j$
$\Rightarrow$ LHS is divisible by pres.
Next, dyrie of RHS in each $x_{i}$ is $N-1$, which is the same as the degree of the LHS $\Rightarrow$

$$
\operatorname{det}\left[x_{i}^{N-j}\right]=\text { Court } \cdot \prod_{9<j}\left(x_{i}-x_{j}\right)
$$

To see Constr $=1$, take $x_{1}^{N-1} x_{2}^{N-2} \ldots x_{N-1}$ coefficient

- In RHS it is that const
- In LKS, it is the product of matrix elements over the wain diagonal.
\{ \}

Note: Same deft can be couprited using row/colormn transformations, bet we have avoided this.

2
\{ \}
(2) $S_{\lambda}\left(x_{1}, \ldots, x_{N}\right)$ - homos. paly in $x_{1}, \ldots x_{N}$ of total defoe $|\lambda|=\sum \lambda_{i}$

So, identity of

$$
S_{\lambda}\left(x_{1}, \ldots, x_{N}\right)=\sum_{\mu<\lambda} s_{\mu}\left(x_{1} \ldots x_{N-1}\right) x_{N}^{(\lambda)-1 \mu)}
$$

is equivalent to

$$
S_{\lambda}\left(\frac{x_{1}}{x_{N}}, \cdots, \frac{x x_{N-1}}{x_{N}}, 1\right)=\sum_{\mu^{k<\lambda}} S_{\mu}\left(\frac{x_{1}}{x_{N}}, \cdots \frac{x_{N-1}}{x_{N}}\right)
$$

3
\{ \}
(3) $\left(\frac{\partial}{\partial x_{i}}\right)^{2}(V f)=$

$$
\begin{array}{r}
=\left(V^{\prime} f+V f^{\prime}\right)^{\prime}=V^{\prime \prime} f+2 V^{\prime} f^{\prime}+V f^{\prime \prime} \\
=f\left(\frac{\partial}{\partial x_{i}}\right)^{2} V+2\left(\frac{\partial}{\partial x_{i}} f\right)\left(\frac{\partial}{\partial x_{i}} V\right)+V\left(\frac{\partial}{\partial x_{j}}\right)^{2} f .
\end{array}
$$

Then $\sum$ over $i$, and divide by $V$.
\{ \}

We have

$$
\begin{aligned}
\frac{\frac{\partial}{\partial x_{i}} V}{V} & =\frac{ \pm \frac{\partial x_{i}}{} \pi_{j \neq i}\left(x_{i}-x_{j}\right)}{\pi_{j \neq i}\left(x_{i}-x_{j}\right)} \\
& =\sum_{j=i+1}^{N} \frac{1}{x_{i}-x_{j}}-\sum_{j=1}^{i-1} \frac{1}{x_{j}-x_{i}} \\
& =\sum_{j=j \neq i} \frac{1}{x_{i}-x_{j}}
\end{aligned}
$$

\{\}

Next, we claim
$\sum_{i}\left(\frac{\partial}{\partial x_{i}}\right)^{2} V=0$
Fudeed, $\frac{\left(\frac{\partial}{\partial x_{i}}\right)^{2} V}{V}$ cantaing terms of


Next, wote thest

$$
\frac{1}{(x-y)(x-z)}+\frac{1}{(y-x)(y-z)}+\frac{1}{(z-x)(z-y)}=0
$$

so $\sum_{i}\left(\frac{\partial}{\partial x_{i}}\right)^{2} V=0$
\{ \}

Finally, we get the answer:

$$
D f=\underbrace{\Delta f}+2 \sum_{i=1}^{\Delta \text { Lpracioun }} \sum_{\substack{n \\ \sum_{i=1}^{N}\left(\partial x_{i}\right)^{2}}} \sum_{j \neq i} \frac{1}{x_{i}-x_{j}}] \frac{\partial f}{\partial x_{i}}
$$

4
\{\}
(4) Follows from

$$
\begin{aligned}
& T_{q, x_{i}}(V f)=T_{q, x_{i}} V \cdot T_{q, x_{i}} f, \\
& \frac{T_{q, x_{i}}}{V}=\prod_{i: i \neq i} \frac{q x_{i}-x_{j}}{x_{i}-x_{j}}
\end{aligned}
$$

[[Problems, $3-1 \mid 6$ problems + T4] ], due $3 / 15$

## 1 Schur polynomials

- 


## 1 Schur polynomials

In the last lecture, we gave two definitions of Schur polynomials, and showed the equivalence of their two definitions:
\{ \}


Def 1. $S_{\lambda}\left(x_{2}, \ldots, x_{N}\right)=\frac{\operatorname{det}\left[x_{i}{ }^{\lambda \rho+N-j}\right]_{i}^{N}}{V(\vec{x})}$

$$
V(\vec{x})=\prod_{1 \leqslant i<j \leqslant N}\left(x_{i}-x_{j}\right)
$$




$$
\lambda^{(1)}<\ldots \lambda \lambda^{(n-1)}<\lambda
$$

## 2 Basic properties of Schur polynomials

## 2 Basic properties of Schur polynomials

Proposition 2.0.1 Schur polynomial $s_{\lambda}\left(x_{1}, \ldots, x_{N}\right)$ is a symmetric polynomial in $x_{1}, \ldots, x_{N}$. It is a homogeneous polynomial, of degree $|\lambda|=\lambda_{1}+\ldots+\lambda_{N}$.

Proof 2.0.1 Straightforward from the definitions.

Proposition 2.0.2. Stability of Schur polynomials We will identify signatures $\lambda=\left(\lambda_{1}, \ldots, \lambda_{N}\right)$ and $\left(\lambda_{1}, \ldots, \lambda_{N}, 0,0, \ldots, 0\right)$ (any finite number of zeroes), where $\lambda_{N} \geq 0$.

With this identification, we have the following stability of Schur polynomials:
\{ \}
Pere. Stability

$$
\begin{aligned}
& S_{\lambda}\left(x_{1}, \ldots, x_{N-1}, 0\right)= \\
& \quad= \begin{cases}S_{\lambda}\left(x_{1}, \ldots, x_{N-1}\right), & \lambda_{N}=0 \\
0, & \text { else }\end{cases}
\end{aligned}
$$

In fact, this stability allows to define symmetric functions, which might be seen as an abstraction meaning "symmetric polynomials in an unspecified number of variables". We will not discuss these just yet.

Proof 2.0.2 \{ \}
Daff. $\operatorname{det}\left[\alpha_{i}^{\lambda_{j} \pi N-j}\right]$ -

$$
\begin{aligned}
&=\operatorname{det}\left[\begin{array}{ll}
x_{N}^{\lambda_{N}^{N N-1}} \ldots x_{N}^{\lambda_{N}^{N}}
\end{array}\right]=0 \\
& \begin{array}{l}
\text { if } \\
\lambda_{N}>0
\end{array} \\
&=\operatorname{det}\left[x_{t}^{\lambda_{j}^{j}+\alpha-j}\right]_{1}^{N-1}= \\
&=\left(x_{1} \ldots x_{N-1}\right) \operatorname{det}\left[x_{i}^{\lambda_{j}+(N-1)-j}\right]_{1}^{N-1} \\
& V\left(x_{1}, \ldots, x_{N-1}, 0\right)-\left(x_{1} \ldots x_{N-1}\right) V\left(x_{1} \ldots x_{N-1}\right)
\end{aligned}
$$

Proposition 2.0.3 The Schur polynomials $s_{\lambda}\left(x_{1}, \ldots, x_{N}\right)$, as $\lambda$ ranges over all nonnegative signatures with $N$ parts, form a basis in the space $\mathcal{S}=\mathbb{C}\left[x_{1}, \ldots, x_{N}\right]^{S_{N}}$ of symmetric polynomials.

Proof 2.0.3 Let us look at the space of polynomials $\mathbb{C}\left[x_{1}, \ldots, x_{N}\right]$. It has basis of monomials, $x_{1}^{\alpha_{1}} \ldots x_{N}^{\alpha_{N}},\left(\alpha_{1}, \ldots, \alpha_{N}\right) \in \mathbb{Z}_{\geq 0}^{N}$. That is, any polynomial is a finite linear combination of monomials.

Now, consider two subspaces, $\mathcal{S}=\mathbb{C}\left[x_{1}, \ldots, x_{N}\right]^{S_{N}}$ of symmetric polynomials, and $\mathcal{A}$ of antisymmetric polynomials. A polynomial is antisymmetric if $f(\sigma x)=$ $\operatorname{sgn} \sigma \cdot f(x)$, where $\sigma \in S_{N}$ acts by permuting the variables.
A basis in $\mathcal{A}$ is formed by antisymmetrized monomials,

$$
a_{\mu}\left(x_{1}, \ldots, x_{N}\right)=\sum_{\sigma \in S_{N}} \operatorname{sgn} \sigma \cdot x_{\sigma(1)}^{\mu_{1}} \ldots x_{\sigma(N)}^{\mu_{N}}
$$

Here $\mu_{1}>\ldots>\mu_{N} \geq 0$ must be strictly ordered. Another way to write $a_{\mu}$ is to use the determinant, $a_{\mu}\left(x_{1}, \ldots, x_{N}\right)=\operatorname{det}\left[x_{i}^{\mu_{j}}\right]_{i, j=1}^{N}$.
Moreover, any antisymmetric polynomial $f \in \mathcal{A}$ is divisible by the Vandermonde $V(x)=\prod_{i<j}\left(x_{i}-x_{j}\right)$, and the ratio is a symmetric polynomial. Therefore, we have $\mathcal{A}=\mathcal{S} \cdot V(x)$.
Passing from the basis in $\mathcal{A}$ to the basis in $\mathcal{S}$, we get Schur polynomials. And they form a basis in $\mathcal{S}$, as desired.

Remark 2.0.4 In fact, the functions $s_{\lambda}$ form an orthogonal basis, with respect to an inner product defined using certain contour integrals over a torus. We will need this fact soon, and then will formulate and prove it.

## 3 Skew Schur polynomials

- 


## 3 Skew Schur polynomials

Schur polynomials form a basis in symmetric polynomials. Viewing $s_{\lambda}\left(x_{1}, \ldots, x_{N}\right)$ as a symmetric polynomial in $\$ \mathrm{x} \_\{\mathrm{k}+1\}, \ldots, \mathrm{x} \_\mathrm{N} \$$ and expanding into the basis of Schur polynomials, we arrive at the following definition:

Definition 3.0.1. Skew Schur polynomials $\}$

$$
\begin{aligned}
& \text { Wef., } \underbrace{S_{\lambda}\left(x_{1}, \ldots, x_{N}\right)}_{\begin{array}{c}
\text { symu. in } \\
x_{k+1}, \ldots, x_{N}
\end{array}}=\sum_{\mu} \underbrace{S_{\lambda / \mu}\left(x_{1}, x_{k}\right)}_{\text {coeff }} \underbrace{S_{\mu}\left(x_{\left.k+1, \ldots, x_{N}\right)}^{S_{1}}\right)}_{\text {basis }} \\
& S_{\lambda / \mu}\left(x_{1, \ldots}, x_{k}\right)=\sum x_{1}^{\left|\lambda^{(1)}\right|-|\mu|} x_{2}^{\mid \lambda^{(2)}-\left(\lambda^{(1)} \mid\right.} \cdots x_{k}^{|\lambda|-\left|\lambda^{(2-1)}\right|} \\
& \frac{x_{k}}{} \lambda^{(k-1)}
\end{aligned}
$$

Skew Schur polynomials are symmetric and homogeneous of degree $|\lambda|-|\mu|$.

## Examples 3.0.2 \{\}

Ex. 1) $S_{(n) /(k)}(x)=x^{n-k}$

2) $S_{\lambda / \mu}(x)=\left\{\begin{array}{cl}x^{|\lambda|-(\mu \mid}, & \text { if } \mu\langle\lambda \\ 0, & \text { otherwise }\end{array}\right.$
3)

$$
\begin{aligned}
& S_{(0,0 \ldots 0)}(\vec{x})=1 \\
& S_{\lambda /(0 \ldots 0)}=S_{\lambda / \varnothing}=S_{\lambda}
\end{aligned}
$$

Proposition 3.0.3 \{ \}

$$
\text { Prop. } \quad \begin{aligned}
S_{\lambda / \mu} & \left(x_{1} \ldots x_{N}, y_{1} \ldots y_{\mu}\right)= \\
& =\sum_{x} S_{\lambda_{x}}\left(x_{1} \ldots x_{N}\right) s_{x / \mu}\left(y_{1} \ldots y_{\mu}\right)
\end{aligned}
$$

Proof 3.0.3 Quite straightforward:
\{ \}


## 4 Cauchy identities

- 


### 4.1 Formulation, examples

Theorem 4.1.1. Cauchy identity \{ \}


Remark 4.1.2 From stability ([[2 Basic properties of Schur polynomials\#Proposition 202 Stability of Schur polynomials|here]]) it follows that the numbers of variables can be different, $x_{1}, \ldots, x_{N}$ and $y_{1}, \ldots, y_{M}$. However, for the first proof it is essential that $M=N$.

Example 4.1.3 When $N=1$, Cauchy identity reduces to the geometric sum:

$$
\sum_{n \geq 0} x^{n} y^{n}=\frac{1}{1-x y}, \quad|x y|<1
$$

- 


### 4.2 Proof via determinants

\{ \}

$$
\begin{aligned}
& M=N . \quad V(x):=\prod_{k i<j \leqslant N}\left(x_{i}-x_{j}\right), \quad l_{j}:=\lambda_{j}+N-j \\
& \text { wis: } \sum_{\lambda_{1} \geq=\lambda_{N} \geqslant 0} \operatorname{det}\left[x_{i}^{\ell_{j}}\right]_{1}^{N} \operatorname{det}\left[y_{i}^{l_{j}}\right]_{1}^{N}= \\
& l_{1}>l_{N} \geq 0 \\
& =V(x) V(y) \prod_{i, j=1}^{N} \frac{1}{1-x_{i} y} .
\end{aligned}
$$

Here it is crucial that $M=N$.
Lemma 4.2.1. Cauchy-Binet identity. For $K \geq N$ :
\{ \}


Note two particular cases, $N=1$ is the definition of the matrix product. And $N=K$ is the fact that $\operatorname{det}(A B)=\operatorname{det} A \cdot \operatorname{det} B$.
For the proof, we refer to [[Problems, 3-1\#1| Problem 1]].

We apply Cauchy-Binet identity to infinite matrices:


$$
\operatorname{det}\left[x_{i}^{e_{j}}\right]=\operatorname{det}\left(A_{L}\right), \quad \operatorname{det}\left[y_{i}^{e_{j}}\right]=\operatorname{det}\left(B_{L}\right)
$$


\{ \}

$$
\begin{aligned}
\Rightarrow \sum_{\substack{|L|=N \\
\text { sidesets of } \\
\mathbb{Z} \geqslant 0}} \operatorname{det}\left(A_{L}\right) \operatorname{det}\left(B_{L}\right) & =\operatorname{det}(A B) \\
& =\operatorname{det}\left(\sum_{k \geqslant 0}\left(x_{i} y_{j}\right)^{K}\right)_{1}^{N} \\
& =\operatorname{det}\left(\frac{1}{1-x_{i} y_{j}}\right)_{q}^{N}
\end{aligned}
$$

It remains to compute an explicit $N \times N$ determinant.
Lemma 4.2.2. Cauchy determinant $\}$


Proof 4.2.2 There are two ways to prove this. We refer to [[Problems, 3$1 \# 2 \mid$ Problem 2]] and [[Problems, 3-1\#3|Problem 3]].
[[\#Lemma $4 \mathbf{4}^{2} 1$ Cauchy-Binet identity|First]] and [[\#Lemma 422 Cauchy determinant|second]] lemma imply the [[4.1 Formulation, examples\#Theorem 411 Cauchy identity|Cauchy identity]].

### 4.3 Skew Cauchy identity and a bijective proof

We give a second proof of the Cauchy identity.
First, generalize it:
Theorem 4.3.1. Skew Cauchy identity $\}$

Thu (Skew canchy identity)


$$
\text { If }\left|k_{i, j j}\right|<1 \quad \forall_{i, j}
$$

$$
\begin{aligned}
& \sum_{\nu} S_{v / \lambda}\left(x_{1} \ldots x_{N}\right) S_{v / \mu}\left(y_{1} \ldots y_{M}\right)= \\
& =\prod_{\substack{1<i \leq N \\
1 \leq j \leq M}} \frac{1}{1-x_{i} y_{j}} \sum_{x} S_{\lambda / x}\left(y_{1}, \ldots, y_{M}\right) s_{\mu / x}\left(x_{1} \ldots x_{N}\right)
\end{aligned}
$$

Remark 4.3.2 \{ \}
Rurk. 1) RHS is a finite sume, LHS infinite
2) $d=\mu=\phi \Rightarrow x=\varnothing \Rightarrow$ Previous Candly id

Proof 4.3.1 \{ \}

Step 1. Enough to show for $M=N=1$

$$
\begin{gathered}
\text { Ex. L8-4 }- \text { justify }^{\prime} \\
\left(\text { using } S_{\lambda / \mu}(x, y)=\sum_{x} s_{y_{x}}(x) S_{x / \mu}(y)\right)
\end{gathered}
$$

See [[Problems, 3-1\#4| Problem 4]].
\{ \}
Step 2.


We will produce a bijection:
\{ \}

$$
\text { Bijection } \begin{aligned}
\{x: & \lambda \succ e<\mu\} \times \mathbb{Z}_{\geqslant 0} \\
& \longleftrightarrow\{\boldsymbol{\omega}: \lambda<0\rangle \mu\}
\end{aligned}
$$

Bijection in a graphical form:
\{ \}







See also [[Problems, $3-1 \# 5 \mid$ problem 5]] for the statement which finalizes this proof.
\# 5 Schur measures and processes
5 Schur measures and processes
Definition 5.0.1. Schur measure $\}$

Def. $\quad S M(\lambda)=\prod_{i j}\left(1-x_{i} y_{j}\right) \cdot S_{\lambda}(\vec{x}) S_{\lambda}(\vec{y})$

1) $\begin{aligned} & \text { Prob measure on on } \\ & \lambda \text { - tuples }\end{aligned} \quad\left(\lambda_{1} \geqslant \ldots \geqslant \lambda_{N} \geqslant 0\right)$
(partitions - Young diagrams

2) $x_{1} \ldots x_{N}, \quad y_{1} \ldots y_{M} \geqslant 0$

- parameters, $x_{i} y_{j}<1 \forall i, j$

3) Sum to 1 by Cauchy

Notation, if parameters are important:

$$
S M_{(\vec{x}, \vec{y})}(\lambda)
$$

Definition 5.0.2. (Ascending) Schur process $\}$

$$
\begin{aligned}
& \begin{array}{r}
=\prod_{i j}\left(1-x_{i} y_{j}\right) \cdot S_{\lambda^{(n)}}\left(x_{1}\right) S_{\lambda^{(2)} / \lambda^{(N)}}\left(x_{2}\right) \ldots S_{\lambda^{(N)} / \lambda^{(N-1)}}\left(x_{N}\right) \\
\cdot S_{\lambda^{(N)}\left(y_{1}, \ldots, y_{\mu}\right)}
\end{array}
\end{aligned}
$$

Prob. meas on

1) $\lambda^{(1)} \alpha \ldots \lambda \lambda^{(N)}$ - interlacing arrays with random top row
2) $x_{1} \ldots x_{N}, \quad y_{1} \ldots y_{M} \geqslant 0$

- parameters, $x_{i} y_{j}<1 \forall i_{j}$

3) Sum to 1 by Cauchy \& def of skew functions

Notation:

$$
S P_{(\vec{x}, \vec{y})}\left(\lambda^{(1)}, \ldots, \lambda^{(N)}\right)
$$

Proposition 5.0.3 \{ \}
$\left.\int P_{(1, \ldots, 1 ;} ; \vec{y}\right)$ is Gibbs (of depth $N$ )
or any $(a, \ldots, a), a>0$.

Proof 5.0.3 \{ \}


We see that the probability weight does not depend on $\lambda^{(1)}, \ldots, \lambda^{(N-1)}$, and depends only on $\lambda^{(N)}$. This is precisely the Gibbs property from [[../Lecture 6, 222/_Lecture 6, 2-22|Lecture 6]] and [[../Lecture 7, 2-24/_Lecture 7, 2-24|Lecture 7]].

Proposition 5.0.4 Under the Schur process $S P_{\left(x_{1}, \ldots, x_{N} ; y_{1}, \ldots, y_{M}\right)}$, the marginal distribution of $\lambda^{(k)}$ is the Schur measure $S M_{\left(x_{1}, \ldots, x_{k} ; y_{1}, \ldots, y_{M}\right)}$.

Proof 5.0.4 (sketch) For simplicity, let $k=N-1$. Other cases are similar. We sum the probability weights of $S P_{\left(x_{1}, \ldots, x_{N} ; y_{1}, \ldots, y_{M}\right)}\left(\lambda^{(1)}, \ldots, \lambda^{(N-2)}, \nu, \lambda^{(N-1)}\right)$ over $\lambda^{(1)}, \ldots, \lambda^{(N-2)}$ and over $\lambda^{(N-1)}$, where $\nu$ is fixed. The first sum does not involve $y_{j}$ 's, and using the definition of the skew Schur polynomials we get $s_{\nu}\left(x_{1}, \ldots, x_{N-1}\right)$. Then we need to sum over $\lambda^{(N)}$. This sum looks as

$$
\sum_{\lambda^{(N)}} s_{\lambda^{(N)} / \nu}\left(x_{N}\right) s_{\lambda^{(N)}}\left(y_{1}, \ldots, y_{M}\right) .
$$

This sum can be computed using the [[4.3 Skew Cauchy identity and a bijective proof\#Theorem 431 Skew Cauchy identity|skew Cauchy identity]], and the result follows. [[Problems, $3-1 \# 6 \mid$ Problem 6]]: finalize the details of the proof.

## Notes and references

1. A fundamental treatise on symmetric functions, I.G. Macdonald, Symmetric functions and Hall polynomials, 1995. We covered some of the material from Ch. I, sections 2-5.
2. Schur measures and processes:

- Okounkov. Infinite wedge and random partitions. https://arxiv.org/abs/math/9907127
- Okounkov and Reshetikhin. Correlation function of Schur process with application to local geometry of a random 3-dimensional Young diagram. https://arxiv.org/abs/math/0107056


## Problems

[[_Lecture 8, 3-1|Lecture 8]]

1
Show the Cauchy-Binet identity:
\{ \}


2
Evaluate the [[4.2 Proof via determinants\#Lemma 422 Cauchy determinant|Cauchy determinant]] via induction on $N$ and row-column operations. Hint:
\{ \}

$$
\frac{1}{1-x_{i} y_{j}}-\frac{1}{1-x_{1} y_{j}}=\frac{y_{j}\left(x_{i}-x_{1}\right)}{\left(1-x_{i} y_{j}\right)\left(1-x_{i} y_{j}\right)}
$$

## 3

Evaluate the [[4.2 Proof via determinants\#Lemma 422 Cauchy determinant|Cauchy determinant]] via induction on $N$ Desnanot-Jacobi identity (also known as Dodgson condensation / Lewis Caroll identity):
\{ \}


That is, show that the right-hand side of the Cauchy identity satisfies the same quadratic relations.

## 4

Show that the skew Cauchy identity (see [[4.3 Skew Cauchy identity and a bijective proof\#Theorem 431 Skew Cauchy identity|here]]) for arbitrary number of variables follows from the skew Cauchy identity for single variables. For this, use the "branching" of skew Schur polynomials.

## 5

Show that the bijection given in [[4.3 Skew Cauchy identity and a bijective proof\#Proof $431 \mid$ this proof]] indeed preserves the powers of $x, y$ as it should.

## 6

Finish the proof of [[5 Schur measures and processes\#Proposition 504 |the marginal distribution under the Schur process]].

## T4

By Ti I will suggest topics for 10-minute talks, where $i$ is a counter throughout all lectures. For the fourth idea, I propose to explain how the bijective proof of the skew Cauchy identity given in [[4.3 Skew Cauchy identity and a bijective proof|this part]] upgrades to the Robinson-Schensted-Knuth correspondence.

Solutions
[[../../Lecture 8, 3-1/Problems, 3-1|Problems]]
[[../../Lecture 8, 3-1/_Lecture 8, 3-1|Lecture]]

1
\{ \}
(1) $L \cdot \operatorname{det}(I d+A B)=\operatorname{det}(I d+B A)$

Pf. if $A, B$ - squpere \& inwertible, then
$B(1+A B) B^{-1}=1+B A$, so they have saue det.

Extends to arb. rectangular matriels by coutimity \&
becanse we cant represent rect wast as segrare, $A \simeq A O D$
\{ \}

$$
n \stackrel{n}{B} k \quad \operatorname{det}(\underbrace{z+A B}_{n \times n})=z^{n-k} \underbrace{\operatorname{det}(z+B A)}_{k \times k}
$$

Taue colf by $z^{n-k}$.

$$
\operatorname{det}(B A)=\sum_{|I|=k, I \leq\{1-n\}} \operatorname{det} B I \operatorname{det} A_{I}
$$

2, 3
\{ \}
(2), (3) Skow that the Rres satisfies the desired sulations.
tedions compntations omitted

4
\{ \}


$$
\text { WTS: } \quad \sum_{V} S_{V / \lambda}\left(x_{1}, x_{2}\right) S_{V / \mu}\left(y_{2}, y_{2}\right)
$$

$$
=\prod_{i j} \frac{1}{1-x_{i} y_{j}} \sum_{x} S_{\lambda)_{x}}\left(y_{1}, y_{2}\right) S_{\mu / x}\left(x_{1}, x_{2}\right) .
$$

$$
\operatorname{LnS}=\sum_{v_{j} \alpha, \beta} \underbrace{S_{v / \alpha}\left(x_{2}\right) S_{v / \beta}\left(y_{2}\right)}_{\text {Aprly } 1 \text { vor. Skew Canchy }} S_{\alpha / \lambda}\left(x_{1}\right) S_{\beta / \mu}\left(y_{1}\right)
$$

\{ \}

$$
\frac{1}{1-x_{2} y_{2}} \sum_{\alpha, \beta, \gamma} s_{\alpha / \gamma}\left(y_{2}\right) s_{\beta / \gamma}\left(x_{2}\right) s_{\alpha / \lambda}\left(x_{2}\right) s_{\beta / \mu}\left(y_{2}\right)
$$

Apply 1 -vas. skew Cavely twice more

$$
\Rightarrow \text { get } \sum_{\gamma, \rho, \tau}
$$

For $\gamma$, apply sum cauvery wist $x_{1}, y_{1}$ the last timer, \&

$$
\rho, \tau \text { - use branching - }
$$

Several case is analogous. is

5
\{ \}
(5) paoers

$$
\begin{array}{ll}
\text { LHS } & x^{|v|} y^{|V|} x^{-|\lambda|} y^{-|\mu|} \\
\operatorname{RKS}: \quad(x y)^{k} x^{|\mu|} y^{|\lambda|}(x y)^{-|x|}
\end{array}
$$

\{ \}
bijection: $V_{1}=\lambda_{1} V \mu_{1}+k$

$$
\begin{gathered}
v_{i}^{0}=\lambda_{i} V \mu_{i}+\lambda_{i-1} \wedge \mu_{i-1}-x_{i-1} \\
\Downarrow \\
|\nu|=k+\underbrace{2|\lambda|+2|\mu|-|x|}
\end{gathered}
$$

maxt vilu at each step
$\Rightarrow$ powers are
presersed $\square$

6
\{ \}

$$
\begin{aligned}
& (6) \frac{1}{z} S_{\lambda^{\prime}}\left(x_{1}\right) S_{\lambda^{2} / \lambda^{\prime}}\left(x_{2}\right) \ldots S_{\lambda^{N} / \lambda^{N-1}}\left(x_{N}\right) \\
& S_{\lambda^{N}}\left(y_{1} \ldots y_{M}\right) \\
& \sum_{\substack{\lambda^{\prime} \cdots \lambda^{k-1} \\
\lambda^{k+1} \ldots \lambda^{N}}} \begin{array}{c}
\sum \text { over } \lambda^{\prime} \ldots \lambda^{k-1} \\
S_{\lambda^{k}}\left(x_{1} \ldots x_{k}\right)
\end{array}
\end{aligned}
$$

\{ \}

$$
\begin{aligned}
& \sum \text { over } \lambda^{k+1}, \ldots, \lambda^{N} \text { is } \\
& \sum S_{\lambda^{N}}^{S_{\lambda^{\prime}}\left(y_{1}-y_{n}\right)} S_{\lambda^{N} / \lambda^{k}}\left(x_{k+1}, \ldots x_{N}\right) \\
& =\underbrace{\prod_{j=+1}^{N} \frac{1}{1-y_{i} x_{j}}}_{\substack{j u s t \text { a } \\
\text { vorvatiziy } \\
\text { coustant }}}, S_{\lambda^{k}}\left(y_{1} \ldots y_{m}\right)
\end{aligned}
$$

[[Problems, 3-3|3 problems]], due 3/17
1 Orthogonality of Schur polynomials

1 Orthogonality of Schur polynomials
Definition 1.0.1 Here we define an inner product on the space of symmetric polynomials, under which Schur polynomials are orthonormal:
\{ \}

$$
\begin{aligned}
& S_{\lambda}\left(x_{1}, \ldots, x_{N}\right) \text { - basis of } \mathbb{G}\left[x_{1}, \ldots, x_{N}\right]^{S_{N}} \\
& \left(\lambda_{1} \geq \ldots \geqslant \lambda_{N} \geqslant 0\right) \\
& \left\langle S_{\lambda,} S_{\mu}\right\rangle=\delta_{\lambda, \mu} \quad \text { on } \mathbb{I}\left[x_{1}, \ldots, x_{N}\right]^{S_{N}} \\
& V \cdot \bar{V} \\
& \langle f, g\rangle=\frac{1}{N!} \int_{\left|z_{i}\right|=1} f \int_{i<j} f(\vec{z}) \overline{g(\vec{z})} \prod_{i}\left|z_{i}-z_{j}\right|^{2} . \\
& \ell(d z) \text { - normalized Lebesgue on towns } \\
& |z|=1 \\
& \Leftrightarrow \text { contono integrals } \frac{1}{2 \pi i} \oint_{|z|=1}^{\square} \frac{d z}{z} \\
& \int z^{k} l(d z)=\frac{1}{2 \pi \pi_{i}^{k}} \oint^{k} \frac{d z}{z}=\delta_{k, 0}
\end{aligned}
$$

Theorem 1.0.2 $\}$

$$
\begin{aligned}
& \text { Thu }\left\langle s_{\lambda}, s_{\mu}\right\rangle=\delta_{\lambda \mu} \quad \text { For } \lambda_{, \mu \mu} \in G T_{N} \\
& \text { (Not necessarily } \\
& \text { manesative) }
\end{aligned}
$$

Proof 1 of Theorem 1.0.2. Via determinants $\}$

$$
\begin{aligned}
& \text { Prof } 1 \text { (via dep's) } \quad l_{1} z \ldots>l_{N}, \mu_{1} \not \ldots>m_{N} \\
& \left\langle S_{\lambda}, S_{\mu}\right\rangle=\frac{1}{N!} \cdot \iint \operatorname{det}[\overbrace{z_{i}+N-j}^{l i j} \operatorname{det}[z_{i}^{-(\overbrace{j+N-j})} \quad l\left(d z^{\vec{z}}\right) \\
& =\frac{1}{N!} \sum_{\sigma, \tau \in S_{N}}(-1)^{b+\tau} \int Z_{1}^{l_{\sigma(1)^{-w} \tau(1)}} \cdots \quad \cdots z_{N}^{l_{\sigma(N)^{-m} \tau(N)}} \\
& =\frac{1}{N!} \sum_{j_{1} \tau}(-1)^{b, \tau} \prod_{j=1}^{N} \delta l_{2(j)}=m_{\tau(j)} \\
& (N!)^{2} \text { terms } \\
& =0, \quad \lambda \neq \rho^{\mu} \\
& =\frac{1}{N!} \cdot N!=1, \quad \lambda=\mu
\end{aligned}
$$

Proof 2 of Theorem 1.0.2. Via difference operators We can prove a weaker statement, orthogonality, using difference operators.

Recall the difference operators $D_{q}$.
\{ \}


One can show that $D_{q}$ are self-adjoint with respect to the inner product $\langle\cdot, \cdot\rangle$, that is, $\left\langle D_{q} f, g\right\rangle=\left\langle f, D_{q} g\right\rangle$. See [[Problems, 3-3\#1| Problem 1]].

Then
\{ \}

$$
\left\langle D_{q} s_{\lambda}, s_{\mu}\right\rangle=\left\langle s_{\lambda}, s_{\mu}\right\rangle \cdot \sum q^{\lambda_{i}+N-i}=\left\langle s_{\lambda}, s_{\mu}\right\rangle \cdot \sum q^{\mu_{i}+s_{-i}}
$$

$$
\Rightarrow \text { if } \sum_{i} q^{\lambda_{i}+N-i}=\sum_{i} q^{\mu_{i}+N-i} \quad \forall q \in \mathbb{C} \text {, }
$$

$$
\text { then it must be } \lambda=\mu
$$

## 2 Continuous specialization

- 


## 2 Continuous specialization

Proposition 2.0.1 We have an integral formula for Schur polynomials:
\{ \}

$$
\begin{aligned}
& S_{\lambda}(\vec{y})=\frac{1}{N!(2 \pi i)^{N}} \oint \frac{d \vec{x}}{\vec{x}} V(\vec{x}) V\left(\frac{1}{\vec{x}}\right) \overrightarrow{S_{\lambda}(\vec{x})} \cdot \prod_{i=1}^{N} \prod_{i=1}^{M} \frac{1}{1-x_{i} y_{j}} \\
& \left|x_{i}\right|=1 \\
& \begin{array}{l}
\text { K §arsund O, } \\
\text { not around poles yo }
\end{array}
\end{aligned}
$$

Proof 2.0.1 Recall Cauchy identity
\{ \}

$$
\begin{aligned}
& \sum_{v} S_{v}(\vec{x}) s_{v}(\vec{y})=\prod_{i j} \frac{1}{1-x_{i} y_{j}} \\
& \vec{y}=\left(y_{1} \ldots y_{M}\right), \quad\left|x_{i} y_{j}\right|<1
\end{aligned}
$$

Multiply it by $\overline{s_{\lambda}(x)}$ and integrate using orthogonality. This picks out the right coefficient:
\{ \}

$$
\left.\left.\frac{t}{(2 \pi i)} N \frac{1}{N!} \oint \right\rvert\, V(\vec{x})\right)^{2} S_{\lambda}(\vec{x}) \overrightarrow{S_{\mu}(\vec{x})} d \vec{x} / \vec{x}=\delta \lambda_{\mu}
$$

This proof requires some justifications of convergence, see [[Problems, 3-3\#2|Problem 2]].

Proposition 2.0.2. Plancherel specialization $\}$
Proof. For any fixed $\lambda, t \in \mathbb{R}_{>0}$,

$$
\lim _{M \rightarrow \infty} \delta_{\lambda}(\underbrace{\frac{t}{M}, \ldots, \frac{t}{M}}_{M}) \text { exists. }
$$

Notation: $S_{\lambda}\left(\rho_{t}\right)$
Co-imbors/Plancherel specialization

Proof 2.0.2 \{ \}

Proof.

$$
\begin{gathered}
S_{\lambda}\left(\rho_{t}\right)=\frac{1}{(2 \pi i)^{N} N!} \oint_{\left|x_{1}\right|=\ldots=\left|x_{N}\right|=1}^{x_{1}} \frac{d \vec{x}}{} \\
\bullet V(x) \cdot V\left(x^{-1}\right) S_{\lambda}\left(x^{-1}\right) \cdot e^{t\left(x_{1}+\ldots+x_{N}\right)}
\end{gathered}
$$

we used

$$
\frac{1}{(1-x t / M)^{M}} \rightarrow e^{x t}
$$

Corollary 2.0.3. Cauchy identity for Plancherel specialization \{ \}

$$
\underline{\text { Cor. } \sum_{\lambda} s_{\lambda}\left(x_{1,}, x_{N}\right) s_{\lambda}\left(\rho_{t}\right)=e^{t\left(x_{1} t-\cdots+x_{N}\right)}}
$$

See [[Problems, 3-3\#3| Problem 3]].

## 3 Push-block dynamics and Schur process - formutation

- 

3 Push-block dynamics and Schur process
\{ \}

$$
\begin{aligned}
& \text { Recall push-block precess } \\
& 1_{0}^{1} 1_{1}^{2} 2^{2}
\end{aligned}
$$

See [[../Lecture 6, 2-22/1.2 Push-block process\#Definition 121 definition]].
Theorem 3.0.1 \{\}

The

1) Push-block process
preserves the class of Gilts measures
2) Starting from packed initial condition, $J_{j}^{(N)}(0)=0 \quad \forall 1 \leq j \leq N$
after time $t \geqslant 0$ the joint' distribution of $\lambda^{(1)}(t), \ldots, \lambda^{(N)}(t)$ is

Scour process $S P_{(\underbrace{1, \ldots 1}_{N} ; \rho_{t})}$.

Note that this Schur process has a rather simple form: \{ \}

$$
\begin{aligned}
& \operatorname{SP}_{(1,1,1} ; \rho_{t)}\left(\lambda_{\nu}^{(1)} \lambda^{(\omega)}\right)= \\
& =e^{-t N} \cdot I_{\left.\left.\lambda^{(1)}<\lambda^{(1)}\right) \ldots\right\rangle \lambda^{(N)}} \cdot S_{\lambda^{(\omega)}}\left(\rho_{t}\right)
\end{aligned}
$$

This result can be proven in a number of ways. I am going to use the machinery of Schur processes and commuting Markov operators.

Corollary 3.0.2 \{ \}

Cor- If TASEP starts from the Step initial configuration $\quad x_{i}(0)=-i$,

$$
i \in \mathbb{Z}_{21}
$$

then after time $t \geqslant 0$,

$$
X_{N}(t)+N \quad \stackrel{d}{=} \lambda_{N}
$$

where $\lambda_{N}$ is the last component of the Sour measure

$$
\begin{gathered}
S M_{\left(1, \cdots, 1 ; \rho_{t}\right)}(\lambda)=e^{-t N} S_{\lambda} \underbrace{(1, \cdots, 1)}_{N} S_{\lambda}\left(\rho_{t}\right) . \\
\lambda=\left(\lambda_{1} \geq-\geqslant \lambda_{N} \geqslant 0\right) .
\end{gathered}
$$

4 Commuting Markov operators
4.1 Commuting operators from Schur polynomials

Let us set up some notation:
\{ \}


Spaces

$$
\begin{aligned}
& G T_{N}=\left\{\lambda_{1} \geq \ldots \geqslant \lambda_{N}\right\} \\
& G T_{N}^{+}=\left\{\lambda_{1} \geq \ldots \geqslant \lambda_{N} \geq 0\right\}
\end{aligned}
$$

Let $x=\left(x_{1}, \ldots, x_{N}\right), \quad \bar{x}=\left(x_{1}, \ldots x_{N-1}\right), y$ -all nonnegative, and $\left|x_{i} y\right|<1$

Definition 4.1.1 \{ \}


Def. $\Lambda$ : Markov operator $G T_{N}^{+} \rightarrow G T_{N-1}^{+}$
defined as $\measuredangle$ dep on $x_{1}, \ldots x_{N}$

$$
\Lambda(\lambda, \bar{\lambda}) \doteqdot \frac{S_{\bar{\lambda}}(\bar{x})}{S_{\lambda}(x)} S_{\lambda / \bar{\lambda}}\left(x_{N}\right)
$$

Markov indeed:

$$
\begin{array}{r}
\sum_{\bar{\lambda}} n(\lambda, \bar{\lambda})=\frac{1}{s_{\lambda}(x)} \sum_{\bar{\lambda}} s_{\bar{\lambda}}\left(x_{1}-x_{N-1}\right) s_{\lambda} /_{\lambda}\left(x_{N}\right) \\
=1
\end{array}
$$

Note $x_{i} \equiv 1 \Rightarrow$ related to Gibbs

Definition 4.1.2 \{ \}

Dep.ow $x_{1} \ldots x_{N}, y$
Def. $P_{N}: G T_{N}^{+} \longrightarrow G T_{N}^{+}$, Markov operator defined as (same for $P_{N-1}$ )
\{ \}


$$
=\frac{1}{\prod_{i}\left(1-x_{i} y\right)} s_{\lambda}(x) / s_{\lambda}(x)
$$

Theorem 4.1.3 We have the following commuting diagram of Markov operators:
\{ \}

(By $\lambda$, etc., we denote the elements of these sets)
In other words, for a fixed $\lambda$, the conditional distributions of $\bar{\nu}$ obtained by applying operators along two routes are the same.

Proof 4.1.3 \{ \}

Proof. wis

$$
\sum_{\bar{J}} \Lambda(\lambda, \bar{J}) P_{N-1}(\bar{\lambda}, \bar{v})=\sum_{v} P_{N}(\lambda, v) \Lambda(v, \bar{v})
$$

Let us write this out:
\{ \}

$$
\begin{aligned}
& \sum_{\bar{\lambda}} \frac{S_{\bar{\lambda}}(\bar{x})}{S_{\lambda}(x)} S_{\lambda / \bar{\lambda}}\left(x_{N}\right)_{S_{\bar{v}}(\bar{x})}^{S_{-\bar{x})}} S_{\bar{V} / \bar{\lambda}}(y) \cdot \prod_{1}^{N-1}(1-x \cdot y) \\
& =\sum^{\sum_{V}} \prod_{1}^{N}\left(1-x_{i} y\right) \frac{S_{v}(x)}{S_{\lambda}(x)} S_{v / \lambda}(y) \underbrace{\frac{S_{\bar{v}}(\bar{x})}{S_{v}(x)} S_{v / \bar{V}}\left(x_{N}\right)}
\end{aligned}
$$

Then there are lots of cancellations:
\{ \}

(Fellows from sew (avery).


### 4.2 From commuting operators to dynamics on interlacing arrays

Let us now shift the focus a little bit. Instead of trying to show that the pushblock dynamics is related to Gibbs property and Schur processes, let us develop the push-block dynamics as a particular case of a multilayer dynamics on interlacing arrays. The ingredients which we use are the commuting operators $\Lambda$ and $P_{N}$ 's.

## Two layers

We begin the discussion with two levels $N-1$ and $N$. Then we will find the solution, a Markov operator $Q$ from $\{(\bar{\lambda}, \lambda): \bar{\lambda} \prec \lambda\} \subset G T_{N-1}^{+} \times G T_{N}^{+}$to itself in the form
\{ \}

$$
\begin{aligned}
& Q\left(\begin{array}{lll}
\lambda & & \nu \\
\bar{\lambda} & \longrightarrow & \bar{\nu}
\end{array}\right)= \\
& =P_{v-1}(\bar{\lambda}, \bar{\nu}) U(\nu \mid \lambda, \bar{\lambda}, \bar{\nu}) \\
& \text { independent move } \bar{\lambda} \rightarrow \bar{v} \\
& \text { according to } P_{N-1} \\
& \text { neore } \Delta \rightarrow \nu \\
& \text { given } \bar{\lambda} \rightarrow \bar{i}
\end{aligned}
$$

There are many good couplings. One can show that they are encoded by functions $U$ satisfying:
\{ \}

Many "good" couplings:
can pick any $U$ satisfying:

1) $\sum_{v} U(0 \mid \lambda, \bar{\lambda}, \bar{j})=1 \quad$ (Marrow upolate)
2) $\sum_{\bar{\lambda}} u(\nu \mid \lambda, \bar{\lambda}, \bar{\nu}) \pi(\lambda, \bar{\lambda}) p_{N-1}(\bar{\lambda}, \bar{\nu})=$

$$
=P_{N}(\lambda, v) \wedge(\nu, \bar{v})
$$

The second condition means that on "Gibbs" distributions, ie., on those compatible with $\Lambda$, the update $\lambda \rightarrow \nu$ is governed by the upper Markov chain $P_{N}$.

For a particular solution (in fact, the simplest one), we pick $U$ in the form \{ \}

$$
\begin{aligned}
& U(\nu \mid \lambda, \bar{\lambda}, \bar{\nu}) \text { index of } \bar{\lambda} \text {. }
\end{aligned}
$$

In other words, we update $\bar{\lambda} \rightarrow \bar{\nu}$ on the lower level first, and then pick $\nu$ as the middle point in the Markov chain $\lambda \rightarrow \nu \rightarrow \bar{\nu}$, conditioned on $\lambda$ and $\bar{\nu}$. This effectively "forgets" the state $\bar{\lambda}$.

## Multiple layers

\{ \}


For any fixed number $N$ of layers, we update $\lambda^{(1)} \rightarrow \nu^{(1)}$ first, then given the update on the first layer, update the second layer using $U$, and so on. On each pair of consecutive levels, we use the two-level mechanism described above.

Theorem 4.2.1 Thus defined Markov operator $Q$ on interlacing arrays with $N$ rows preserves the class of Gibbs measures.

Proof 4.2.1 This follows from the properties of $U$. The update looks like: \{ \}

$$
\begin{aligned}
Q\left(\left\{\lambda^{(1)}\right\} \rightarrow\right. & \left.\left\{\nu^{(1)}\right\}\right)= \\
= & \left.P_{1}\left(\lambda^{(1)} \rightarrow \nu^{(1)}\right) U\left(\nu^{(2)}\right) \lambda^{(1)}, \lambda^{(2)}, \nu^{0}\right) \\
& 0 U\left(\nu^{(3)} \mid \lambda^{(2)}, \lambda^{(3)}, v^{(2)}\right) \\
0 \ldots & \cdots\left(\nu^{(N)} \mid \ldots\right)
\end{aligned}
$$

The action on Gibbs measures looks like:
\{ \}

$$
\begin{aligned}
& \left\{\lambda^{(i)}\right\} \\
& \left.P_{1}\left(\lambda^{(1)} \rightarrow \nu^{(1)}\right) U\left(\nu^{(2)}\right) \lambda^{(1)}, \lambda^{(2)}, \nu^{0)}\right) \\
& 0 U\left(v^{(3)} \mid \lambda^{(2)}, \lambda^{(3)}, v^{(2)}\right) \\
& o \cdots \cdots \cdots(U)
\end{aligned}
$$

Sum over $\lambda^{(1)}$, turn $P_{1}$ into $P_{2}$,

$$
\Lambda\left(\lambda^{(2)}, \lambda^{(1)}\right) \rightarrow \Lambda\left(\nu^{(2)}, \nu^{(1)}\right)
$$

\& continue on
$\Rightarrow$ end up with

$$
\begin{aligned}
& \sum_{(N)} P_{N}\left(\lambda^{(N)} \rightarrow v^{(N)}\right) M\left(\lambda^{(N)}\right) 0 \\
& \left.\lambda^{(N)} \nu^{(N-1)}\right) \ldots \Lambda\left(v^{(2)} v^{(1)}\right)
\end{aligned}
$$

Notes and references

1. The proof of orthogonality via difference operators extends almost without changes to Macdonald polynomials. See I.G. Macdonald, Symmetric
functions and Hall polynomials, 1995, Chapter VI.9.
2. The idea of commuting Markov operators is originally due to $P$. Diaconis and J.A. Fill, Strong stationary times via a new form of duality, Ann. Probab. 18 (1990), 1483-1522.
3. Application commuting Markov operators to interlacing arrays is due to A. Borodin and P. Ferrari, Anisotropic growth of random surfaces in $2+1$ dimensions, Comm. Math. Phys. 325 (2014), 603-684, arXiv:0804.3035 [math-ph]. See also section 2 in A. Borodin and L. Petrov, Nearest neighbor Markov dynamics on Macdonald processes, (2013), arXiv:1305.5501 [math.PR] for a more general construction.

## Problems

[[_Lecture 9, 3-3|Lecture 9]]

## 1

Recall the inner product
\{ \}

$$
\langle f, g\rangle \underset{N!}{\langle } \int_{\left|z_{i}\right|=1} \quad \int_{i<j} f(\vec{z}) \overrightarrow{g(\vec{z})} \prod_{i}\left|z_{i}-z_{j}\right|^{2} .
$$

Show that the difference operator $D_{q}$ is self-adjoint: $\left\langle D_{q} f, g\right\rangle=\left\langle f, D_{q} g\right\rangle$.
Hint: Recall $D_{q}=V^{-1} \circ \sum_{i=1}^{N} T_{q, x_{i}} \circ V$. Therefore, in the inner product, under the integral we can write

$$
\left(D_{q} f\right) V \overline{V g}=\left(\sum_{i=1}^{N} T_{q, x_{i}}(V f)\right) \overline{V g}
$$

Therefore, it remains to show that $T_{q, x_{i}}$ is self-adjoint (for each $i$ ).

## 2

Justify that the integration and the summation can be swapped in the application of orthogonality to Cauchy identity (see [[2 Continuous specializa-
tion\#Proposition 20 1| Proposition 2.0.1]]).

3
Prove the version of the Cauchy identity for the Plancherel specialization: \{ \}

$$
\text { Cor- } \sum_{\lambda} s_{\lambda}\left(x_{1,-,} x_{N}\right) s_{\lambda}\left(\rho_{t}\right)=e^{t\left(x_{1}+\cdots+x_{N}\right)}
$$

[[Problems, $3-8 \mid 6$ problems]], due $3 / 22$

## 1 Setup

### 1.1 Schur polynomials - two key properties

There are two fundamental properties of the Schur polynomials: the branching rule

Proposition 1.1.1 \{ \}

$$
S_{\lambda}\left(x_{1}-x_{N}\right)=\sum_{\mu=\mu 々 \lambda} S_{\mu}\left(x_{1} \ldots x_{N-1}\right) S_{\lambda / \mu}\left(x_{N}\right)
$$

Proposition 1.1.2 and the skew Cauchy identity
\{ \}

$$
\begin{aligned}
& \sum_{V} S_{v / \lambda}(x) S_{v / \mu}(y)=\frac{1}{1-x y} \sum_{x} S_{\mu / x}(x) S_{\lambda / x}(y) \\
& \lambda_{\left.\sum_{x}^{x}\right]_{\mu}^{v} y}^{v}
\end{aligned}
$$

We proved both results in previous lectures

### 1.2 Commuting Markov operators

Recall that we employ the notation for the space of signatures:

$$
G T_{N}^{+}=\left\{\lambda=\left(\lambda_{1} \geq \ldots \geq \lambda_{N} \geq 0\right): \lambda_{i} \in \mathbb{Z}\right\}
$$

Definition 1.2.1 We defined two Markov operators:
\{ \}

$$
\begin{aligned}
& \lambda_{\lambda}\left(\lambda, \bar{\lambda}=S_{\lambda / \bar{\lambda}}\left(x_{N}\right) \frac{S_{\lambda}\left(x_{1}-x_{N-1}\right)}{S_{\lambda}\left(x_{1}-x_{N}\right)}\right. \\
& P_{N}(\lambda, v)=\prod_{\substack{0 \\
j=1}}^{N}\left(1-x_{1} y\right) \cdot S_{v / \lambda}\left(y_{1}\right) \frac{S_{v}\left(x_{1}-x_{N}\right)}{S_{\lambda}\left(x_{1}-x_{N}\right)}
\end{aligned}
$$

Example 1.2.2 \{ \}

$$
\begin{aligned}
P_{1}(l \rightarrow n) & =(1-x, y) \cdot y^{n-e)^{\sec } \frac{s_{n}\left(x_{1}(y)\right.}{} \frac{\left(x_{1}\right)}{s_{e}\left(x_{1}\right)}} \\
n \geqslant l & =\underbrace{\left(1-x_{1} y\right)\left(y x_{1}\right)^{n-l}}_{\text {Geom. distribution }}
\end{aligned}
$$

Theorem 1.2.2 These operators form a commutative diagram:
\{ \}


We proved this result in the previous lecture
1.3 Markov dynamics Q on two levels

Definition 1.3.1 Using $P_{N-1}$ and $P_{N}$ commuting with the $\Lambda$ 's, we define: \{ \}

$$
\begin{aligned}
& \text { Define } Q\left[\begin{array}{lll}
\lambda & \left.\frac{0}{\lambda}\right]
\end{array}\right] \text { as } \\
& =P_{N-1}(\bar{\lambda}, \bar{\nu}) \cdot U(\nu \mid \lambda, \bar{\nu}) \text {, where } \\
& U(\nu \mid \lambda, \bar{\nu}) \text { is tue middle point } \\
& \text { distribution in } \\
& d \xrightarrow{\rho_{N}}(\nu) \xrightarrow[\nu]{\nu}
\end{aligned}
$$



This is a simultaneous one-step update of the pair $(\bar{\lambda}, \lambda)$ to $(\bar{\nu}, \nu)$, which is compatible with the whole scheme of commuting Markov operators.
-

### 1.4 Multilevel dynamics

Definition 1.4.1 In a similar fashion to the two-level case, we can define the multilevel update:
\{ \}

$$
\begin{aligned}
& Q\left[\begin{array}{ccc}
\lambda^{(N)} & v^{(N)} \\
\vdots & \rightarrow & \vdots \\
\lambda^{(1)} & v^{(1)}
\end{array}\right]= \\
& =P_{1}\left(\lambda^{(1)} \rightarrow v^{(1)}\right) U_{2}\left(v^{(2)} \mid \lambda^{(1)}, v^{(1)}\right) 0 \\
& \\
& \left.0 U_{3}\left(v^{(3)}\right) \lambda^{(2)} v^{(2)}\right) \cdots-U_{N}\left(v^{(N)} \mid \lambda^{(N-1)}, v^{(N-1))}\right)
\end{aligned}
$$

Here $P_{1}$ is a rather simple object, it is a geometric random walk (see [[1.2 Commuting Markov operators\#Example 12 2|here]]).

## 2 Action on Gibbs measures

- 


### 2.1 Action on abstract Gibbs measures

Definition 2.1.1 Recall that an abstract Gibbs measure is defined as follows: \{ \}
$\downarrow$
$G T_{1}^{+}$


That is, we take an arbitrary probability distribution $M$ on the $N$-th floor $G T_{N}^{+}$, and attach to it a chain of $\Lambda$ 's.

Theorem 2.1.2 The multilevel process $Q$ preserves the class of abstract Gibbs measures. Moreover, the action on a Gibbs measure as above changes $M$ as

$$
M \mapsto \widetilde{M}(\nu)=\sum_{\lambda \in G T_{N}^{+}} M(\lambda) P_{N}(\lambda, \nu)
$$

Proof 2.1.2 The key fact is the followin refinement of the commutation relation:
\{ \}

$$
\begin{aligned}
& \sum_{\bar{\lambda}} u_{N}(v \mid \lambda, \bar{v}) \Lambda(\lambda, \bar{\lambda}) P_{N-1}(\bar{\lambda}, \bar{\nu}) \\
&= P_{N}(\lambda, v) \Lambda(v, \bar{v}) \\
& \begin{array}{l}
\text { sum over } \nu, \\
\text { get } \quad P_{N} \Lambda=\Lambda P_{N N-1}
\end{array}
\end{aligned}
$$

We apply it repeatedly:
\{ \}

$$
\begin{aligned}
& \begin{array}{ll}
\sum_{\lambda^{(1)} \lambda^{(N)}} & M\left(\lambda^{(N)}\right) \\
& \ldots \\
\ldots & \Lambda\left(\lambda^{(N)} \lambda^{(N)} \lambda^{(v-1)}\right) \ldots \\
& \\
& \Lambda\left(\lambda^{(2)} \lambda^{(1)}\right) .
\end{array} \\
& \text { - } \left.P_{1}\left(\lambda^{(1)}, v^{(7)}\right) H_{2}\left(\nu^{(2)}\right] \lambda^{(1)} v^{(1)}\right) \ldots \\
& \text { 为 } \\
& u_{N}\left(v^{(N)} \mid \lambda^{(\nu-1)}, v^{(N-1)}\right)
\end{aligned}
$$

In the fist step, the sum over $\lambda^{(1)}$ involves only the highlighted terms, and they produce $P_{2}\left(\lambda^{(2)}, \nu^{(2)}\right) \Lambda\left(\nu^{(2)}, \nu^{(1)}\right)$.
Similarly, on the next step, $P_{2}, U_{3}$, and $\Lambda\left(\lambda^{(3)}, \lambda^{(2)}\right)$ leads to $P_{3}$ and $\Lambda\left(\nu^{(3)}, \nu^{(2)}\right)$.
This process goes in the same manner till $\lambda^{(N)}$. The final summation over $\lambda^{(N)}$ involves $M$ and $P_{N}$, and produces the action $M \mapsto \widetilde{M}$ as in the claim. This completes the proof.
-

### 2.2 Application to Schur processes

Theorem 2.2.1 \{ \}

$$
\begin{aligned}
& \frac{1}{Z} S_{\lambda}(1)\left(x_{1}\right)-S_{\lambda}^{(N)} / \lambda^{(N-1)}\left(x_{N}\right) . \\
& \downarrow-S_{\lambda^{(N)}}\left(y_{1}-y_{M}\right) \\
& \text { Time. Action on } S P_{\left(x_{1}, \ldots, x_{N} ; y_{1}, \ldots, y_{M}\right)} \\
& \left.Q^{(\tilde{y})}: \operatorname{SP}_{\left(x_{1}, \ldots, x_{N}\right.}^{0} y_{1}, \ldots, y_{M}\right) \longrightarrow \\
& \rightarrow \int_{\left(x_{1}, \ldots, x_{N} 0 y_{1}, \ldots, y_{M}, y\right)}^{N}
\end{aligned}
$$

Here $\widetilde{y}$ is the parameter which was denoted by $y$ in $P_{N}$.
Proof 2.2.1 \{ \}
Proof. SP is "Gibbs" wot maps $\Lambda$, so it suffices to check the action of $P_{N}$ on marginal $\lambda^{(N)}$ :


That is, let us now check the action only on the $N$-th level:
\{ \}



$$
=S M_{\left(x_{1} \ldots x_{N} ; y_{1 \ldots} y_{M}, \tilde{y}\right)}(v)
$$

## 3 Continuous time limit

### 3.1 Continuous time limit of the push-block process

We claim that the continuous time limit of the multilevel dynamics as $y=d t \rightarrow 0$ is the push-block process. Here we set $x_{i} \equiv 1$, and need to replace the variable $\widetilde{y}$ which enters $P_{N}$ by the continuous specialization $\rho_{d t}$. By [[Problems, 3$8 \# 4 \mid$ Problem 4]], this is the same as simply taking the expansion of all the Markov operators into series in $\widetilde{y}$. Therefore, we can simply replace $\widetilde{y}$ by the specialization $\rho_{d t}$ everywhere in the computations.
Example 3.1.1. Case $\mathbf{N}=1$. $\}$

$$
\begin{aligned}
& (\operatorname{Case} N=1) \quad \lambda^{(1)}=l, V^{(1)}=n, y \\
& P_{1}(l \rightarrow n)=(1-y) \frac{S_{l}(1)}{S_{n}(1)} S_{n / l}(g)=(1-j) y^{n-l} I_{n>l} \\
& \text { A }_{l l}(l \mid n,-,-) \text { (geom-distribution) } \\
& y=d t \longrightarrow 0 \text { : }
\end{aligned}
$$

Proposition 3.1.2 In the general case, using [[Problems, 3-8|Problems]], we can show that: $\}$

$$
P_{N}(\lambda, v)=\left\{\begin{array}{l}
1+0(d t), \quad \lambda=a \\
O(d t),|v|=|\lambda|+1, \quad \nu>\lambda \\
O\left(d d^{2}\right) \uparrow, \text { else }
\end{array}\right.
$$

- all because of factor.

$$
\begin{aligned}
& \text { has a } \oint \\
& \text { formula }
\end{aligned}
$$

$$
\Rightarrow \text { in the "stitched" dynamics, }
$$

$$
\begin{aligned}
& \text { at mort one particle should } \\
& \text { jump indequendenty of a time }
\end{aligned}
$$

Theorem 3.1.3 In the limit $d t \rightarrow 0$, the push-block component $U(\nu \mid \lambda, \bar{\nu})$ becomes:

1. If $\bar{\nu} \prec \lambda$ :

$$
U(\nu \mid \lambda, \bar{\nu})= \begin{cases}d t, & |\nu|=|\lambda|+1, \lambda \prec \nu, \bar{\nu} \prec \nu \\ 1-O(d t), & \nu=\lambda \\ O\left(d t^{2}\right), & \text { else }\end{cases}
$$

In the multilevel dynamics $Q$, this corresponds to an independent jump of any unblocked particle at rate 1 if $\bar{\lambda}=\bar{\nu}$ (there were no jumps at lower levels); or to no move if $\bar{\lambda}=\bar{\nu}$ and $\bar{\nu} \prec \lambda$ (if there were moves at lower levels). Note that the condition $\bar{\nu} \prec \nu$ corresponds to blocking: if a move $\lambda \rightarrow \nu$ would violate the interlacting, then this move is blocked.
2. If $\bar{\nu} \nprec \lambda$, we have $U(\nu \mid \lambda, \bar{\nu})=1$ for the unique $\nu$ which would restore the interlacing $\nu \succ \lambda, \nu \succ \bar{\nu}$.
\{ \}


Proof 3.1.3 \{ \}

$$
U^{\text {Recall }} U^{(y)}(v \mid \lambda, \bar{v})=\frac{p_{v}^{(y)}(\lambda, v) \wedge(v, \bar{v})}{\sum_{\bar{\lambda}} \Lambda(\lambda, \bar{\lambda}) P_{N-1}^{(y)}(\bar{\lambda}, \bar{v})}
$$

For the first case, the denominator is of a constant order, and so in the numerator we pick something of constant order and of order $d t$. We have

$$
\sum_{\bar{\lambda}} \Lambda(\lambda, \bar{\lambda}) p_{N-1}^{(y)}(\bar{\lambda}, \bar{\lambda})=
$$

=only terms $\sim$ comet matter (no dr)

$$
=\Lambda(\lambda, \bar{\lambda})+O(d t)
$$

\{ \}

$$
\begin{aligned}
& \text { Then } \frac{n(v, \bar{\lambda})}{\Lambda(\lambda, \bar{\lambda})} P_{N}(\lambda, \nu)= \\
& =\frac{S_{\lambda}\left(1^{N-1}\right)}{S_{v}\left(1^{N}\right)} \cdot \frac{S_{\lambda}\left(1^{N}\right)}{S_{\bar{\lambda}}^{-\left(I^{N-1}\right)}} \cdot(1-d t)^{N} \frac{S_{v}\left(1^{N}\right)}{S_{\lambda}\left(1^{N}\right)} S_{v / \lambda}(d t) \\
& \left.I_{v>\lambda} \quad \Rightarrow \text { get (dts) if } \mid v\right)=|\lambda|+1 .
\end{aligned}
$$

\{ \}


For the second case,
\{ \}
$\bar{D} \& \lambda$, we still have

(v) there is a unique $V$ which restores inserlacing

$$
\bar{\lambda} \xrightarrow{1 \text { move }} \bar{v}
$$


\{ \}

$$
\begin{aligned}
& \text { Here in } \sum_{\bar{\lambda}} P_{N-1}(\bar{\lambda}, \bar{\gamma}) \wedge(\lambda, \bar{x}), \bar{\lambda}^{\lambda} \text { is } \\
& \frac{P_{N}(\lambda, v) \wedge(\nu, \bar{v})}{\Lambda(\lambda, \bar{\lambda}) P_{N-1}(\bar{\lambda}, \bar{v})}=
\end{aligned}
$$

$$
\begin{aligned}
& \text { It need to be congensated } \\
& \Rightarrow \text { testing reeckonisu! }
\end{aligned}
$$

This implies our [[../Lecture 9, 3-3/3 Push-block dynamics and Schur process\#Theorem 301 |main result]] from the previous lecture, and the [[../Lecture 9, 3-3/3 Push-block dynamics and Schur process\#Corollary 30 2|corollary]] about connection to TASEP.

In particular, the action of the push-block process on the Schur process $S P_{\left(1, \ldots, 1 ; \rho_{t}\right)}$ for some other time $t^{\prime}$ produces the Schur process $S P_{\left(1, \ldots, 1 ; \rho_{t+t^{\prime}}\right)}$.
-

### 3.2 Continuous time version of $\mathbf{P} \_\mathbf{N}$

Recall the operator of the marginal Markov process on $G T_{N}^{+}$:
\{ \}

$$
P_{N}(\lambda, v)=\prod_{i=1}^{N}\left(1-x_{i} y\right) \cdot \frac{S_{V}\left(x_{1}-x_{N}\right)}{S_{\lambda}\left(x_{1}-x_{N}\right)} S_{V / \lambda}(y)
$$

This is the marginal evolution of the $N$-th level, on Gibbs measures.
Theorem 3.2.1 For $y=\rho_{t}$, the Markov process $P_{N}$ becomes a Markov jump process with the following jump rates:
\{ \}


This theorem follows from lemma:
Lemma 3.2.2 \{ \}

$$
\underbrace{S_{\lambda}(1, \ldots, 1)}_{\|}=\prod_{1 \leqslant i<j \leqslant N} \frac{\lambda_{i}-\lambda_{j}+j-j}{j-i}
$$

\# of arrays w.top row $\lambda$.

Proof 3.2.2 \{ \}

\{ \}

\{ \}


Example 3.2.3 So, $P_{N}$ becomes a particle system, in which particles jump to the right by one, provided that the desination is not occupied. However, this process is not the same as TASEP, and they have very different jump rates. In TASEP, the jump rates are "local", while in $P_{N}$, the jump rates depend on
the whole configuration.
\{ \}


## Notes and references

1. The idea of commuting Markov operators is originally due to $P$. Diaconis and J.A. Fill, Strong stationary times via a new form of duality, Ann. Probab. 18 (1990), 1483-1522.
2. Application commuting Markov operators to interlacing arrays is due to A. Borodin and P. Ferrari, Anisotropic growth of random surfaces in $2+1$ dimensions, Comm. Math. Phys. 325 (2014), 603-684, arXiv:0804.3035 [math-ph]. See also section 2 in A. Borodin and L. Petrov, Nearest neighbor Markov dynamics on Macdonald processes, (2013), arXiv:1305.5501 [math.PR] for a more general construction.

## Problems

[[_Lecture 10, 3-8|Lecture 10]]

## 1

Let $\lambda, \nu \in G T_{N}^{+}$. Show that if $\lambda \prec \nu$ (interlace) and $|\lambda|=|\nu|$, then $\lambda=\nu$.

## 2

Establish the following contour integral formula for the skew Schur polynomials: \{ \}

$$
\begin{aligned}
& S_{\lambda / \mu}\left(y_{1}, \ldots, y_{m}\right)=\frac{1}{N!_{0}\left(2 \pi_{1}\right)^{N}} \oint_{\substack{\text { around } 0 \\
\text { and not } y_{j}^{-1}}} \frac{d x_{1}, d x_{N}}{x_{1} \ldots x_{N}} . \\
& \qquad \prod_{0}\left(x_{i}-x_{j}\right)\left(x_{i}^{-1}-x_{j}^{-1}\right) S_{\lambda}\left(x_{1}^{-1}, \ldots, x_{N}^{-1}\right) S_{\mu}\left(x_{\left.1, \ldots, x_{N}\right)} \prod_{\substack{1 \leqslant i \leqslant N \\
1 \leqslant i \leqslant j \leqslant M}} \frac{1}{1-x_{i} y_{j}}\right.
\end{aligned}
$$

Hint: use the skew Cauchy identity and orthogonality as in [[../Lecture $9,3-3 / 2$ Continuous specialization\#Proposition 20 1|here]].

## 3

Show that for the continuous specialization $\rho_{t}$ we have \{ \}

$$
\begin{aligned}
& S_{\lambda / \mu}\left(\rho_{t}\right)=\frac{1}{N!\left(a n_{1}\right)^{N}} \oint_{\text {around } 0} \oint_{1} \frac{d x_{1-}-d x_{N}}{x_{1}-x_{N}} . \\
& 0 \prod_{1 \leq 1<j \leqslant N}\left(x_{i}-x_{j}\right)\left(x_{i}^{-1}-x_{j}^{-1}\right) S_{\lambda}\left(x_{1}^{-1}, \ldots, x_{N}^{-1}\right) S_{\mu}\left(x_{1, \ldots}, x_{N}\right) e^{t\left(x_{1}+\ldots+x_{N}\right)}
\end{aligned}
$$

## 4

For the continuous specialization $\rho_{d t}$, as $d t \rightarrow 0$, we have the following expansion:

$$
s_{\lambda / \mu}\left(\rho_{d t}\right)= \begin{cases}1+O(d t), & \lambda=\mu \\ O(d t), & |\lambda|=|\mu|+1 \\ O\left(d t^{2}\right), & \text { else }\end{cases}
$$

Hint: use the contour integral formula from the previous problem.

## 5

Prove [[3.1 Continuous time limit of the push-block process\#Proposition 31 2|Proposition 3.1.2]].

## 6

Show that the Schur process $S P_{(1, \ldots, 1) ; \rho_{0}}$ with the continuous specialization $\rho_{t}$ at $t=0$ is the delta measure at the densely packed configuration $\lambda_{j}^{(k)}=0$ for all $k, j$.
[[Problems, 3-10|3 problems]], due 3/24

## 1 TASEP and Schur measures

- 


## 1 TASEP and Schur measures

Let us recall what was proven over the course of the past lectures.
Theorem 1.0.1. Schur measure \{ \}

$$
\begin{array}{r}
\text { Start TASEP from step } I C, \quad x_{i}(0)=-i \\
\ldots, \ldots x \times x, \\
i=1,2,3, \ldots
\end{array}
$$

Then:

$$
X_{N}(t)+N \stackrel{D}{=} \lambda_{N},
$$

$$
\text { where } \lambda_{N} \text { is the last consent of }
$$

$$
\lambda \sim S M_{\left(1,->1 ; \rho_{t}\right)}^{N}
$$

$$
P_{\text {rob }}(\lambda)=e^{-N t} S_{\lambda}(\underbrace{1,-, 1}_{N}) S_{\lambda}\left(\rho_{t}\right), \quad t \in \mathbb{R}_{\geqslant 0}
$$

Theorem 1.0.2. Schur process Under the assumptions of the previous theorem,
\{ \}

$$
\begin{aligned}
& \left\{x_{1}(t)+1, x_{2}(t)+2, \ldots, x_{N}(t)+N\right\} \\
& \stackrel{\varnothing}{=}\left\{\lambda_{1}^{(1)}, \lambda_{2}^{(2)}, \lambda_{3}^{(3)}, \ldots, \lambda_{N}^{(N)}\right\}
\end{aligned}
$$

## 2 Density of a Schur measure

- 


### 2.1 Expectation via q-difference operators

Now we use the $q$-difference operators $D_{q}^{(x)}$ in the variables $x_{1}, \ldots, x_{N}$, to extract observables of Schur measures.

Recall:
\{ \}

$$
\begin{aligned}
& T_{q, x} f(x)=f(x \cdot q) \\
& D_{q}^{(x)}=\sum_{i=1}^{N} \prod_{j \neq i}^{q \mid x_{i}-x_{j}} \frac{x_{i}-x_{j}}{} T_{q, x_{i}} \\
& D_{q}^{(x)} S_{\lambda}(x)=\left(\sum_{i=1}^{N} q^{\lambda_{i}+N-i}\right) S_{\lambda}(x)
\end{aligned}
$$

Proposition 2.1.1 \{ \}

$$
\begin{aligned}
& \text { Let } x=\left(x_{1}, \ldots, x_{N}\right), \quad y=\left(y_{1}, \ldots, y_{M}\right) \\
& \left|x_{i} y_{j}\right|<1 \quad \forall i, j \\
& \sum_{\lambda} s_{\lambda}(\vec{x}) s_{\lambda}(\vec{y})=\Pi(\vec{x}, \vec{y}), \quad \Pi(\vec{x}, \vec{y})=\prod_{i j} \frac{1}{1-x_{i} y_{j}} \\
& \sum_{\lambda}\left(\sum_{i=1}^{N} q^{\lambda_{i}+N-i}\right) s_{\lambda}(\vec{x}) s_{\lambda}(\vec{y})=D_{q}^{(x)} \Pi(\vec{x}, \vec{y}) .
\end{aligned}
$$

Proof 2.1.1 This is straightforward - we can apply the operator under the summation over $\lambda$. We assume that $|q|<1$, so that the series with the $q$ dependent prefactor converges.

Proposition 2.1.2 The action of the $q$-difference operator has a form of a contour integral:
\{ \}

$$
\begin{aligned}
& \text { Let }\left|x_{i} y_{j}\right|<1 \text { for all } i_{i j} \\
& \sum_{\lambda}\left(\sum_{i=1}^{N} q^{\lambda i+N-i}\right) S_{\lambda}(x) S_{\lambda}(y)= \\
& =\prod(x, y) \frac{1}{2 \pi i} \oint \prod_{j=1}^{N} \frac{4 z-x_{j}}{z-x_{j}} \frac{1}{4 z-z} . \\
& \text { (viz) around } x_{L_{j}, \chi_{N}} \text { aud not } \prod_{j=1}^{M} \frac{1-z y_{j}}{1-4 z y_{j}} d z
\end{aligned}
$$

Proof 2.1.2 Let us take the integral and evaluate it as a sum of residues at all $z=x_{i}$. We have
\{ \}


The last product over $1 \leq j \leq M$ is
\{ \}


We see that this completely matches the action of $D_{q}^{(x)}$.
Corollary 2.1.3. Expectation over a general Schur measure $\}$


Corollary 2.1.3. Expectation over a Schur measure with continuous specialization $\}$

$$
\begin{aligned}
& \prod_{\operatorname{SM}(1, \ldots, 1 ; \rho t)}^{\sim N}\left(\sum_{i=1}^{N} a^{\lambda_{i}+N-1}\right)= \\
& =\frac{1}{2 \pi i} \oint_{\substack{\text { around } \\
1}}\left(\frac{q z-1}{z-1}\right)^{N} \frac{1}{q z-z} e^{(q-1) z t} d z
\end{aligned}
$$

2.2 Density function

Now we can employ the second integration, now over $q$, to extract the coefficients by any fixed power of $q$. This produces the density function:

Proposition 2.2.1 \{ \}

$$
\begin{gathered}
\text { For } x \in \mathbb{Z}_{\geq 0}, \\
\mathbb{P}\left(\exists j ; \quad \lambda_{j}+N-j=x\right) \\
=\frac{1}{2-i} \oint_{\text {around } 0} \frac{d q}{q^{\alpha+1}} \mathbb{E}_{S M\left(1,-1 ; p_{t}\right)}\left(\sum_{i=1}^{N} q^{\lambda_{i}+N-i}\right)
\end{gathered}
$$

This is a double integral:
\{ \}

$$
=\frac{1}{(2 \pi i)^{2}} \oint \frac{d q}{q^{x+1}} \oint d z\left(\frac{q z-1}{z-1}\right)^{N} \frac{1}{q z-z} e^{(q-1) z t}
$$



Proof 2.2.1 This is straightforward, by interchanging integration over $q$ and expectation. Since the $q$ integration contour is around 0 , we can make $|q|$ small, so there are no convergence issues.

Example 2.2.2. Notation for density $\}$

$$
\begin{aligned}
& N=6 \\
& \lambda=(5,5,3,2,1,0) \\
& \downarrow \\
& \left\{\lambda_{i}+N-i\right\}=(10,9,6,4,2,0)
\end{aligned}
$$

We denote for any $x \in \mathbb{Z}_{\geq 0}$ :
\{\}

$$
\mathbb{P}\left(\exists j: \lambda_{j}+N-j=x\right)=r_{N, t}(x)
$$

- density function of this random $N$-point configuration.

Claim 2.2.3 (to be proven) $\}$

$$
x / N=x \in \mathbb{R}_{\geqslant 0}
$$

In limit $N \rightarrow \infty, t \rightarrow \infty$ if $\frac{t}{N}=\tau \in \mathbb{R}_{>0}$,


3 Asymptotic of density
3.1 Change of variables

Recall the density
\{ \}

$$
r_{N, t}(x)=
$$

$$
\left.=\frac{1}{(2 \pi i)^{2}}\right\} \frac{d q}{q^{x+1}} \oint d z\left(\frac{q z-1}{z-1}\right)^{N} \frac{1}{q z-z} e^{(q-1) z t}
$$



Via a simple change of variables $q=w / z, d q=d w / z$, where $w$ is a new variable around 0 , we can write:
Proposition 3.1.1 \{\}


### 3.2 Oscillatory integrals

In the regime as $x, t, N$ proportionally go to infinity, the three components under the integral,

$$
(z / w)^{x}, \quad((w-1) /(z-1))^{N}, \quad e^{t(w-z)}
$$

generically explode or decay to zero exponentially fast.
Indeed, as $N \rightarrow \infty$,

$$
\left(|z| e^{i \varphi}\right)^{N}=|z|^{N} e^{i N \varphi} \rightarrow \begin{cases}\infty, & |z|>1 \\ 0, & |z|<1\end{cases}
$$

and when $|z|=1$, the behavior is purely oscillatory.
Therefore, for a nontrivial asymptotics we need to find the right balance between all the components.
Let us consider a simpler case of a single integral.

Proposition 3.2.1. Stationary phase / Laplace method For $f$ smooth, bounded, $\mathbb{C}$ - valued, and with finitely many global maxima, we have the following asymptotics as $N \rightarrow \infty$ :

$$
\int_{-\infty}^{+\infty} e^{N f(x)} d x \sim \operatorname{Poly}(N) \cdot \exp \left\{N \cdot \max _{x \in \mathbb{R}} \operatorname{Re} f(x)\right\}
$$

where $\operatorname{Poly}(N)$ is some factor which grows at most polynomially.
Proof 3.2.1 See [[Problems, 3-10\#2|Problem 2]].
We will apply the idea of this result to double oscillating integrals (like we see in the density function). This is usually referred to as a saddle point / steepest descent method.

### 3.3 Double integrals and asymptotics of the density. Complex conjugate case

Let us set $t=N \tau, x=\lfloor N \chi\rfloor$, and let $N \rightarrow \infty$. Here $\tau, \chi \geq 0$ are the scaled time and space, respectively.

Then, the integrand in $r_{N, t}(x)$ given by [[3.1 Change of variables\#Proposition 3 1 1|Proposition 3.1.1]] takes the form
$\frac{1}{w(w-z)} \exp \left[N\left\{\frac{\lfloor\chi N\rfloor}{N}(\log z-\log w)+\log (w-1)-\log (z-1)+\tau(w-z)\right\}\right]$.
The quantity $\frac{\lfloor\chi N\rfloor}{N}$ can be replaced by $\chi$, as asymptotically this is irrelevant (on the scale we're working at).

Definition 3.3.1 Define the function

$$
S(w):=-\chi \log w+\log (w-1)+\tau w
$$

Remark 3.3.2 Any branches of the logarithms work, as were taking their exponents or will be working with their real parts and derivatives. All these operations do not depend on the branch.
So, we have $\}$

$$
P_{N, t}(x)=\frac{1}{\left(2 \pi_{i}\right)^{2}} \int_{\int}^{(w-z) w} e^{N[S(w)-S(z)]}
$$

Let us study the asymptotics of the integrand $e^{N(S(w)-S(z))}$, which is dictated by looking at maxima and minima of $\operatorname{Re} S(z)$, along the contours.

Here is an example of a surface $\operatorname{Re} S(z)$, for some particular $\tau$, $\chi$.
\{ \}


The $w$ and $z$ contours are drawn.
Clearly, for this configuration of the contours, we have $\operatorname{Re} S(w)-\operatorname{Re} S(z)>0$ for $z, w$ on the contours, which suggests that the density might be going to infinity.

In fact, this is not the case. One reason for this is that the density is always between zero and one. Another reason is that while the integral of the absolute value might be large, the signs present in the integrand might lead to cancellation of exponentially large terms.
So, our goal is to move the $w$ contour to minimize $\operatorname{Re} S(w)$, and move the $z$ contour to maximize $\operatorname{Re} S(z)$. At the same time, the contours cannot cross 0 and 1 (but might cross each other; possibly at a cost of picking residues).

Let us show how to place the integration contours so that to see the asymptotics of the integral. Let us look for saddle points of $\operatorname{Re} S(z)$, which are the same as the critical points of $S(z)$ :
\{ \}

$$
\begin{aligned}
& S^{\prime}(z)=\tau-\frac{x}{z}+\frac{1}{z-1}=0 \\
& z_{1,2}=\frac{x+\tau-1 \pm \sqrt{D}}{2 \tau} \\
& D=\tau^{2}+(x-1)^{2}-2 \tau(x+1)
\end{aligned}
$$

In the above picture $D<0$, which corresponds to two complex conjugate saddles $z_{c}, \bar{z}_{c}$ (where $z_{c}$ is in the upper half plane), and clearly $\operatorname{Re} S\left(z_{c}\right)=\operatorname{Re} S\left(\bar{z}_{c}\right)$.

In this piece we consider only the case $D<0$.

Definition 3.3.3 Let the region $\mathcal{L}$ be the $(\tau, \chi)$ region in which $D<0$. This is readily seen to be

$$
(\sqrt{\tau}-1)^{2}<\chi<(\sqrt{\tau}+1)^{2}
$$

In $\mathcal{L}$, the region where $\operatorname{Re}\left(S(z)-S\left(z_{c}\right)\right)>0$ looks as follows:
Contour configuration 1
\{ \}


Here the old contours $z, w$ are also given.
Let us move the $z, w$ contours to the new position as follows:
Contour configuration 2
\{ \}


On the new contours we have $\operatorname{Re}(S(w)-S(z)) \leq 0$, and via a little careful analysis one can show that the double contour integral over these new contours goes to zero.

However, it is not true that $r_{N, t}(x)$ equals this double contour integral over the new contours which goes to zero. Indeed, in the process of moving the contours, we cross the pole at $z=w$, which leads to a residue:
\{ \}


The residue at $w=z$ is simply equal to $1 / z$, see [ [3.1 Change of variables\#Proposition 31 1|Proposition 3.1.1]].

Putting this all together, we have proven the following result:

Theorem 3.3.4 In the regime $t=\tau N, x=\lfloor\chi N\rfloor$ and $N \rightarrow+\infty$, where $(\tau, \chi) \in \mathcal{L}$ (i.e., $\left.(\sqrt{\tau}-1)^{2}<\chi<(\sqrt{\tau}+1)^{2}\right)$, the density function of the Schur measure $S M_{\left(1, \ldots, 1 ; \rho_{t}\right)}$ converges to:

$$
r_{\tau}(\chi)=\frac{1}{2 \pi i} \int_{\bar{z}_{c}(\tau, \chi)}^{z_{c}(\tau, \chi)} \frac{d z}{z}
$$

where the integration contour from $\bar{z}_{c}$ to $z_{c}$ crosses the real line to the right of 0 . This does not finish the proof of TASEP's limit shape, which will be continued in [[../Lecture 12, 3-15/_Lecture 12, 3-15|Lecture 12]]. There are two things remaining: 1. Complete the analysis of the double contour integral in the case when there are two real saddle points (instead of two complex conjugate) 2 . Recover the TASEP limit shape parabola from these computations.

## Notes and references

1. The approach of Schur measure asymptotics via contour integrals is pioneered by Okounkov in Symmetric functions and random partitions
(2003), arXiv:math/0309074 [math.CO], section 3.
2. Another approach to the same asymptotic analysis goes via orthogonal polynomials and their associated difference operators, arXiv:math/0610240 [math .PR]
3. The use of difference operators to extract density follows the Macdonald processes work (arXiv:1111.4408 [math.PR]), see also arXiv:1310.8007 [math .PR]

## Problems

[[_Lecture 11, 3-10|Lecture 11]]

1

1. Show that the expectation $\}$

$$
\mathbb{E}_{S M\left(1, \ldots, 1 ; \rho_{t}\right)}\left(\sum_{i=1}^{N} q^{\lambda_{i}+N-i}\right)
$$

is finite for arbitrary $q \in \mathbb{C}$.
2. When the above Schur measure is replaced by a general one $S M_{(\vec{x}, \vec{y})}$, show that the same expectation exists for $|q|<1$.

2

1. Prove [[3.2 Oscillatory integrals\#Proposition 32 1|Proposition 3.2.1]].
2. Compute the polynomial factor $\operatorname{Poly}(N)$ in this proposition, assuming for simplicity that $f(x)$ has a unique global maximum. You will see that the factor $\operatorname{Poly}(N)$ in fact goes to zero and does not even grow.

## 3

1. Justify the location of the contours $\operatorname{Re} S(z)=\operatorname{Re} S\left(z_{c}\right)$ separating the blue and the white regions in [[3.3 Double integrals and asymptotics of the density. Complex conjugate case\#Contour configuration 1|this figure]], in the case when $(\sqrt{\tau}-1)^{2}<\chi<(\sqrt{\tau}+1)^{2}$.
2. Show that the contours $z, w$ in the double integral can be moved, without crossing any poles except $w=z$, to the new locations in [[3.3 Double integrals and asymptotics of the density. Complex conjugate case\#Contour configuration $2 \mid$ this figure]]
[[Problems, 3-15|2 problems]], due $3 / 29$

## 1 Schur / TASEP / density

- 


## 1 Schur-TASEP-density function

Here we recall the main ingredients that we proved so far.
1.0.1 Schur and TASEP matching $\}$

$$
\begin{align*}
& \text { Start TASEP from step IC } \\
& \quad x_{i}(0)=-i, \quad \quad i=1,2,3, \ldots \\
& \text { Then } \quad x_{N}(t)+N \stackrel{d}{=} \lambda_{N}, \\
& \text { where } \lambda=\left(\lambda_{1} \geqslant \cdots \geqslant \lambda_{N} \geqslant 0\right) \text { is from } \\
& \qquad S M\left(1, \ldots, 1 ; \rho_{t}\right) . \\
& \left.P(\lambda)=e^{-t N} S_{\lambda}(1 \ldots)\right) S_{\lambda}\left(\rho_{t}\right)
\end{align*}
$$



$$
\begin{aligned}
& \sum_{\lambda} s_{\lambda}\left(x_{1}, \ldots, x_{N}\right) s_{\lambda}\left(p_{t}\right)=e^{t\left(x_{1}+\ldots+x_{N}\right)} \\
& D_{q} s_{\lambda}=\left(\sum_{i=1}^{N} q^{\lambda_{i}+N-i}\right) s_{\lambda} \\
& \Rightarrow E_{S M}\left[\sum_{i=1}^{N} q^{\lambda_{i}+N-i}\right]=\left.\frac{D_{q}^{(x)} e^{t\left(x_{1}+\ldots+x_{N}\right)}}{e^{t\left(x_{1}+\ldots+x_{N}\right)}}\right|_{x_{i}=1} \\
& =\frac{1}{2 \pi i} \oint_{\text {around } x_{i}} \prod_{j=1}^{N} \frac{q^{z z-x_{j}}}{z-x_{j}} e^{(q-1) t z} \frac{d z}{(q-1) z} \\
& \left(\& \text { set } x_{j}=1 \quad \forall j\right)
\end{aligned}
$$

1.0.3 Density function $\}$
$x \in \mathbb{Z}$

$$
\begin{aligned}
& P_{N, t}(x)=\circledast\left(\exists j \text { such that } \lambda_{j}+N-j=x\right) \\
& \left(\int \operatorname{in}_{n} q \text {, change var. } q=w / z\right) \\
& =\frac{1}{\left(d \tau_{i}\right)^{2}} \oint \oint \frac{d w d z}{\omega(w-z)} \exp \{N(S(\omega)-S(z))\} \\
& S(z)=\frac{t}{N} z+\log (z-1)-\frac{x}{N} \log (z)
\end{aligned}
$$

Note that the integral has the following singularities:

- At infinity there are essential singularities
- Pole at $w=0$
- Pole at $z=1$
- Simple pole at $w=z$

There are no other singularities. In particular, we can drag the $z$ contour through 0 , and/or the $w$ contour through 1 .

## 2 Density asymptotics and the parabola

- 


### 2.1 Formulation

Theorem 2.1.1 \{ \}

$$
\begin{aligned}
& \text { Thu. Let } \quad \frac{t}{N} \rightarrow \tau, \quad \frac{x}{N} \rightarrow \chi \quad(\tau, x \geq 0) \\
& \text { Then } \\
& r_{N, t}(x) \longrightarrow r_{\tau}(x)=\left\{\begin{array}{l}
\frac{1}{2 \pi k} \int_{\bar{z}_{c}}^{z_{c}} \frac{d \tau}{z}, \\
\text { if }(\sqrt{\tau}-1)^{2}<x<(\sqrt{\tau}+1)^{2} \\
\tau<1 \\
1, x<(\sqrt{\tau}-1))^{2}, \tau \leqslant 1 . \\
0, \text { else }
\end{array}\right. \\
& 0 \sum_{z_{c}}^{z_{c}} \\
& z_{c}=\frac{x+\tau-1+\sqrt{D}}{2 \tau}, D=\tau_{\tau}^{2}+(x-1)^{2}-2 \tau(x+1)<0
\end{aligned}
$$

Example 2.1.2 The limiting density tells us that the density of the point configuration $\left\{\lambda_{j}+N-j\right\}$ looks as follows in the two regimes:
\{ \}

$$
\tau \leqslant 1, \quad \lambda_{N} \sim 0
$$


2.1.3 From arc integral to the argument \{ \}

$$
\begin{aligned}
& \text { Note } \frac{1}{2 \pi i} \int_{r e^{-i \varphi}}^{r e^{i \varphi}} \frac{d z}{z}=\frac{1}{2 \pi i} \int_{-\varphi}^{\varphi} \frac{i r e^{i t} d t}{r e^{i t}} \\
& z(t)=r e^{i t},-\varphi<t<\varphi \\
& d z=i r e^{i t} d t \\
& =\frac{1}{2 \pi} \int_{c}=P e^{i} \\
& -\varphi
\end{aligned}
$$

### 2.2 TASEP parabola from density asymptotics

Let us leave contour integrals for a moment, and show how the parabola limit shape of the TASEP height function is obtained from the result of [[2.1 Formulation\#Theorem 21 1|Theorem 2.1.1]].
2.2.1. Asymptotics of the $\mathbf{N}$-th particle We have
\{ \}

$$
\begin{aligned}
& r_{N, T N}\left(x_{N}\right) \rightarrow 0 \text {, if } \quad x<(\sqrt{\tau}-1)^{2} \\
& \text { last } \chi \text { for which } r>0 \text { is } \\
& \begin{array}{l}
x \simeq(\sqrt{\tau}-1)^{2} \quad \begin{array}{r}
\tau<1, x=0, \\
\text { particle didint }
\end{array} \\
\left.\frac{\lambda_{N}}{N} \simeq(\sqrt{\tau}-1)^{2}, \tau \geq 1 \quad \begin{array}{rl}
\text { yet start } \\
\text { to trove }
\end{array}\right)
\end{array} \\
& \text { Back to TASEP: } \quad \lambda_{N}-N \stackrel{d}{=} x_{N}(t) \text {, so } \\
& \frac{X_{N}(\tau N)}{N} \simeq-1+(\sqrt{\tau}-1)^{2}=\tau-2 \sqrt{\tau}
\end{aligned}
$$

(we assume that $\tau>0$, because for $\tau \leq 1$ the particle $x_{N}$ has not yet started moving, and so its location is determined)
The theorem thus gives:
\{ \}
look at height function again

$\Rightarrow$ particle $x_{N}$ is at $\simeq N(\tau-2 \sqrt{\tau})$
2.2.2. Homogeneity trick $\}$
$\Rightarrow$ particle $X_{\text {LIN }}$ is at

$$
X_{\lfloor C N J}(\tau N)=c N\left(\frac{\tau}{c}-2 \sqrt{\frac{\tau}{c}}\right)=N(\tau-2 \sqrt{\tau C})
$$

(as if time is $c N \frac{\tau}{c}$ )

$$
\frac{\uparrow}{\frac{\tau}{c} \geqslant 1}
$$

(if $\frac{\tau}{c}<1, X_{\lfloor C N\rfloor}$ did not start to move get)
2.2.3 From particle to the density $\}$

$\Rightarrow 1=\sqrt{\frac{\tau}{c}} \rho(\tau, \tau-2 \sqrt{c \tau})$
$\Rightarrow \rho(\tau, \underbrace{\tau-2 \sqrt{c t}}_{x})=\sqrt{\frac{c}{\tau}} / x(c)=\frac{(\tau-x)^{2}}{4 t}$
\{ \}




$$
|x|<\tau \quad \text { (proof of forme for for }
$$ the livest shame:)

$$
\int_{x}^{\tau} \rho(\tau, y) d y=\frac{(\tau-x)^{2}}{4 \tau}
$$

$$
\begin{aligned}
& \& \text { height funct. } \\
& \text { is } h(\tau, x)=2 \frac{(t-x)^{2}}{4 \tau}+x=\frac{t^{2}+x^{2}}{2 t} \text {. }
\end{aligned}
$$

We see that this gives the desired $\rho(\tau, \chi)$, and the parabola height function.
2.2.4. Remark We have checked that $\rho(\tau, \chi)$ satisfies the Burgers equation, but to rigorously conclude that this is the TASEP limiting density we need local invariance of the TASEP distributions, and also some PDE theory.

This proof of the formula for $\rho(\tau, \chi)$ is more "elementary" as it relies only on the analysis of the exact formula for the density.
2.3 Density - last steps of the proof

Recall density:
\{ \}

$$
\begin{aligned}
& P_{N_{1} t}(x)=P(\exists j \text { such twat } \lambda j+N-j=x) \\
& (\delta \ln q \text {, change var. } q=\omega / z) \\
& =\frac{1}{\left(\alpha x_{i}\right)^{2}} \oint \oint \frac{d \omega d z}{\omega(w-z)} \exp \{N(S(\omega)-S(z))\} \\
& S(z)=\frac{t}{N}+\log (z-1)-\frac{x}{N} \log (z)
\end{aligned}
$$

We will indicate the necessary transformations which bring the density to the desired limit.
2.3.1. Case 1, complex conjugate critical points \{ \}


We did this computation in detail last time.
2.3.2 Cases aa, 2b, ec leading to density $\mathbf{0}$ or 1 Let us define three further regions:

- 2a: $\tau>1,(\sqrt{\tau}-1)^{2}>\chi ;$
- 2b: $(\sqrt{\tau}+1)^{2}<\chi$;
- 2c: $\tau<1, \chi<(\sqrt{\tau}-1)^{2}$

Depending on these cases, there are three different locations of the real critical points.

Lemma 2.3.3. aa In the case ea, two critical points $z_{1}, z_{2}$ satisfy

$$
0<z_{1}<z_{2}<1, \quad \operatorname{Re} S\left(z_{1}\right)<\operatorname{Re} S\left(z_{2}\right)
$$

Lemma 2.3.4. 2b In the case 2 b , two critical points $z_{1}, z_{2}$ satisfy

$$
1<z_{1}<z_{2}, \quad \operatorname{Re} S\left(z_{1}\right)>\operatorname{Re} S\left(z_{2}\right)
$$

Lemma 2.3.5. 2c In the case 2c, two critical points $z_{1}, z_{2}$ satisfy

$$
z_{1}<z_{2}<0, \quad \operatorname{Re} S\left(z_{1}\right)>\operatorname{Re} S\left(z_{2}\right)
$$

Proofs of these lemmas involve an analysis of concrete functions, and we leave them as exercises.
2.3.6. 2 a - new contours and density asymptotics $\}$

2.3.7. 2 b - new contours and density asymptotics $\}$

\{ \}

2.3.8. Rc - new contours and density asymptotic $\}$


After all these contour moves and asymptotics of the density, the proof of [[2.1 Formulation\#Theorem $211 \mid$ Theorem 2.1.1]] is completed. Thus, we have established the parabola limit shape for the TASEP height function
-

### 2.4 A remark on the limit shape of Young diagrams

$$
\lambda=\left(\lambda_{1} \geqslant \ldots \geqslant \lambda_{N} \geqslant 0\right)
$$



$$
\begin{array}{ll}
N, t \rightarrow \infty, & t=\tau N \\
\text { Shape of } & \lambda_{?}^{?}
\end{array}
$$

It is enough to use [[2.1 Formulation\#Theorem 21 1|Theorem 2.1.1]] together with the remark on the $\arg \left(z_{c}\right) / \pi$ asymptotics of the density. This (almost) produces the result of Diane (2001). Here is a picture from that paper.
\{ \}


## 3 Towards fluctuations in TASEP

- 


### 3.1 Approach with single q

Let us now discuss the question of asymptotic fluctuations in TASEP \{ \}

$$
x_{L[N]}(\tau N)=N(\tau-2 \sqrt{\tau C})+\underbrace{\substack{ \\\text { Jolaksson }}}
$$

\{\}
We know, $x_{N}+N \stackrel{d}{=} \lambda_{N}$, last past
\&

$$
P_{N, t}(x)=\mathbb{P}\left(\exists j \text { such that } \lambda_{j}+N-j=x\right)=\oint \oint
$$

but, how to mare sure $j=N$ in $\mathbb{P}$ ?
i.e. Want $\mathbb{P}\left(\lambda_{N}=x\right)$ ?

$$
\mathbb{P}\left(x_{N}+N>x+N\right)=\mathbb{P}\left(\lambda_{N}>x\right)
$$

$=\mathbb{P}\left(\forall N \leqslant x\right.$, there is no $j$ sit. $\left.\lambda_{j}+N-j=y\right)$

$$
\uparrow
$$

nontrivial dependence on $y$,
So, not just $\sum_{y} \prod_{N, t}(y) \cdots \prod_{y} r_{N, t}(y)$

Let us indicate a possible approach using the fact that the single- $q$ integral for the expectation over the Schur measure has $q$ as a free parameter.
\{ \}

$$
\mathbb{E}\left(\sum_{j} q^{\lambda} \lambda_{j}+N_{j}^{0}\right)=\oint
$$

but how to ensue that we pick the minimal power of $q$,

$$
\phi^{\lambda_{N}} \quad ?
$$

$$
\text { Maybe knowing } \lambda_{N}=N(\sqrt{\tau}-1)^{2}+\xi
$$

Tave

$$
\left[\mathbb{E}\left(\sum_{j}^{\left(\sum_{j} q^{\lambda j+N-j}\right)}\right]^{q^{-N(\sqrt{E}-1)^{2}}} \quad \& \operatorname{let} q \rightarrow 0\right.
$$

This expectation should be a combination of 0,1 , and $\infty$ with some weights. Maybe there is a way of working this approach towards at least the right order of the fluctuations $N^{1 / 3}$ ?

### 3.2 Multiple contour integral approach

\{ \}

Recall

$$
\mathbb{E}\left(\sum 4^{\lambda_{j}+N-j}\right)=\left.\frac{D_{g}^{x} e^{t \sum x_{i}}}{e^{t \sum x_{i}}}\right|_{x_{i} \equiv 1}
$$

\{ \}

Then toy:

$$
\begin{aligned}
& \mathbb{E}\left[\left(\sum q_{1}^{\lambda_{j}+N-j}\right) \ldots\left(\sum_{\ldots}^{\lambda_{j}+N-j}\right)\right] \\
& =\left.\frac{D_{q_{1}}^{(x)} \ldots-D_{q_{m}}^{(x)} e^{t \sum x_{i}}}{e^{t \sum x_{i}}}\right|_{x_{i}=1}=\oint_{m} . \oint
\end{aligned}
$$

\{ \}

Then we could extract powers of
$q_{1}, \ldots, q_{u} \& a_{k}$ for the
same probability
$\mathbb{P}\left(\forall y \leqslant x\right.$, there is no $j$ sit. $\left.\lambda_{j}+N-j=y\right)$

1) $\{$ no particle at each $y \leq x\}$
is liver inckesion-exclusion,
so in principle
worms for extracting
coefficients
\{ \}
2) Need $y=0,1, \ldots x$
and $x$ is proportional to $N$


In fact, there is hope, and it's based on two "magic" steps:

First, the $2 x$-fold contour integral can be rewritten as $x \times x$ deteminant of double integrals. This is still a deteminant of growing size, but... Second, there is a certain structure of this growing determinant (it is in fact equal to a certain Fredholm determinant), which allows to analyze it asymptotically.

## Notes and references

1. Paper by Philippe Biane - https://arxiv.org/abs/math/0006111

- A determinantal point process approach to this setting https://arxiv.org/abs/math/0610240

2. Vershik-Kerov's paper on the limit shape for $\tau \ll 1-\mathrm{http}: / / \mathrm{www} . m a t h n e t . r u / \mathrm{eng} / \mathrm{dan} 40430$
3. Logan-Shepp's paper on the limit shape for $\tau \ll 1$ - https://www.sciencedirect.com/science/article/pii/0001

## Problems

[[_Lecture 12, 3-15]]

## 1

Prove Lemmas 2a, 2b, 2c in [[2.3 Density - last steps of the proof|here]] on the locations of the critical points of $S$.

## 2

Using [[2.1 Formulation\#Theorem 21 1|Theorem 2.1.1]], draw the curves for the limit shapes of Young diagrams under the Schur measre $S M_{\left(1, \ldots, 1 ; \rho_{\tau N}\right)}$ in the limit regime $N \rightarrow \infty$. This is a family of curves parametrized by $\tau$.

## T5

Think whether the approach using the single- $q$ integral could be developed into an asymptotic fluctuation result by somehow extracting the smallest power of $q$ in the expectation. See [[3.1 Approach with single q|this part]] for details.
[[Problems, 3-17|2 problems]], due $3 / 31$

## 1 Fluctuations

- 


## 1 Fluctuations

Recall that we have shown the following for TASEP, started from the step initial configuration:

$$
x_{\lfloor c N\rfloor}(\tau N)=N(\tau-2 \sqrt{\tau c})+o(N)
$$

Our next objective is to get a handle on the fluctuations $o(N)$.
We recall that

$$
x_{N}(t) \stackrel{d}{=} \lambda_{N}-N, \quad \lambda=\left(\lambda_{1} \geq \ldots \geq \lambda_{N} \geq 0\right) \sim S M_{\left(1, \ldots, 1 ; \rho_{t}\right)}
$$

The main property of the Schur measures which is of interest to us is their determinantal structure. More precisely, we will show today that the random subset of $\mathbb{Z}$ defined as

$$
\mathcal{S}=\left\{\lambda_{1}+N-1, \lambda_{2}+N-2, \ldots, \lambda_{N-1}+1, \lambda_{N}\right\}
$$

forms a determinantal point process.

## 2 Determinantal point processes on a discrete space

- 


## 2 Determinantal point processes

Let the space be $\mathbb{Z}_{\geq 0}$ or $\mathbb{Z}$.
Definition 2.0.1 A random subset $\mathcal{S} \subset \mathbb{Z}$ is called a determinantal point process if for any $m$ and any distinct points $a_{1}, \ldots, a_{m} \in \mathbb{Z}$ we have

$$
\operatorname{Prob}\left(\mathcal{S} \supset\left\{a_{1}, \ldots, a_{m}\right\}\right)=\operatorname{det}\left[K\left(a_{i}, a_{j}\right)\right]_{i, j=1}^{m}
$$

for some function $K(a, b)$. The function $K$ is called the determinantal correlation kernel.

Remark 2.0.2 For $m=1$ we have $K(x, x)=\operatorname{Prob}(\mathcal{S} \supset\{x\})$, which is the density function we already worked with.

Remark 2.0.3. Repelling property Assume that $K$ is Hermitean symmetric, that is, $K(x, y)=\overline{K(y, x)}$. Then

$$
\operatorname{Prob}(\mathcal{S} \supset\{x, y\})=K(x, x) K(y, y)-|K(x, y)|^{2} \leq K(x, x) K(y, y)
$$

This signifies that particles in a determinantal point process repel each other.
Recall this example:
\{ \}


Example 2.0.4 Take the Bernoulli process on $\mathbb{Z}$, where each location $x \in \mathbb{Z}$ is included in $\mathcal{S}$ independently with probability $p$. Then
\{ \}


$$
K(a, b)=p \cdot \delta_{a=b}
$$

is a DPP.

Remark 2.0.5 A kernel is not defined uniquely, $\frac{f(x)}{f(y)} K(x, y)$ defines the same process (where $f$ is a nonvanishing function).

Example 2.0.6. Some other kernels Discrete sine: \{ \}

$$
\begin{aligned}
& R(x, y)=\left\{\begin{array}{l}
\frac{\sin (\varphi(x-y))}{\pi(x-y)}, x \neq y \\
\varphi / \pi, \quad x=y
\end{array}\right. \\
& x, y \in \mathbb{Z} \\
& \text { Transl. rus. on } \mathbb{Z} \\
& \varphi / \pi=\text { density of it }
\end{aligned}
$$

See [[Problems, 3-17\#1|Problem 1]].
Another correlation kernel, for example:
\{ \}

$$
\begin{aligned}
K(x, y)=\frac{2 \sqrt{x y}}{x+y} & \frac{\sin \left(\frac{\pi}{2}(x-y)\right)}{\pi(x-y)} \\
& x, y \in \mathbb{Z}_{\geqslant 1}
\end{aligned}
$$

Example 2.0.7. Kernel for the process on the right figure here: \{ \}

(here the process is in $\mathbb{C}$ ). We have

$$
K(z, w)=\frac{1}{\pi} e^{z \bar{w}-\frac{1}{2}\left(|z|^{2}+|w|^{2}\right)}
$$

## 3 Determinantal structure of Schur measure -

### 3.1 Operators

We're going to work with Schur measures with general parameters $\vec{x}, \vec{y}$.
Recall that the application of a single operator $D_{q}$ in the $\vec{x}$ variables produces: \{ \}

$$
\left.\begin{array}{l}
\quad \sum_{\lambda} s_{\lambda}(\vec{x}) s_{1}(\vec{y})=\pi(\vec{x}, \vec{y})=T \frac{1}{1-x_{i} y_{j}} \\
\mathbb{E}_{S M(\vec{x}, \vec{j})}\left(\sum_{i=1}^{N} q^{\lambda i}+N-i\right.
\end{array}\right)=0
$$

Now we apply two such operators:
\{ \}

$$
\begin{aligned}
& \mathbb{E}_{S M(\vec{x}, \vec{y})}\left(\sum_{i=1}^{N} q_{1}^{\lambda_{i}+N-i}\right)\left(\sum_{i=1}^{N} q_{2}^{\lambda_{i}+N-i}\right) \\
& = \\
& \underbrace{\frac{1}{n(\vec{x}, \vec{y})}}\left[D_{q_{2}}^{(x)}\right. \text { applied to: } \\
& \underbrace{\prod(\vec{x}, \vec{y})}_{\text {product }} \cdot \frac{1}{2 \pi i} \oint_{\text {dependence on } x_{j}}^{\prod_{j=1}^{N} \frac{4 z-x_{j}}{z-x_{j}}} \frac{1}{q z-z} \cdot \prod_{j=1}^{M} \frac{1-z y_{j}}{1-q z y_{j}} d_{z}]
\end{aligned}
$$

We observe that before the application of the second operator we have the dependence on the variables $x_{i}$ in a product form.
Let us observe a general fact:
Lemma 3.1.1 For nice $f$ which is holomorphic inside the integration contour and does not have any poles/zeros at $x_{i}, q x_{i}$, we have:
\{ \}

Lever, $\frac{D_{q}^{(x)}\left[f\left(x_{1}\right)_{1}, f\left(x_{N}\right)\right]}{f\left(x_{1}\right) \ldots f\left(x_{N}\right)}$
$=\frac{1}{\text { ai }} \prod_{\substack{\text { around } \\ x_{i}}} \prod_{j=1}^{N} \frac{4 z-x_{j}}{z-x_{j}} \frac{1}{q z-z} \cdot \frac{f(q z)}{f(z)} d z$

Proof 3.1.1 The proof repeats [[../Lecture 11, 3-10/2.1 Expectation via qdifference operators\#Proof 21 2|Proof of L11 - Proposition 2.1.1]], almost exactly.

Now let us use this lemma to apply the second operator. We get:

Lemma 3.1.2 \{ \}

$$
\begin{aligned}
& \mathbb{E}_{S M(\bar{y}, \bar{j})}\left(\sum_{i=1}^{N} q_{1}^{\lambda_{i}+N-i}\right)\left(\sum_{i=1}^{N} q_{2}^{\lambda_{i}+N-i}\right) \\
& \|
\end{aligned}
$$

$$
\begin{aligned}
& x_{i} \\
& \cdot \frac{4_{1} z-q_{2}}{z-q_{2}} \cdot \frac{z-w}{q_{1}^{z-w}} \cdot \prod_{i=1}^{M} \frac{\left(1-z y_{i}\right)\left(1-w y_{i}\right)}{\left(1-q_{1} z_{i} y_{i}\right)\left(1-q_{2} y_{i}\right)}
\end{aligned}
$$

Note: The yellow boxed formula is something which did not appear in the density function, but it comes from $g\left(q_{2} w\right) / g(w)$, for $g(w)=\frac{q_{1} z-w}{z_{1}-w}$.

Now let us generalize to many operators. The proof of that is straightforward:
Proposition 3.1.3 \{ \}

$$
\begin{aligned}
& \mathbb{E}_{S M(2, j)}\left(\sum_{i=1}^{N} q_{1}^{\lambda_{i}+N_{-i}}\right) \ldots\left(\sum_{i=1}^{N} q_{m}^{\lambda_{i}+N_{-}-i}\right)
\end{aligned}
$$

$$
\begin{aligned}
& \underset{x_{i}}{\operatorname{arax}} \prod_{i=1}^{\pi} \prod_{i=1}^{\mu} \frac{\left(1-z_{x} y_{i}\right)}{\left(1-q_{\alpha} z_{y} y_{i}\right)}
\end{aligned}
$$

### 3.2 Extracting coefficients

Integrating in $q_{i}$ 's raised to suitable powers, we can extract the correlation functions:
\{ \}

Correl function:

$$
\begin{aligned}
& \mathbb{P}\left(\left\{\lambda_{i}+N-i\right\} \geq\left\{a_{1}, \ldots, a_{m}\right\}\right) \\
& =\text { Apply } \frac{1}{(2 \pi i)^{m}} \oint_{\text {around } 0}--\oint_{0} \frac{d q_{1}}{q_{1}^{a_{1}+1}} \cdots \frac{d q_{m}}{q_{m}^{a_{m}+1}} \\
& \mathbb{E}_{S M(\bar{x}, \vec{j})}\left(\sum_{i=1}^{N} q_{1}^{\lambda_{i}+N-i}\right) \ldots\left(\sum_{i=1}^{N} q_{m}^{\lambda_{i}+N-i}\right) . \\
& \text { \{ \} } \\
& \text { Chase var's } q_{i}=\omega i / a_{i} \\
& d g_{i}=d w_{i} / z_{i}
\end{aligned}
$$

We thus have:
Proposition 3.2.1 \{ \}

Proposition, with $z_{d}$ int around $x_{1} \ldots x_{N}$ $\omega_{\alpha}$ int around $O$

$$
\begin{aligned}
& \| P\left(\left\{\lambda_{i}+N-1\right\} \geq\left\{a_{1}, \ldots, a_{m}\right\}\right)=
\end{aligned}
$$

$$
\begin{aligned}
& \prod_{\alpha} \prod_{i=1}^{M} \frac{1-z_{\alpha} y_{i}}{1-w_{\alpha} y_{i}} \quad \prod_{\alpha<\beta}^{\left(z_{\alpha}-w_{\beta}\right)\left(w_{\alpha}-z_{\beta}\right)}
\end{aligned}
$$

3.3 Getting determinantal structure
\{ \}

$$
\frac{1}{(2 \pi i)^{2}} \int^{\Gamma \sum_{j}^{2 m}} \prod_{\alpha=1}^{m} \frac{d z_{\alpha} d w_{\alpha}}{a_{\alpha+1}} z_{\alpha}^{a_{\alpha}}\left(w_{\alpha}-z_{\alpha}\right) \prod_{\alpha}^{N} \frac{w_{j}-x_{j}}{N}
$$

$$
\prod_{\alpha} \prod_{i=1}^{M} \frac{1-z_{\alpha} y_{i}}{1-w_{\alpha} y_{i}} \quad \prod_{\alpha<\beta}^{\left(z_{\alpha}-w_{\beta}\right)\left(w_{\alpha}-z_{\beta}\right)}
$$

Let us look at the previous formula. Everything not highlighted in red depends on $z_{\alpha}, w_{\alpha}$ in a product form! Moreover, in the highlighted part we can recognize a determinant!

We employ the Cauchy determinant that we proved earlier ([[../Lecture 8, 3-1/4.2 Proof via determinants\#Lemma 422 Cauchy determinant|here]]):
\{ \}

$$
\operatorname{det}\left[\frac{1}{w_{i}-z_{j}}\right]_{i, j=1}^{N}=(-1)^{\frac{N(w-1)}{2}} \cdot \frac{V(\vec{z}) V(\vec{w})}{\prod_{i, j}\left(w_{i}-z_{j}\right)}
$$

Theorem 3.3.1 For the general Schur measure we have determinantal correlatons:
\{ \}

$$
\begin{aligned}
& \mathbb{P}\left(\left\{\lambda_{i}+N-i\right\} \geq\left\{a_{1}, \ldots, a_{m}\right\}\right)= \\
& =\operatorname{det}\left[\left(a_{0}^{0}, a_{0}^{6}\right)\right] \begin{array}{l}
n \\
1, \\
u
\end{array}=1 \\
& \forall(a, b)=\frac{1}{(a+i)^{2}} \oint \frac{d w d z}{w(w-z)} \frac{z}{w b} \cdot \prod_{j=1}^{w} \frac{w-x_{j}}{z-x} \\
& \text { ? } \\
& \prod_{j=1}^{M} \frac{1-z y_{j}}{1-w y_{j}}
\end{aligned}
$$

Proof 3.3.1 \{\}


Terms highlighted in light blue have product form:
\{ \}

$$
\begin{aligned}
& \text { of the form } \prod_{\alpha} f_{\alpha}\left(z_{\alpha}\right) g_{\alpha}\left(\omega_{\alpha}\right), \\
& f_{\alpha}(z)=z^{a_{\alpha}} \prod_{j=1}^{N} \frac{1}{z-x_{j}} \prod_{j=1}^{M}\left(1-z y_{j}\right) \\
& g_{\alpha}(\omega)=\omega^{-a_{\alpha}-1} \prod_{j=1}^{N}\left(\omega-x_{j}\right) \prod_{j=1}^{M} \frac{1}{1-\omega_{\alpha} y_{j}}
\end{aligned}
$$

So we can put them inside the determinant, and get:

$$
\operatorname{det}\left[\frac{f_{\alpha}\left(z_{\alpha}\right) g_{\beta}\left(w_{\beta}\right)}{w_{\beta}-z_{\alpha}}\right]_{i, j=1}^{m} .
$$

Then, one readily sees that the $2 m$-fold integral of a determinant is the same as the determinant of double integrals. This completes the proof.
-

### 3.4 Determinantal structure for TASEP

\{ \}


### 3.5 Other uses of Schur measures and processes

Beyond TASEP / push-block process, Schur measures and processes appear in a number of settings:

Plane partitions \{ \}


We put "boxes" in the corner, and the probability of a configuration is $q^{\text {number of boxes }}$

## Domino tilings $\}$



We tile this figure by dominoes $1 \times 2$ or $2 \times 1$.

Products of random matrices In a limit, Schur measures describe the eigenvalue distribution of a spectrum of a product of random matrices.

See, for example, Borodin, Gorin and Strahov ([[_Lecture 13, 3-17\#Notes and references|refs]]).

## 4 Edge points, gap probabilities

4.1 Which probability we need from Schur measure For TASEP:
\{ \}

$$
\begin{aligned}
& \mathbb{P}\left(x_{N}(t)>x+N\right)=\mathbb{P}\left(\lambda_{N}>x\right) \\
& =\mathbb{P}\left(\text { No particles of }\left\{\lambda_{i}+N-i\right\}\right. \text { oft } \\
& \{0,1, \ldots, x\})
\end{aligned}
$$



Therefore, we are now interested in a so-called gap probebility in a Schur measure.
4.2 Complementation and gap probability
\{ \}


Proposition 4.2.1 \{ \}

$$
\Rightarrow \text { - DPP 隹 }
$$

Proof 4.2.1 $\}$



$$
+\sum_{i<j} P\left(S \rightarrow a_{i}, a_{j}\right) \ldots
$$

Next, we express each term as a sum of determinants.
\{ \}

11

$$
\begin{gathered}
1-\sum_{i} \operatorname{det}_{1 \times 1} k\left(a_{i}, a_{i}\right)+\frac{1}{2!} \sum_{i, j}\left(\operatorname{det}_{2 \times 2} k\right)\left(a_{i}, a_{j}\right) \\
-\frac{1}{3!} \sum_{i, j, k}(\operatorname{det} 3 \times 3 k)\left(a_{i}, a_{i}, a_{k}\right) \\
+\ldots
\end{gathered}
$$

This is the determinant of $I d-K$ :
\{ \}

$$
\begin{aligned}
& =\operatorname{det}[1-K]_{m_{x m}} a_{1} a_{1}, a_{m} \\
& =\operatorname{det}\left[\delta_{i j}-k\left(a_{i}, a_{j}\right)\right]_{1}^{m}
\end{aligned}
$$

Now, let us apply this to TASEP: \{ \}

$$
\begin{aligned}
& P\left(X_{N}(t)>x+N\right) \\
&= \operatorname{det}\left(1-K^{(\text {SM })}\right)_{\{0,1, \ldots, x\}} \\
&(\text { size } x+1)
\end{aligned}
$$

For the asymptotics of this quantity (a determinant of growing size), we need another techniue - Fredholm determinants

### 4.3 Fredholm determinants

Let us just give a definition of a Fredholm determinant of a "nice" (locally trace class) kernel $K(x, y), x, y \in \mathbb{R}$.

Let $A$ be a subset of $\mathbb{R}$ of finite measure.

Definition 4.3.1. Fredholm determinant $\}$

$$
\begin{aligned}
& \operatorname{det}(1-K)_{A}:= \\
&=1-\int_{A} K(x, x) d x+\frac{1}{2!} \iint_{A A} \operatorname{det}_{2 \times 2}^{K} d x d y \\
&-\frac{1}{3!} \iiint_{A A A} \operatorname{det}_{3 \times 3} K x d y d z
\end{aligned}
$$

(this is a convergent infinite series)

## Notes and references

1. Correlation functions of Schur measures and processes via $q$-difference operators is a somewhat later addition to the theory, due to Amor Aggarwal. Correlation Functions of the Schur Process Through Macdonald Difference Operators, https://arxiv.org/abs/1401.6979
2. Other approaches to correlation functions are:

- via fermionic operators in Okounkov's papers https://arxiv.org/abs/math/9907127, https://arxiv.org/abs/math/0107056
- via linear algebra (manipulations with determinants) in Borodin-Rains https://arxiv.org/abs/math-ph/0409059

3. Surveys on determinantal processes:

- Alexander Soshnikov. Determinantal random point fields. https://arxiv.org/abs/math/0002099
- J. Ben Hough, Manjunath Krishnapur, Yuval Peres, Bálint Virág. Determinantal Processes and Independence. https://arxiv.org/abs/math/0503110
- Alexei Borodin. Determinantal point processes. https://arxiv.org/abs/0911.1153

4. Alexei Borodin, Vadim Goring, Eugene Strahov. Product matrix processes as limits of random plane partitions. https://arxiv.org/abs/1806.10855

## Problems

[[_Lecture 13, 3-17|Lecture]]

## 1

Show that the discrete sine kernel
\{ \}

$$
R(x, y)=\left\{\begin{array}{l}
\frac{\sin (\varphi(x-y))}{\pi(x-y)}, x \neq y \\
\varphi / \pi, \quad x=y
\end{array}\right.
$$

$$
x, y \in \mathbb{Z}
$$

$$
\begin{aligned}
& \text { Transl. Tow on } \mathbb{Z} \\
& \varphi_{n}=\text { density of it }
\end{aligned}
$$

could define a point process: 1. Show that it is positive definite, that is, all diagonal minors $\operatorname{det}\left[K\left(x_{i}, x_{j}\right)\right]_{i, j=1}^{m}$ are nonnegative. 2. Show that these minors are at most one. Hint: here you can use the complementation principle and pass from $K$ to $\mathbf{1}-K$.

## 2

Is the determinantal kernel $K(a, b)$ for the Schur measure Hermitean symmetric?
[[Lecture 14, 3-22/Problems, 3-22|4 problems]], due April 5
Here we finish the proof of the TASEP fluctuations, and discuss the results.

## 1 Recall the formulas we have for TASEP

- 


### 1.1 Kernel

We proved the formula
\{ \}

$$
\begin{aligned}
& K(a, b)=\frac{1}{(2 \pi)^{2}} \circlearrowleft \frac{d \omega d z}{\omega(\omega-z)} \frac{z^{a}}{\omega^{b}}\left(\frac{\omega-1}{z-1}\right)^{N} e^{t \mid \omega-z)} \\
& K(a, b)=0, \quad b<0
\end{aligned}
$$

One can check that $K(a, b)=0$ for $b<0$, so the Fredholm determinant $\operatorname{det}(1-K)_{\{0,1, \ldots, x\}}$ can be replaced by the Fredholm determinant on $\mathbb{Z} \cap(-\infty, x]$.
-

### 1.2 Fredholm determinants, general discussion

Recall that the Fredholm determinant in general is defined as \{ \}

It makes sense for locally trace class operators:
\{ \}

$$
\underbrace{\int_{A} K(x, x) d x}_{=\mathbb{E}(\neq \text { particles in } A)}<\infty \quad \text { for bid } A
$$

A useful estimate showing that the series for Fredholm determinants often converges, is the Hadamard's bound:
\{ \}

$$
\begin{gathered}
\left|a_{i j}\right| \leqslant B \Rightarrow \\
\text { Hadouard's bound: }\left|\operatorname{det}_{n \times n} A\right| \leq \underbrace{B^{n} n^{n / 2}}_{\text {less than } n!}
\end{gathered}
$$

1.3 How to compute Fredholm asymptotics
\{ \}


$$
\begin{aligned}
& \tilde{k} \sim \text { limit of } k \text { in } h^{1 / 3}-u b c \mid \text { of }|x N|
\end{aligned}
$$

2 Kernel asymptotics and Fredholm determinants
2.1 The part with the single critical point

Throughout the computations today we assume that $\tau>1$.
We have proved the following:
\{ \}

$$
\begin{aligned}
& K(x, x) \text {, proved : } N \rightarrow \infty \text {, } \\
& t=\tau N \rightarrow \infty \\
& x=\lfloor\chi N\rfloor \rightarrow \infty
\end{aligned}
$$

Exercise 2.1.1 \{ \}

Exercise: In same regime, $\quad \Delta x \in \mathbb{Z}$

$$
\begin{aligned}
& x=\lfloor\nsim N\rfloor, \quad y=\lfloor\not \subset N\rfloor+\Delta x \\
& \underline{\underline{K(x, y)} \rightarrow \frac{1}{2 \pi_{k}^{0}} \int_{\bar{z}_{c}}^{z_{c}} \frac{d \omega}{\omega \Delta x+1}}=\frac{\frac{\operatorname{sit}(\varphi \Delta x)}{\varphi=\arg z_{c}}}{\lambda \Delta x}
\end{aligned}
$$

See [[Problems, 3-22\#2| Problem 2]]

This discrete sine kernel determinantal process is a fundamental object, as it appears in asymptotics of many models.

A continuous analogue of the discrete sine kernel process on $\mathbb{R}$ is useful in describing spacings between zeroes of the Riemann zeta function $\zeta(s)$ on the critical line $\operatorname{Re}(s)=\frac{1}{2}$.
-

### 2.2 Edge and critical point behavior

\{ \}

Let $\tau>1$.


Let us recall the notation:
\{ \}

$$
\begin{aligned}
& S(z)=[z+\log (z-1)-x \operatorname{leg}(z) \\
& S^{\prime}(z)=0 \Rightarrow z_{c}, \bar{z}_{c} \text { inside } \\
& \qquad(\sqrt{\tau}-1)^{2}<x<(\sqrt{\tau}+1)^{2} \\
& (\& \text { real roots entsidl })
\end{aligned}
$$

At the boundary betwen "two complex conjugate" and "two real" regimes, the critical points merge and become the double critical point.


We consider the behavior around only one double critical point, $\chi^{*}=(\sqrt{\tau}-1)^{2}, \tau>1$.
2.3 Expansion around the double critical point \{ \}


Now we investigate local and global behavior of the sign of the real part of $S$ to determine how to move contours so that the contribution of everything outside a neighborhood of the critical point goes to zero.
\{ \}

\{ \}


We see that we can move the contours in a desired way.
2.4 Asymptotics of the kernel

The expansion and manipulation with contours proves the following result
Theorem 2.4.1 The kernel
\{ \}

$$
\left.K(a, b)=\frac{1}{(2 \pi i)^{2}}\right\} \frac{d \omega d z}{\omega(w-z)} \frac{z^{a}}{\omega^{b}}\left(\frac{w-1}{z-1}\right)^{N} e^{t \mid w-z)}
$$

has the following asymptotics:

$$
\exp \left[-\frac{\tilde{w}^{3}}{3}+\frac{\tilde{z}^{3}}{3}+\frac{r}{\partial z^{*}} \vec{z}-\frac{s}{\delta z^{*}} \tilde{\omega}\right]\left(1+O\left(N^{-1 / 3}\right)\right)
$$

in the regime
\{ \}

$$
\begin{aligned}
& a=\lfloor\chi N\rfloor+r N^{1 / 3} \\
& b=\lfloor\not X N\rfloor+S N^{1 / 3} \quad s, r \in \mathbb{R} \\
& x=\chi^{*}=(\sqrt{\varepsilon}-1)^{2}
\end{aligned}
$$

Proof 2.4.1 \{ \}

$$
K(a, b)=\frac{1}{(2 \pi i)^{2}} \int \frac{d w d z}{\omega(w-z)} e^{N\left(\frac{\pi \omega+\log (w-1)-\chi \log w}{(\omega(w)-s(z))} \frac{z^{r N^{1 / 3}}}{\omega N^{1 / 3}}\right.}
$$



Move the contours like this, so only a small neighborhood of $\chi^{*}$ matters.

$$
\begin{aligned}
& \text { Take } N^{-1 / \sigma} \rightarrow N^{-1 / 3} \\
& Z= Z^{*}+\frac{\tilde{Z}}{\sigma} N^{-1 / 3}, \quad w=z^{*}+\frac{\tilde{U}}{\sigma} N^{-1 / 3} \\
& \forall
\end{aligned} \quad\left(\sigma=\sqrt[3]{\frac{2 \tau^{2}}{\sqrt{\tau}-1}}\right)
$$

We have the following three expansions under the integral:
(1) $S(z)=S\left(z^{*}\right)-\frac{b^{3}}{3}\left(\frac{\tilde{z}}{b}\right)^{3} N^{-1}+\ldots$

$$
=S^{*}\left(z^{*}\right)-\frac{\tilde{z}^{3}}{3} N^{-1}+\cdots
$$

(2) $z^{P N^{1 / 3}}=e^{r N^{1 / 3} \log z}=$

$$
\begin{aligned}
& =e^{r N^{1 / 3}}\left[\log z^{*}+\frac{z-z^{*}}{z^{*}}+\cdots\right] \\
& =\left(z^{*}\right)^{r N^{1 / 3}} \exp \left[r \cdot \frac{\tilde{z}}{z^{*} b}+\theta\left(N^{-1 / 3}\right)\right]
\end{aligned}
$$

$\left(1+O\left(N^{-1 / 3}\right)\right)$
(3) $\frac{d \omega d z}{\omega(\omega-z)}=\frac{N}{6} \frac{d \tilde{\omega} d \tilde{z}}{z^{*}(\tilde{\omega}-\tilde{z})}$

Putting this all together, we arrive at the desired asymptotics.
2.5 Airy kernel
\{ \}



We have proven the following asymptotic of the kernel:

Theorem 2.5.1 \{ \}

$$
\begin{array}{r}
K(a, b) \\
\sim
\end{array} \frac{N^{-1 / 3}}{\sim} K^{N_{i}^{*}}\left[-\frac{p}{b z^{*}},-\frac{s}{b z^{*}}\right]
$$

- 


### 2.6 Fredholm determinant asymptotics

Using the Fredholm determinants, asymptotics of the kernel, and some more
estimates, we have
Theorem 2.6.1 \{ \}

$$
\begin{gathered}
\operatorname{Prob}\left(X_{N}(t)>N+\left\lfloor(\sqrt{\tau}-1)^{2} N\right\rfloor-P \sigma z^{*} N^{1 / 3}\right) \\
\quad \downarrow N \rightarrow \infty \\
\operatorname{det}\left(1-K^{A i}\right)_{(r,+\infty)}
\end{gathered}
$$

Remark 2.6.2 Note that by the very nature of the Fredholm determinant $\operatorname{det}\left(1-K^{A i}\right)_{(r,+\infty)}$, it is reasonable to guess that as $r \rightarrow \pm \infty$, the Fredholm determinant converges to 1 and 0 , respectively. This is to be expected of a cumulative distribution function.

Proof 2.6.1 \{ \}

$$
\begin{aligned}
& \operatorname{det}\left(1-k_{N, t}\right)_{\{0,1, \ldots, x\}}^{\left.t^{(v \tau}-1\right)^{2} N-}-r b z^{t} N^{n / s} \\
& 1+\sum_{n=1}^{\infty} \frac{(-1)^{n}}{n!}\left(\sum_{0}^{x} \ldots \sum_{0}^{x}\right) \operatorname{det}{ }_{n \times n} k_{N, t}
\end{aligned}
$$

We also need to estimate the kernel $K(x, y)$ for $y$ away from the neighborhood
of $\left\lfloor\chi^{*} N\right\rfloor$. This may be done as follows:

$$
|K(x, y)|<C\left(z^{*}\right)^{(r-s) N^{1 / 3}} \frac{e^{-c_{1} N^{1 / 3}}+e^{c_{2} N^{-1 / 3}\left(y-\left\lfloor\chi^{*} N\right\rfloor\right)}}{\left\lfloor\chi^{*} N\right\rfloor-y+1}
$$

if $y-\left\lfloor\chi^{*} N\right\rfloor<-s N^{1 / 3}$ for some $s>0$.
This estimate may be proven by further analysis of the double contour integral formula for the kernel.

3 Discussion of the asymptotics
-
3.1 Tracy-Widom distribution

Definition 3.1.1. Tracy-Widom GUE distribution $\}$

$\left.\begin{array}{r}\text { cumselative } \\ \text { distr. function }\end{array}\right]$ (not too trivial but believabk from scaling
$K^{A_{i}^{0}(r, s)}=\frac{1}{\left(2 \pi r_{i}^{0}\right)^{2}} \iint \frac{e^{\frac{u^{3}}{3}-\frac{v^{3}}{3}-r u+s v}}{u-v} d u d u$


Let discuss one more formula for the kernel, and one more formula for $F_{2}(r)$.
Lemma 3.1.2 \{ \}

$$
\begin{aligned}
& \frac{L \text { emma }}{\text { Lit }} k^{A_{i}}(x, y)=\frac{A_{i}(x) A_{i}^{\prime}(y)-A_{i}^{\prime}(x) A_{i}(y)}{x-y} \\
& A_{i}(x)=\frac{1}{2 \pi} \int_{-\infty}^{+\infty} e^{i t^{3} / 3+i t x} d t \\
& \xrightarrow[\longrightarrow]{\longrightarrow}=\frac{1}{\pi} \text { (reg.) } \int_{-\infty}^{+\infty} \cos \left(\frac{t^{3}}{3}+t x\right) d t
\end{aligned}
$$

$A i(x)$ is called the Airy function

Proof 3.1.2. Proof sketch $\}$

$$
\begin{aligned}
& \text { Proof, } e^{-r u+s v}=\frac{1}{s-r}\left[\frac{\partial}{\partial u}+\frac{\partial}{\partial v}\right] e^{-r u+s v} \\
& e^{\frac{u^{3}}{3}-\frac{v^{3}}{3}}\left[\frac{\partial}{\partial u}+\frac{\partial}{\partial v}\right] e^{-r u+s v} \\
& u-v \quad \text { - eliminates } \\
& \Downarrow \text { ley parts } \\
& \frac{1}{s-r} \underbrace{\left.\frac{1}{\left(2 \pi v^{2}\right.}\right)} \underbrace{\int \exp [-] \cdot(u+v)} \\
& \underbrace{\int e^{\sigma}}_{A_{i}}, \underbrace{\int v \cdot e^{\nabla}}_{A i}+\int-\int
\end{aligned}
$$

On the Airy function:
\{ \}
$\xrightarrow{\text { Ai }(x)=}$



$$
\begin{aligned}
& \text { a solutions, } \\
& A_{i}^{i}(x) \rightarrow 0
\end{aligned}
$$

$$
\text { at }+\infty
$$

Theorem 3.1.3. Tracy-Widom distribution via Painleve II \{ \}

$$
\begin{aligned}
& \text { Theorem (form diff analysis of } \log \left(\operatorname{det}\left(1-K^{A i}\right)\right) \text { ) } \\
& F_{2}(r)=\exp \left(-\int_{r}^{\infty}(x-r) q^{2}(x) d x\right) \\
& q^{\prime \prime}(x)=x q(x)+2 q^{3}(x) \quad \text { Painteve } \frac{\mathbb{I}}{} \\
& q^{\prime}(x) \sim A P(x), x \rightarrow \infty \quad(I, \ldots, \underline{V I})
\end{aligned}
$$

(without proof; this is proven via differentiation of the logarithm of the Fredholm determinant)
\{ \}


### 3.2 KPZ universality

Most stochastic particle systems with strong dependence are believed (and proven) to be in the so-called KPZ (Kardar-Parisi-Zhang) universality class. The name for the class comes from a stochastic PDE introduced in the work of Kardar-Parisi-Zhang in 1986.

There are several versions of fluctuation distribution (which are the GUE TracyWidom distribution and its relatives):
\{ \}

| Six Geometries of KPZ Universality |  |  |  |
| :---: | :---: | :---: | :---: |
| Geometry | Limit shape | Fluctuations | KPZ/SHE |
| - Wedge | $\begin{aligned} & \bar{h}(T, T X)= \\ & \begin{cases}-X & X<-1 \\ T^{\frac{1+X^{2}}{2}} & \|X\| \leq 1 \\ X & X>1\end{cases} \end{aligned}$ | - One pt: $F_{\text {GUE }}$ [9, 93] <br> - Multi pt: Airy ${ }_{2}$ [137, 94, 29] | $\mathcal{Z}(0, X)=\delta_{\{X=0\}}$ <br> - Converges:[5] <br> - One pt:[5, 147] <br> (bounds and stats) |
| - Brownian | $\bar{h}(T, T X)=T / 2$ | - One pt: $F_{0}$ [11, 69] <br> - Multi pt: Airy stat [10] | $\mathcal{Z}(0, X)=e^{B(X)}$ <br> - Converges:[23] <br> - One pt:[15, 49] (bounds, NO stats) |
| - Flat | $\bar{h}(T, T X)=T / 2$ | - One pt: $F_{\mathrm{GOE}}$ [12, 13, 70, 146] <br> - Multi pt: Airy ${ }_{1}$ [31, 32] | $\mathcal{Z}(0, X)=1$ <br> - Converges:[23] <br> - One pt: OPEN <br> (NO bounds / stats) |
| - Wedge $\rightarrow$ Brownian | $\begin{aligned} & \bar{h}(T, T X)= \\ & \begin{cases}-X & X<-1 \\ T \frac{1+X^{2}}{2} & X \in[-1,0] \\ T / 2 & X>0\end{cases} \end{aligned}$ | - One pt: $\left(F_{\mathrm{GOE}}\right)^{2}$ [11, 8, 136, 19] <br> - Multi pt: Airy ${ }_{2 \rightarrow \mathrm{BM}}$ [90, 46] | $\begin{aligned} & \mathcal{Z}(0, X)= \\ & e^{B(X)} \mathbf{1}_{X \geq 0} \end{aligned}$ <br> - Converges:[49] <br> - One pt:[49] (bounds and stats) |
| - Wedge $\rightarrow$ Flat | $\begin{aligned} & \bar{h}(T, T X)= \\ & \begin{cases}-X & X<-1 \\ T \frac{1+X^{2}}{2} & X \in[-1,0] \\ T / 2 & X>0\end{cases} \end{aligned}$ | - Multi pt: Airy $_{2 \rightarrow 1}$ [33] | $\mathcal{Z}(0, X)=\mathbf{1}_{X \geq 0}$ <br> - Converges: here <br> - One pt: OPEN <br> (NO bounds / stats) |
| - Flat $\rightarrow$ Brownian | $\bar{h}(T, T X)=T / 2$ | - Multi pt: Airy ${ }_{1 \rightarrow B M}$ [34] | $\begin{aligned} & \mathcal{Z}(0, X)= \\ & \mathbf{1}_{X<0}+e^{B(X)} \mathbf{1}_{X \geq 0} \end{aligned}$ <br> - Converges: here <br> - One pt: OPEN <br> (NO bounds / stats) |

(table from Corwin's KPZ survey, see the [[_Lecture 14, 3-22\#Notes and references|refs]])

## Notes and references

1. Asymptotics via contour integrals (in a simpler case): Andrei Okounkov. Symmetric functions and random partitions. https://arxiv.org/abs/math/0309074
2. Survey on KPZ universality: Ivan Corwin. The Kardar-Parisi-Zhang equation and universality class. https://arxiv.org/abs/1106.1596
3. A proof of the representation of the GUE Tracy-Widom dis-
tribution through the Painleve II equation may be found in Craig A. Tracy, Harold Widom. Airy Kernel and Painleve II. https://arxiv.org/abs/solv-int/9901004

## Problems

[[_Lecture 14, 3-22|Lecture]]

## 1

Show that for a determinantal process on $\mathbb{Z}$ or $\mathbb{R}$ with correlation kernel $K(x, y)$, we have (for finite / bounded set $A$ )

$$
\mathbb{E}(\text { number of points of the point process belonging to } A)=\sum_{x \in A} K(x, x)
$$

for $\mathbb{Z}$, and

$$
\mathbb{E}(\text { number of points of the point process belonging to } A)=\int_{A} K(x, x) d x
$$

for $\mathbb{R}$.

## 2

Show that the kernel $K(x, y)$ of the Schur measure given [[1.1 Kernel|here]] converges to the discrete sine kernel:

As $x=\lfloor\chi N\rfloor, y=\lfloor\chi N\rfloor+\Delta x, \Delta x \in \mathbb{Z}$ fixed, $N \rightarrow+\infty, t=\tau N$, and $(\sqrt{\tau}-1)^{2}<\chi<(\sqrt{\tau}+1)^{2}$, we have
\{ \}

$$
\begin{aligned}
& \text { Exercise: In same regime, } \quad \Delta x \in \mathbb{Z} \\
& x=\lfloor x N\rfloor, \quad y=\lfloor\not \subset N\rfloor+\Delta x \\
& \underline{\underline{K(x, y)} \rightarrow \frac{1}{2 \pi_{c}^{0}} \int^{z_{c}} \frac{d \omega}{\omega \Delta x+1}=\frac{\sin (\varphi \Delta x)}{\psi \Delta x}} \\
& \bar{z}_{c} \quad \Delta x \in \mathbb{Z}
\end{aligned}
$$

## 3

In the analysis, we considered only the case of one double critical point, $\chi^{*}=$ $(\sqrt{\tau}-1)^{2}, \tau>1$. There are two more regimes:

- $0<\tau<1, \chi=(\sqrt{\tau}-1)^{2}$
- $\tau>0, \chi+(\sqrt{\tau}+1)^{2}$.

Determine the asymptotics of the kernel in these two other cases.

## 4

Explain why the terms $\left(z^{*}\right)^{(r-s) N^{1 / 3}}$ (in red in [[2.4 Asymptotics of the kernel\#Theorem 241 |this theorem]]) do not matter for the asymptotics of the process, and can be ignored.
[[Lecture 15, 3-24/Problems, 3-24|3 problems]], due April 7
\{ \}
!
Note: No class Mar 29
(UVA ,reading day")

Next class - wednesday, Mar 31.

This lecture motivates and introduces the six vertex model and the stochastic six vertex model

1 Models in statistical mechanics
1.1 Lozenge tilings
\{ \}

\{ \}


See also [[Problems, 3-24\#2|Problem 2]] for a particular case of the enumeration of lozenge tilings.

For large lozenge tilings, we have many universal asymptotics - for example, local behavior is given by the same distribution as for the push-block dynamics. \{ \}

\{ \}

$$
\begin{aligned}
& \begin{aligned}
2 d \text { config }= & \text { taus } 1 . \text { in } v \\
& \text { ergoelic Gibbs }
\end{aligned} \\
& z_{c} \text { rueay.on } \\
& \int_{z_{c}} \omega^{(x-1}(1-\omega)^{a t-1} d \omega \text { tilings of } \mathbb{Z}^{2} \\
& z_{c} \underbrace{t}_{x} \text { (2-poraweter geveraliz. } \\
& \begin{array}{ll}
z_{c}=r e^{i \varphi} & \text { Of }(*) \\
\text { Fluctuations at } 2 d & \text { local equilibrial }
\end{array} \\
& \text { are Not Tracy-Widom } N^{1 / 3} \\
& \text { Bather, they ore described by } \\
& \text { Goussion Free Field., grow as } \log N
\end{aligned}
$$

### 1.2 Square ice

We start by taking a bijection of lozenge tilings with nonintersecting lattice paths.
\{ \}


There are five local configurations of paths at a vertex. We call them $a_{1}, b_{1}, c_{1}, b_{2}, c_{2}$.
\{ \}


Thus, we can generalize the lozenge model to a following stat-mech model with Boltzmann weights:

For uniform lozenges, all are $=1$

$$
\begin{gathered}
P(\text { coufig })=\frac{1}{z} a_{1}^{\#(1-)} b_{1}^{\#(-\infty)} c_{1}^{\#(-d)} \\
(\text { Boltzmann weights) }
\end{gathered}
$$

Here $Z$ is the partition function, which in the particular case reduces to the MacMahon's triple product:
\{ \}

$$
\begin{aligned}
& a_{1}=b_{1}=c_{1}=b_{2}=c_{2}=1 \\
& \Rightarrow \quad Z=\prod_{i=1}^{a} \prod_{i=1}^{b} \prod_{k=1}^{c} \frac{i+j+k-1}{i+j+k-2}
\end{aligned}
$$

Now we introduce a complication and allow for one more vertex type, $a_{2}$. \{ \}


This leads to pictures which are no longer interpretable as lozenge tilings. \{ \}


SNot a beige picture
anymore)

$$
\not \approx \text { (partition function) }
$$

is much more complicated

This model is called the six vertex model
\{ \}


The six vertex model has a very interesting history in physics and mathematics. I will not attempt to survey it, see the lecture notes mentioned in the [[_Lecture $15,3-24 \#$ Notes and references $\mid$ refs $]$ ].

One of the most important boundary conditions in the six vertex model is the domain wall boundary condition:
\{ \}

Izergin-korepin 1980s

There is a determinantal formula for the partition function. This formula is much more complicated than the explicit products we saw for lozenge tilings.
There is a particular case when the determinant simplifies, $a_{1}=a_{2}=b_{1}=b_{2}=$
$c_{1}=c_{2}=1:$
\{ \}


## 2 Stochastic six vertex model

- 


### 2.1 Definition of the stochastic six vertex model

Within particle systems, a particular subfamily of six vertex models is much easier to study. These are the stochastic six vertex models.
\{ \}


1


$1-b_{1}$
$b_{2}$
$1-b_{2}$

In this case, by making endpoints of our paths free, we can sample the stochastic six vertex model by just running a Markov chain.

And the partition function becomes simply equal to 1 .
\{ \}

(decide where they go randomly, as a markov chain)
all contig, free boundary at exit

$$
=1 \quad \text { (simple past } f .)
$$

So, we consider the half-domain wall boundary condition:
\{ \}


-

### 2.2 Degeneration of the stochastic six vertex model

There are several degeneration of the stochastic six vertex model.
be $=\mathbf{0} \quad\{ \}$

\{ \}

stay $b_{1}$, go $1-b_{1}$ (oke step right \& particles push each other.
(distr. time Push TASEP.

This is a discrete time PushTASEP
be $=0$, and limit to continuous time $\}$

$$
\begin{aligned}
& \text { (1a) 1-b, } \rightarrow 0, \quad t=[\tau / 1-b\rfloor \\
& \Rightarrow \text { partides jup at rase \& } \\
& \text { in cont. time, \& push }
\end{aligned}
$$

ASEP / TASEP near the diagonal \{ \}

2.3 Stationary model

We can construct a stationary version of the stochastic six vertex model.
Theorem 2.3.1 $\}$


As it is not too clear which arrow jumps first, we need to explain what is the dynamics. In other words, we construct the stationary model.

Lemma 2.3.2 If $\left(1-b_{2}\right) \alpha(1-\beta)=\left(1-b_{1}\right) \beta(1-\alpha)$, then having independent Bernoulli inputs from below and from the left, the vertex model's output produces independent Bernoulli outputs with the same distributions.
\{ \}


Proof 2.3.2 \{ \}

Probabilities of the outside arrow config $(1-\alpha)(1-\beta)$

\{\}


$$
\begin{aligned}
& b_{2} \alpha(1-\beta)+\left(1-b_{1}\right) \beta(1-\alpha)=\text { want } \\
& \left(1-b_{2}\right) \alpha(1-\beta)=\left(1-b_{1}\right) \beta(1-\alpha)
\end{aligned}
$$

\{\}

(also true under the condition) $D$

Proof 2.3.1 Lemma proves that Bernoulli distributions at the horizontal and vertical pieces lead to a compatible family of measures on quadrants which can be made going to $-\infty$.
\{ \}

\{ \}


## Notes and references

1. Some historical account on the six vertex model is in the introduction to N .

Reshetikhin. Lectures on the integrability of the 6-vertex model. https://arxiv.org/abs/1010.5031
2. Some relevant wikipedia articles:

- https://en.wikipedia.org/wiki/Ice-type_model
- https://en.wikipedia.org/wiki/Lieb\'s_square_ice_constant
- https://en.wikipedia.org/wiki/Alternating_sign_matrix

3. Nature paper (2015) claiming to have discovered square ice in thin layers between sheets of graphene: https://www.nature.com/articles/nature14295

## Problems

[[_Lecture 15, 3-24|Lecture]]
1
Show that the number of tilings of a figure like this
\{ \}

is given by $s_{\lambda}(1,1, \ldots, 1)$ for a suitable $\lambda=\left(\lambda_{1} \geq \lambda_{2} \geq \ldots \geq \lambda_{N} \geq 0\right)$.
2
\{ \}


$$
\begin{aligned}
& \mathbb{H}^{\text {tilings }}=\prod_{i=1}^{a} \prod_{j=1}^{b} \prod_{k=1}^{c} \frac{i+j+k-1}{i+j+k-2} \\
& (\text { for this example, } 1176)
\end{aligned}
$$

Compute the current of the discrete time and the continuous time PushTASEP in stationarity (see [[2.2 Degeneration of the stochastic six vertex model|this part of the lecture]] for their definitions as degeneration of the stochastic six vertex model).
[[Lecture 16, 3-31/Problems, 3-31|4 problems]], due April 14

## 1 Recall stochastic six vertex model

1 Recall stochastic six vertex model
\{ \}


1

$1 \quad 1-b_{1}$

$1-b_{2}$


Recall that we discussed the stationary six vertex model last time:
\{ \}


This allows in principle to get the hydrodynamics for the model; however, we will not spend time on this, but will get to the discussion of exact formulas.

## 2 How to solve the stochastic six vertex model

- 


### 2.1 Height function of S6V

\{ \}

$$
\begin{aligned}
& \text { Key object - height function } \\
& h(x, y)=\text { \#t paths passing to the right } \\
& \text { (or below, which is equivalent) } \\
& y \square x \text { of the cell }(x, y)
\end{aligned}
$$

$$
\begin{aligned}
& \text { • }
\end{aligned}
$$

2.2 Goal for this lecture


## 3 Hall-Littlewood vertex model

Also called "the model of deformed bosons"
-

### 3.1 Vertex weights

First, let us explain how to translate our signatures $\lambda$ into vertex model language. We take vertex models where there can be arbitrary many arrows vertically, but at most one horizontally at each edge.
\{ \}
allow multiple paths vertically


Fix the parameter $0 \leq t<1$. Let $u$ be another parameter which may depend on the vertex.

Definition 3.1.1. Red vertices \{ \}
Does.



g

$u\left|1-t^{g+1}\right|$|  |
| :--- |
| $l$ |$\quad(g \geqslant 0)$

Proposition 3.1.2 \{\}


Proof 3.1.2

See [[Problems, 3-31\#2|Problem 2]].
3.2 Hall-Littlewood polynomials

Definition 3.2.1 \{ \}

Def. Hall-Littlewoed polynomial


Without normalization, we use the notation $F_{\lambda}$ :
\{ \}


Proposition 3.2.2 \{ \}

$$
\text { Pro. (1) } P_{\lambda} \text { - polynomial in } u_{i} \text { (V) }
$$

$$
\begin{aligned}
& \text { (2) Homogeveans in } u_{1}, \ldots, u_{N} \\
& \| \ggg \text { degree }|\lambda|=\sum_{i=1}^{N} \lambda_{i} \mid
\end{aligned}
$$


(3) Lexicographically top term is

$$
\begin{aligned}
& u_{1}^{\lambda_{1}} u_{2}^{\lambda_{2}} \cdots u_{N}^{\lambda_{N}} \\
& \text { (explains the prefactor } \frac{1}{\prod_{i=0}(t, t)_{m_{i}(\lambda)}} \text { ) }
\end{aligned}
$$

Note that we're not yet proving that $P_{\lambda}$ is symmetric in the variables.

## Proof 3.2.2

1. The polynomiality is clear because all weights are polynomial, and the partition function is of a finite sum
2. Homogeneity follows from the fact that the weight $u$ is attached to a horizontal outgoing arrow, so the total power of each monomial in $u_{i}$ is $|\lambda|$
3. The lexicographically maximal monomial corresponds to a unique path configuration $\}$


Dividing by the normalizing factor, we get the desired result

See also the stability property of the $P_{\lambda}$ 's, [[Problems, $3-31 \# 1 \mid$ Problem 1]].
-
3.3 Comparison between Schur and Hall-Littlewood \{ \}


Today we'll discuss symmetry and Cauchy identity. For the Cauchy identity, we will introduce another, blue HL vertex model.

## 4 Consequences of Yang-Baxter equations

- 


### 4.1 Symmetry

Definition 4.1.1 Define the red cross vertex weights $R_{z}$ as follows:
\{ \}


Theorem 4.1.2. Yang-Baxter equation (YBE) We have the following equality of partition functions:
\{ \}


Proof 4.1.2 The proof is a direct verification of an identity involving summing over $k_{1}, k_{2}, k_{3}$.
\{ \}


Here is code
\{ \}

```
HL weights (RED)
man). w[u_][i1_, j1_, i2_, j2_] := If [i1 + j1 == i2 + j2 && i1 \geq 0&& i2 \geq0&& 1>= j1 \geq0&& 1>= j2 \geq0,
    If[i1 == i2 && j1 = j2 == 0, 1, 0] + If[i1 == i2 && j1 == j2 == 1,u, 0] + If[i1 +1 == i2 && j1 == 1&& j2 == 0, 1 - t^ i2, 0] +
        If[i1-1 == i2 && j1 == 0 && j2 == 1,u, 0], 0]
Crossweights
```



```
    X[z][i1, j1, i2, j2], 0]
40% X[z_][0, 0, 0, 0]:= 1;
    X[z_][1, 1, 1, 1]:= 1;
    x[z_][1, 0, 0, 1]:= z(1-t)/(1-zt);
    X[z_][1, 0, 1, 0] := (1-z)/(1-zt);
    X[z_][0, 1, 0, 1] := t (1-z) / (1-z t);
    X[z_][0, 1, 1, 0]:=(1-t)/(1-zt);
YBE
M(0)= m:= 3
wom- Table[Sum[w[v][i3, k1, k3, j1] \w[u][k3, k2, j3, j2] \ R[u/v][i1, i2, k1, k2],{k1, 0, 1},{k2, 0, 1},{k3, 0,m+1}]-
            Sum[w[v][k3, i1, j3, k1] \w[u][i3, i2, k3, k2] }\timesR[u/v][k1, k2, j1, j2], {k1, 0, 1},{k2, 0, 1}, {k3, 0,m+1}],
            {i1, 0, 1},{i2, 0, 1},{i3, 0, m},{j1, 0, 1},{j2, 0, 1}, {j3, 0, m}] // Simplify
amp. {{{{{{0, 0, 0, 0}, {0, 0, 0, 0}}, {{0, 0, 0, 0}, {0, 0, 0, 0}}},
            {{{0, 0, 0, 0}, {0, 0, 0, 0}}, {{0, 0, 0, 0}, {0, 0, 0, 0}}}, {{{0, 0, 0, 0}, {0, 0, 0, 0}},
            {{0, 0, 0, 0},{0, 0, 0, 0}}}, {{{0,0, 0, 0},{0, 0, 0, 0}}, {{0, 0, 0, 0},{0, 0, 0, 0}}}},
            {{{{0, 0, 0, 0},{0, 0, 0, 0}}, {{0, 0, 0, 0}, {0, 0, 0, 0}}}, {{{0,0,0,0},{0,0,0,0}},
            {{0, 0, 0, 0}, {0, 0, 0, 0}}}, {{{0, 0, 0, 0}, {0, 0, 0, 0}}, {{0, 0, 0, 0}, {0, 0, 0, 0}}},
            {{{0, 0, 0, 0}, {0, 0, 0, 0}}, {{0, 0, 0, 0}, {0, 0, 0, 0}}}}},
        {{{{{0, 0, 0, 0}, {0, 0, 0, 0}}, {{0, 0, 0, 0}, {0, 0, 0, 0}}},
            {{{0,0,0,0},{0,0,0,0}},{{0,0,0,0},{0,0,0,0}}},{{{0,0,0,0},{0,0,0,0}},
            {{0, 0, 0, 0}, {0, 0, 0, 0}}}, {{{0, 0, 0, 0}, {0, 0, 0, 0}}, {{0, 0, 0, 0}, {0, 0, 0, 0}}}},
        {{{{0, 0, 0, 0}, {0, 0, 0, 0}}, {{0, 0, 0, 0}, {0, 0, 0, 0}}}, {{{0, 0, 0, 0}, {0, 0, 0, 0}},
            {{0,0,0, 0}, {0, 0, 0, 0}}}, {{{0, 0, 0, 0}, {0, 0, 0, 0}}, {{0, 0, 0, 0},{0, 0, 0, 0}}},
            {{{0,0,0,0},{0,0,0,0}},{{0,0,0,0},{0,0,0,0}}}}}}
```

See also:
Mathematica file • pdf

Corollary 4.1.3 \{ \}

$$
\begin{array}{r}
P_{\lambda}\left(u_{1}, \ldots, u_{N}\right) \text { is symuetrie } \\
\text { in } u_{1}, \ldots, u_{N} .
\end{array}
$$

Proof 4.1.3 Via YBE, we have the equality of two partition functions \{ \}


This allows to swap $u_{i}$ and $u_{i+1}$, resulting in the symmetry.
4.2 Blue HL vertex model

Definition 4.2.1. Blue HL vertex weights \{ \}


Note that in the blue vertices, the paths go down and right. And in the red vertices, the paths go up and right

Definition 4.2.2. Q_lambda $\}$


$$
\lambda=\left(\lambda_{1} \geqslant \ldots \geqslant \lambda_{N} \geqslant 0\right)
$$

4.3 Red-blue YBE
$\underset{\sim}{\text { Definition 4.3.1. Red-blue cross vertex weights }}$ We define the weights $\widetilde{R}_{z}$ :
\{ \}


Theorem 4.3.2. Red-blue YBE \{ \}


## Proof 4.3.2 \{ \}

HL weights (BLUE)
$\sigma$
move wstar[ $\left.v_{-}\right]\left[i 1_{-}, j 1_{-}, i 2_{-}, j 2_{-}\right]:=\mathbf{I f}[i 1+j 2=i 2+j 1 \& \& i 1 \geq 0 \& \& i 2 \geq 0 \& \& 1>=j 1 \geq 0 \& \& 1>=j 2 \geq 0$, $\operatorname{If}[i 1=i 2 \& \& j 1=j 2=0,1,0]+\operatorname{If}[i 1=i 2 \& \& j 1=j 2=1, v, 0]+\operatorname{If}[i 1-1=i 2 \& \& j 1=1 \& \& j 2=0,1$, 0$]+$ If $\left.\left[i 1+1=i 2 \& \& j 1=0 \& \& j 2=1, v\left(1-t^{\wedge} i 2\right), 0\right], 0\right]$

Weights $R$-tilde
minn Rtilde[z_][i1_,j1_,i2_,j2_]:=1f[i1+j2=i2+j1\&\&1>=i120\&\&1>=i2z0\&\&1>=j1z0\&\&1>=j2z0, $\mathrm{Xt}[z][i 1, j 1, i 2, j 2], 0]$
merin $=X t\left[z_{-}\right][0,0,0,0]:=1 ;$
$\mathrm{Xt}\left[z_{-}\right][1,1,1,1]:=\mathrm{t}$;
$\mathrm{xt}\left[z_{-}\right][0,0,1,1]:=(1-t) z /(1-z)$;
Xt [ $\left.z_{-}\right][1,0,1,0]:=(1-t z) /(1-z) ;$
$\mathrm{Xt}\left[z_{-}\right][0,1,0,1]:=(1-\mathrm{t} z) /(1-z) ;$
$\mathrm{Xt}\left[z_{-}\right][1,1,0,0]:=(1-\mathrm{t}) /(1-z)$;
YBE 2
mon- Table[Sum[wstar [v][i3, k1, k3, j1] $\times w[u][k 3, k 2, j 3, j 2] \times R t i l d e[u v][i 1, i 2, k 1, k 2],\{k 1,0,1\}$, $\{k 2,0,1\},\{k 3,0, m+1\}]-\operatorname{Sum}[w s t a r[v][k 3, i 1, j 3, k 1] \times w[u][i 3, i 2, k 3, k 2] \times R t i l d e[u v][k 1, k 2, j 1, j 2]$, $\{k 1,0,1\},\{k 2,0,1\},\{k 3,0, m+1\}\},\{i 1,0,1\},\{i 2,0,1\},\{i 3,0, m\},\{j 1,0,1\},\{j 2,0,1\},\{j 3,0, m\}] / /$ Simplify
anm $\{\{\{\{\{\{0,0,0,0\},\{0,0,0,0\}\},\{\{0,0,0,0\},\{0,0,0,0\}\}\}$,
$\{\{\{0,0,0,0\},\{0,0,0,0\}\},\{\{0,0,0,0\},\{0,0,0,0\}\}\},\{\{\{0,0,0,0\},\{0,0,0,0\}\}$, $\{\{\theta, \theta, \theta, 0\},\{\theta, \theta, \theta, \theta\}\}\},\{\{\{0, \theta, \theta, \theta\},\{\theta, 0, \theta, \theta\}\},\{\{\theta, \theta, \theta, 0\},\{\theta, \theta, 0,0\}\}\}\}$, $\{\{\{\{\theta, \theta, \theta, \theta\},\{\theta, \theta, \theta, \theta\}\},\{\{\theta, \theta, \theta, \theta\},\{\theta, \theta, \theta, \theta\}\}\},\{\{\{\theta, \theta, \theta, \theta\},\{\theta, \theta, \theta, \theta\}\}$, $\{\{0,0,0,0\},\{0,0,0,0\}\}\},\{\{\{0,0,0,0\},\{0,0,0,0\}\},\{\{0,0,0,0\},\{0,0,0,0\}\}\}$, $\{\{\{0,0,0,0\},\{0,0,0,0\}\},\{\{0,0,0,0\},\{0,0,0,0\}\}\}\}\}$,
$\{\{\{\{\{0,0,0,0\},\{0,0,0,0\}\},\{\{0,0,0,0\},\{0,0,0,0\}\}\}$,
$\{\{\{0,0,0,0\},\{0,0,0,0\}\},\{\{0,0,0,0\},\{0,0,0,0\}\}\},\{\{\{0,0,0,0\},\{0,0,0,0\}\}$,
$\{\{0, \theta, 0,0\},\{0,0,0,0\}\}\},\{\{\{0,0,0,0\},\{0,0,0,0\}\},\{\{\theta, 0,0,0\},\{0,0,0,0\}\}\}\}$, $\{\{\{\{0,0,0,0\},\{0,0,0,0\}\},\{\{0,0,0,0\},\{0,0,0,0\}\}\},\{\{\{0,0,0,0\},\{0,0,0,0\}\}$, $\{\{0,0,0,0\},\{0,0,0,0\}\}\},\{\{\{0,0,0,0\},\{0,0,0,0\}\},\{\{0,0,0,0\},\{0,0,0,0\}\}\}$,
 $\{\{\{0,0,0,0\},\{0,0,0,0\}\},\{\{0,0,0,0\},\{0,0,0,0\}\}\}\}\}\}$

See also:
Mathematica file • pdf

### 4.4 Cauchy identities

Lemma 4.4.1. Skew Cauchy identity For $|x y|<1$, we have \{ \}
$\sum_{\nu} F_{\nu / \lambda}(x) Q_{\nu / \mu}^{*}(y)=\frac{1-t_{x y}}{1-x y} \sum_{x} F_{\mu / x}(x) Q_{\lambda / x}^{*}(y)$

\{ \}


Proof 4.4.1 We also consider equality of two partition functions: \{ \}


Add the empty cross vertex on the left:
\{ \}


Move it over, and get the other side of the skew Cauchy identity:
\{ \}


Proposition 4.4.2. Full Cauchy identity We have two equivalent identities (provided that $\left|u_{i} v_{j}\right|<1$ for all $i, j$ ):
\{ \}

$$
\begin{aligned}
& \sum_{\lambda} P_{\lambda}\left(u_{1}, w_{N}\right) Q_{\lambda}\left(\sigma_{I} \ldots v_{k}\right)=\lambda_{i, j} \frac{1-t u v_{j} V_{j}}{1-i_{i} v_{j}}
\end{aligned}
$$

Important notes about the function $\mathbf{Q}$ In L16 and L17 I made a mistake in the definition of the function $Q_{\lambda}$. The correct $Q_{\lambda}$ can be reconstructed as follows, from the Cauchy identities:

$$
\begin{aligned}
& \sum_{\lambda} F_{\lambda}\left(u_{1}, u_{N}\right) \frac{1}{(t, t)_{N}} Q_{\lambda}^{*}\left(v_{1} \ldots v_{M}\right)= \\
& =\sum_{\lambda} P_{\lambda}\left(u_{1} \ldots u_{N}\right) \underbrace{\text { call this }}_{\frac{\prod_{i \geqslant 0}(t, t)_{m_{i}(\lambda)}}{(t, t)_{N}} Q_{\lambda}^{*}\left(v_{1} \ldots v_{M}\right)} \\
& \Rightarrow Q_{\lambda}\left(v_{1} \ldots v_{M}\right) \\
& \Rightarrow \sum_{\lambda} P_{\lambda}(\vec{u}) Q_{\lambda}(\vec{v})=\prod_{i j} \frac{1-t u_{i} v_{j}}{1-u_{i} v_{j}}
\end{aligned}
$$

I have edited all the notes so that the old function $Q_{\lambda}$ is now denoted by $Q_{\lambda}^{*}$. The function $Q_{\lambda}$ is, by definition, dual to $P_{\lambda}$ in the sense of the Cauchy identity.

Proof 4.4.2 We will prove the identity with $F$. Again, this is an equality of two partition functions.

\{ \}
(2)

(3) Result:

$$
\begin{array}{r}
\square(1-t)\left(1-t^{2}\right)-\left(1-t^{3}\right) \\
3=N
\end{array}
$$

## 5 From HL vertex model to stochastic six vertex model

- 


### 5.1 Random step

The goal now is to use the skew Cauchy identity to upgrade the height function random field to a random field of $\lambda$ 's. This means that we need to come up with a rule of randomly selecting $\nu$ given $\lambda, \varkappa, \mu$ :
\{ \}

\{ \}


By "compatible with skew Cauchy" we mean
\{ \}

$$
\begin{aligned}
& F_{\nu / \lambda}(x) Q_{\nu / \mu}^{*}(y)=\quad \text { for all } \nu, \lambda, \lambda / x \\
& =\frac{1-t x_{x y}}{1-x_{y}} \sum_{x} F_{\mu_{1 / x}}(x) Q_{1 / x}^{*}(y) \cdot U(\nu / x, \mu, \lambda)
\end{aligned}
$$

This property, upon summing over $\nu$, leads to the skew Cauchy: \{ \}

$$
\sum_{v} F_{\nu / \lambda}(x) Q_{V / \mu}^{*}(y)=\frac{1-t x_{y}}{1-x y} \sum_{x} F_{r_{/ x}}(x) Q_{\lambda_{x}}^{*}(y)
$$

### 5.2 Constructing random steps from couplings of YBE

\{ \}


We reduce the sampling of $\nu$ to a sequence of sampling of the $\nu_{i}$ 's using the YBE on each step.
\{ \}
Local raulom step:


## A general principle

Any identity with nonnegative terms can be made into a Markov chain (in multiple ways, in fact). This is known as "bijectivisation", "coupling", or "probabilistic bijection".
\{ \}


$$
w(A) p_{1}+w(B) \rho_{2}=\tilde{w}(C)
$$

and so ou, all idutities live tis

Example 5.2.1 $\}$
Example $\quad 1+4=2+3$


Example 5.2.2 \{ \}

$$
\begin{aligned}
& E x \\
& \omega(A)=\begin{array}{l}
\omega \\
\\
\text { Uuigue } \\
\text { one lowpling }
\end{array} \\
& A \stackrel{D}{\substack{C \\
P_{1}}} \begin{array}{c}
C P_{1} \\
\hline
\end{array} \\
& \frac{p_{1}=\tilde{\omega}(C) / \omega(A)}{\left.1-P_{1}=\tilde{\omega} / D\right) / \omega(A)}
\end{aligned}
$$

To conclude, a coupling exists; and if there is a singleton in the left-hand side, then the coupling is unique.

This discussion proves the following result:

Theorem 5.2.1. Main theorem. The desired $U(\nu \mid \lambda, \varkappa, \mu)$ exists, on the leftmost column 0 it is unique and looks like this:
\{ \}

\{ \}

prob $=\alpha$
\{ \}



$$
p r o b=1-\beta
$$

The probabilities $\alpha, \beta$ are
\{ \}

$$
\}
$$



### 5.3 Finalizing the result

Now let us connect [[5.2 Constructing random steps from couplings of YBE\#Theorem 5 21 Main theorem|our theorem]] to the stochastic six vertex model.

The signatures should be understood as if in the square


Then the 6 cases in [[5.2 Constructing random steps from couplings of YBE\#Theorem 521 Main theorem|the theorem]] correspond to the following cases in the stochastic six vertex model.
\{ \}


Therefore, we get the following theorem

Theorem 5.3.1. S6V to HL coupling. Take the stochastic six vertex model, with inhomogeneous parameters $u_{1}, u_{2}, \ldots$ along the vertical, and $v_{1}, v_{2}, \ldots$ along the horizontal directions. The stochastic six vertex model updates the vertex at $(x, y)$ with probabilities

$$
b_{1}\left(u_{y}, v_{x}\right)=\frac{1-u_{y} v_{x}}{1-t u_{y} v_{x}}, \quad b_{2}\left(u_{y}, v_{x}\right)=t \frac{1-u_{y} v_{x}}{1-t u_{y} v_{x}} .
$$

Then the height function of this stochastic six vertex model (with domain wall like boundary conditions in $\mathbb{Z}_{\geq 0}^{2}$, i.e., paths enters at each site on the left boundary and nothing enters from below) has the following equality in distribution:

$$
h(x, y) \stackrel{d}{=} m_{0}\left(\lambda^{(x, y)}\right)=y-\ell\left(\lambda^{(x, y)}\right)
$$

where $\lambda^{(x, y)}$ is the random signature distributed according to the Hall-Littlewood measure

$$
\operatorname{Prob}(\lambda)=\prod_{i=1}^{x} \prod_{j=1}^{y} \frac{1-u_{j} v_{i}}{1-t u_{j} v_{i}} P_{\lambda}\left(u_{1}, \ldots, u_{y}\right) Q_{\lambda}\left(v_{1}, \ldots, v_{x}\right)
$$

## Notes and references

1. Macdonald's book: I.G.Macdonald. Symmetric Functions and Hall Polynomials (Oxford Classic Texts in the Physical Sciences), 2nd ed. 1995.
2. The closest explanation of how Cauchy identity / YBE leads to Markov chains is in Alexey Bufetov, Leonid Petrov. Yang-Baxter field for spin Hall-Littlewood symmetric functions. https://arxiv.org/abs/1712.04584 (this is a more general setting, but the HL case is recovered by putting $s=0$ everywhere)

## Problems

[[_Lecture 16, 3-31|Lecture]]

## 1

Show that the Hall-Littlewood polynomials defined as partition functions normalized by $\Pi 1 /(t ; t)_{m_{i}}$ (in the lecture) satisfy the stability property, where $\lambda=\left(\lambda_{1}, \lambda_{2}, \ldots, \lambda_{N-1}, \lambda_{N}\right):$
$P_{\left(\lambda_{1}, \ldots, \lambda_{N-1}, \lambda_{N}\right)}\left(u_{1}, u_{2}, \ldots, u_{N-1}, 0\right)= \begin{cases}P_{\left(\lambda_{1}, \ldots, \lambda_{N-1}\right)}\left(u_{1}, u_{2}, \ldots, u_{N-1}\right), & \lambda_{N}=0, \\ 0, & \text { otherwise } .\end{cases}$

Show that the partition function for the Hall-Littlewood polynomial $P_{\lambda}$ reduces to the Schur polynomial $s_{\lambda}$ for $t=0$. For that, present a weight-preserving bijection between path configurations on the red lattice, and interlacing arrays (which are the model for Schur polynomials).

## 3

Let $Q_{\lambda}^{*}$ be the partition function on the blue lattice, as defined in the lecture. Show that

1. $Q_{\lambda}^{*}\left(v_{1}, \ldots, v_{M}\right)$ is symmetric in $v_{1}, \ldots, v_{M}$.
2. We have

$$
Q_{\lambda}^{*}\left(v_{1}, \ldots, v_{M}\right)=b_{\lambda} P_{\lambda}\left(v_{1}, \ldots, v_{M}\right)
$$

where the constant $b_{\lambda}$ does not depend on $v_{1}, \ldots, v_{M}$.

## 4

Consider the HL vertex model on the cylinder:
\{ \}


Show that the partition function is symmetric in the variables $u_{i}$.
Hint. Use the fact that the red cross weights satisfy the following cancellation property:
\{ \}

[[Problems, 4-5|2 problems]], due April 19

## 1 Recall definitions and results

### 1.1 Stochastic six vertex model

\{\}


### 1.2 Hall-Littlewood polynomials

[^0]


\[

$$
\begin{aligned}
& (g \geqslant 0) \\
& 43310 \\
& 1 \uparrow \uparrow \uparrow\}
\end{aligned}
$$
\]

$$
\text { Here }(t, t)_{k}=(1-t)\left(1-t^{2}\right) \ldots\left(1-t^{k}\right)
$$

Q $Q_{\lambda}^{*}$ polynomials
\{ \}


Cauchy identity $\}$

$$
\begin{gathered}
\sum_{\lambda} P_{\lambda}\left(x_{1} \ldots x_{N}\right) Q_{\lambda}\left(y_{1} \ldots y_{M}\right)=\prod_{i j} \frac{1-t x_{i} y_{j}}{1-x_{i} y_{j}} \\
\left|x_{i} y_{j}\right|<1 \text { all } i, j
\end{gathered}
$$

### 1.3 Coupling between S6V and HL measures

The coupling between S6V and HL measures goes through the height function \{ \}

$$
\begin{aligned}
& x=0 \text { case } \\
& \underbrace{b_{1}}_{x-1}(x, y)=\frac{1-u_{x} v_{y}}{1-t u_{x} v_{y}} \begin{array}{l}
\text { inhomogeneous } 56 \mathrm{~V} \\
\text { (show) }
\end{array}
\end{aligned}
$$

We consider an inhomogeneous model with parameters $u_{1}, u_{2}, \ldots$ in the horizontal, and $v_{1}, v_{2}, \ldots$ in the vertical direction.

Here is an example simulation:
\{ \}


Theorem 1.3.1 (proved in L16) \{ \}

Theorem. $\forall x, y$-fixed,

$$
\begin{aligned}
& h(x, y) \stackrel{d}{=} y-l\left(\lambda^{(x, y)}\right)=m_{0}\left(\lambda^{(x, y)}\right) \\
& d^{(x, y)} \sim H L \text { measure } \frac{1}{Z} P_{\lambda}\left(v_{1}, ., v_{y}\right) Q_{\lambda}\left(u_{1}, \ldots, u_{x}\right)
\end{aligned}
$$

Remark 1.3.2 Since $Q_{\lambda}=b_{\lambda} P_{\lambda}$, we may as well have written $\frac{1}{Z} P_{\lambda}\left(u_{1}, \ldots, u_{x}\right) Q_{\lambda}\left(v_{1}, \ldots, v_{y}\right)$.

Remark 1.3.3 Joint distributions of the height function $h(x, y)$ are also available through Hall-Littlewood processes. Well, not all joint distributions, but only those along down-right paths. Since asymptotic analysis of Hall-Littlewood processes (and of these joint distributions) is much more involved, we do not focus on this more general coupling.

Recall the proof $\}$

\{ \}


$$
p \text { rob }=b_{1}
$$

$$
p r o b=1-b_{1}
$$

\{ \}


## 2 Hall-Littlewood polynomials

### 2.1 Formulation and easy case

Recall the HL polynomials defined as partition functions:
\{ \}

$$
=Q_{\lambda}\left(v_{1} \ldots v_{M}\right) \cdot \frac{(t, t)_{N}}{\prod_{i \geqslant 0}(t, t)_{m_{0}(\lambda)}}
$$

Theorem 2.1.1 \{ \}

$$
\begin{aligned}
& \text { Theorem. }
\end{aligned}
$$

$$
\begin{aligned}
& \text { (2) } \left.\prod_{i=0}(t, t)\right)_{m i}(x) P_{d}\left(x_{1} \cdots x_{n}\right)
\end{aligned}
$$

$$
\begin{aligned}
& \theta_{1}=Q_{\lambda}\left(x_{1}, \ldots x_{N}\right) \cdot\left(x_{1}\right)_{n_{0, N}}
\end{aligned}
$$

Proof 2.1.1. Easy case Identities 2, 4 follow from definitions of $P_{\lambda}$ and $Q_{\lambda}$ through $F_{\lambda}$ and $Q_{\lambda}^{*}$, respectively.
Let us establish 3.
\{ \}


Check:

$$
\omega^{*}\left(i_{1}, j_{1} ; i_{1}, j_{2}\right)=\frac{(t, t)_{i_{2}}}{\left(t_{1} t\right)_{i_{1}}} \omega\left(i_{2}, j_{1} ; i_{1}, j_{2}\right)
$$

That is, the red and the blue weights are related in this way. Checking this relation:
\{ \}

Indeed

$$
\omega^{*}\left(-\|_{j+1}^{g} \cdots\right)=1=\frac{(t, t)_{g}}{(t, t)_{g+1}} w\left(-\|_{j}^{g+1}\right)
$$

$$
w^{*}\left(\cdot \|_{g}^{g+1}\right)=\left(1-t^{g+1}\right) v=\frac{(t ; t)_{g+1}}{(t ; t)_{g}} \omega\left(\| \|_{g+1}^{g}\right)
$$

\{ \}

$z\left(\int_{N}^{\lambda} \int_{N}^{\lambda}\right) \cdot \frac{(t, t)_{N}}{\prod_{i \geqslant 0}^{i}(t, t)_{m_{i}}(\lambda)}$
\{ \}


$$
\xrightarrow{0 \text {-th colmmin -replace: }} \underset{\substack{\text { (multiply by this) }}}{\longrightarrow}
$$

### 2.2 Proof part 1

Step 0 Consider the space $V$ and define 4 operators $A, B, C, D$ in it: \{ \}

$$
\begin{aligned}
& V=l^{2}=\operatorname{span}\left(e_{0}, e_{1}, e_{2}, e_{3}, \ldots\right) ; e_{k} \leftrightarrow \pi^{k} \\
& A(u)+\cdots \quad B(m)-\quad(l u) F D(m) \frac{1}{t} \\
& \text { egg. } \quad B e_{k}=w\left(-\|_{k}^{k+1}\right) \cdot e_{k+1}=\left(1-t^{k+1}\right) e_{k+1}, \ldots
\end{aligned}
$$

Vertically attaching vertices means product of operators, and horizontally attaching vertices means tensor product.
\{ \}


Step 1. Express F via operators $\}$

Step 1 In $V \otimes V \otimes V \otimes \cdots$, define

$$
\begin{aligned}
& \text { In } V \otimes V \otimes V \otimes \cdots, \text { detinue } \\
& e_{\phi}=e_{0} \otimes e_{0} \otimes e_{0} \otimes \cdots \operatorname{lon}^{t} \text { all } e_{0} \\
& e_{\lambda}=e_{\operatorname{mol}(\lambda)} \otimes e_{m_{1}(\lambda)} \otimes e_{m_{2}(\lambda)} \otimes \cdots
\end{aligned}
$$

Then $\quad F_{\lambda}=\left\langle e_{\lambda}, B\left(u_{N}\right) B\left(u_{N-1}\right) \ldots B\left(u_{1}\right) e_{\phi}\right\rangle$


Step 2. Action of B on a tensor product \{ \}

$$
\begin{gathered}
\text { Ste 2. } B\left(v_{1} \otimes v_{2}\right)=B v_{1} \otimes A v_{2}+D v_{1} \otimes B v_{2} \\
B \begin{array}{ccc}
1 & D & B \\
v_{1} & v_{2} & \\
\cdots & v_{1} & v_{2}
\end{array}
\end{gathered}
$$

Step 3. Yang-Baxter equation for products Here and below we use notation $A_{i}=A\left(u_{i}\right), B_{i}=B\left(u_{i}\right), C_{i}=C\left(u_{i}\right), D_{i}=D\left(u_{i}\right)$.

$$
\begin{aligned}
& \text { Step 3. YBE } \Rightarrow \\
& B\left(u_{1}\right) D\left(u_{2}\right)=\frac{u_{1}-u_{2}}{t u_{1}-u_{2}} D\left(u_{2}\right) B\left(u_{1}\right)+\frac{(1-t) u_{2}}{u_{2}-t u_{1}} B\left(u_{2}\right) D\left(u_{1}\right) \\
& B\left(u_{1}\right) A\left(u_{2}\right)=\frac{u_{1}-u_{2}}{u_{1}-t u_{2}} A\left(u_{2}\right) B\left(u_{1}\right)+\frac{(1-t) u_{2}}{u_{1}-t u_{2}} B\left(u_{2}\right) A\left(u_{1}\right) \\
& \text { Indeed: } \\
& \underbrace{D_{2}}_{B_{1}-D_{2}}+\underbrace{B_{B_{2}}}_{z=\frac{1-z}{1-1, z}}
\end{aligned}
$$

\{ \}


$$
B v_{1} \otimes A v_{2}+D v_{1} \otimes B v_{2}
$$

### 2.3 Proof part 2

Step 4. Action of B on a tensor product of two spaces YBE and tensor action implies that the product of the $B_{i}$ 's expresses as


Step 4. $\quad B_{i}=B\left(u_{i}\right), D_{i}=D\left(u_{i}\right)$
YB $\Rightarrow B_{N}-B_{1}\left(e_{0} \otimes e_{0}\right)$ - linear cub, of


We will show that $\mathcal{I} \cap \mathcal{K}=\varnothing$, which would imply that $\mathcal{I}=\mathcal{L}$ and $\mathcal{K}=\mathcal{J}$. \{ \}

Assume $l \in \operatorname{In} K$

$$
B_{N} \cdots B_{2} B_{1}
$$

$$
B_{1}\left(v_{1} \otimes v_{2}\right)=\underbrace{B_{1} v_{1} \otimes A_{1} v_{2}}+\underbrace{D_{1} v_{1} \otimes B_{1} v_{2}}
$$

we want to camise $B_{1}$ to the left in beth $v_{1}$ and $v_{2}$ actions

- Cannot tare $D_{1} \otimes B_{1}$ because $D$ is already to the right \& does not enter the commuting
- also count take $B_{1} \otimes A_{1}$ because $A_{1}$ already to the right.

$$
\Rightarrow 1 \notin \operatorname{Ink}
$$

\{\}
$B_{i}$ commute $\Rightarrow I \cap K=\varnothing$.

$$
B_{N} \ldots B_{2} B_{1} \leadsto B_{N} \cdots B_{1} B_{2}
$$

Step 5. Computation of the coefficients \{\}
Step 5.

$$
\begin{aligned}
& B_{N}-B_{1}\left(e_{0} \otimes e_{0}\right)= \\
& =\sum_{K \in\left\{1_{1}, \ldots N\right\}} \frac{\widetilde{C}_{k}\left(u_{1} \cdot u_{N}\right)}{} \quad \prod_{i \in K} B_{i} \prod_{l \& K} D_{l} e_{0} \otimes \\
& \otimes \prod_{i \notin K} B_{i} \prod_{l \in K} A e^{e_{0}}
\end{aligned}
$$

sym in $u_{1} \ldots u_{N}$ \& $K$

$$
\tilde{C}_{6 \pi}\left(b^{-1} u\right)=\vec{C}_{k}^{(u)}
$$

Step 6. Computation of the coefficients \{
Step 6.

$$
\tilde{C}_{k}\left(u_{1} \ldots u_{N}\right)=\prod_{\alpha=1}^{i} \prod_{\beta=r+1}^{N} \frac{u_{\beta}-t u_{\alpha}}{u_{\beta}-u_{\alpha}}
$$

Because we just open parentheses in

$$
\begin{aligned}
& \prod_{i}\left(B_{i} \otimes A_{i}+D_{i} \otimes B_{i}\right) \text { \& comutle }
\end{aligned}
$$

$$
\begin{aligned}
& \begin{aligned}
& A(u) B(v)=\frac{v-t u}{v-u} B(v) A(u)+\frac{(t-1) u}{v-u} \frac{B(u) A(v)}{l} \\
\Rightarrow & \widetilde{C}_{k}=\prod_{\substack{\alpha \in \pi \\
\beta \notin \pi}} \frac{u_{\beta}-t u \alpha}{u_{\beta}-u_{\alpha}} \quad \text { ignore }
\end{aligned}
\end{aligned}
$$

$\qquad$

Step 7. Final expression for the action on two tensor factors \{ \}

Step 7. Action on $e_{0}$ of $A_{1} D$
is diagonal $\Rightarrow$

2.4 Proof part 3

Step 8. Splitting into many tensor factors $\}$

$$
\begin{aligned}
& \text { Step 8. Action on } e_{0} \otimes \ldots e_{0} \otimes \cdots \\
& \text { produces many splitting, } \\
& \text { So } \\
& \begin{array}{l}
B_{N} \otimes \cdots B_{1}\left(l_{0} \otimes e_{0} \otimes \cdots\right) \\
=\sum_{K_{0} \cup J_{1} \cup \ldots} \prod_{0 \leqslant i<j}\left[d\left(J_{j}\right) a\left(K_{i}\right) \prod_{\substack{\alpha \in K_{i} \\
\beta \in J_{j}}} \frac{u_{\beta}-t u_{\alpha}}{u_{\beta}-u_{\alpha}}\right]
\end{array} \\
& =\{1 . n\} \\
& \text { disjoint } \\
& \underbrace{B\left(K_{6}\right)}_{\zeta} e_{0} \otimes B\left(K_{1}\right) e_{0} \otimes \cdots \\
& \prod_{i \in X_{0}} B\left(u_{i}\right)
\end{aligned}
$$

Now we will express this sum as a sum over permutations

Step 9 \{ \}

Step 9. $a$ - eigar of $A_{j}$ on eo

$$
=1
$$

$d$ - eigen r. of Dou eo

$$
\begin{gathered}
=u_{j} \\
B(k) e_{0}=(\# \|)=(t, t)_{|k|} e_{|k|} \\
\text { so }\left|k_{j}\right|=n_{j}(\lambda)
\end{gathered}
$$

\{\}

$$
\begin{aligned}
& \Downarrow \\
& F_{\lambda}\left(u_{1}, \ldots, u_{N}\right)= \\
& =\sum_{\substack{K_{0} \cup K_{1} \cup \ldots \\
=\{\mid \cdots, N\}}} u_{\substack{r_{1}} u_{N}^{r_{N}} \prod_{0 \leq i<j} \prod_{\substack{\alpha \in K_{i} \\
\beta \in K_{j}}} \frac{u_{\beta}-t u_{\alpha}}{u_{\beta}-u_{\alpha}} i f f}^{i \in J_{j}}
\end{aligned}
$$

Example
\{ \}


Step 10. Finalizing the proof $\}$

Step 10. Read off the coif by

$$
\begin{gathered}
e_{m_{0}} \otimes e_{m_{1}} \otimes \ldots \otimes e_{m_{\lambda_{1}}} \\
\prod_{\substack{\alpha \in K_{i} \\
\beta \in J_{j}}} \frac{u_{\beta}-t u_{\alpha}}{u_{\beta}-u_{\alpha}}=\prod_{\substack{u_{\beta}-t u_{\alpha} \\
1 \leqslant \alpha<\beta \leqslant N \\
r_{\beta}-u_{\alpha}}}^{\Gamma_{\alpha}<\Gamma_{\beta}}
\end{gathered}
$$

\{ \}




For the proof of this identity, see [[Problems, 4-5\#1|Problem 1]] and [[Problems, $4-5 \# 2$ |Problem 2]].

## Notes and references

1. Macdonald's book: I.G.Macdonald. Symmetric Functions and Hall Polynomials (Oxford Classic Texts in the Physical Sciences), 2nd ed. 1995.

- Chapter III contains most formulas on Hall-Littlewood polynomials and symmetric functions
- Chapter III.3,5 develops a proof of the symmetrization formula through the Hall algebra (involved in the study of abelian p-groups)
- A potential another proof of the symmetrization formula might be developed by looking at identity III.(2.14), if one manages to prove the same identity at the vertex model level.

2. A verification style proof of the symmetrization formula (in a more general fashion) may be found in A. Borodin. On a family of symmetric rational functions, https://arxiv.org/abs/1410.0976, Theorem 5.1.
3. The proof presented here follows Algebraic Bethe ansatz ideology, and follows the one in A. Borodin, L. Petrov. Higher spin six vertex model and symmetric rational functions, https://arxiv.org/abs/1601.05770, Theorem 4.14.1.
4. On Algebraic Bethe ansatz see, e.g., Korepin, v. and Bogoliubov, N. and Izergin, A. Quantum inverse scattering method and correlation functions, 1993. In particular, see formula VII.(5.9) and Appendix VII. 2

## Problems

[[_Lecture 17, 4-5|Lecture]]

## 1

Show that

$$
\sum_{i=1}^{N} \prod_{j \neq i} \frac{t x_{i}-x_{j}}{x_{i}-x_{j}}=\frac{1-t^{N}}{1-t}
$$

Hint. Express the left-hand side as a contour integral over a contour containing the poles $x_{i}$, and then compute it in another manner.

## 2

Show that

$$
\sum_{\sigma \in S_{N}} \sigma\left(\prod_{1 \leq i<j \leq N} \frac{x_{i}-t x_{j}}{x_{i}-x_{j}}\right)=\frac{(t ; t)_{N}}{(1-t)^{N}}
$$

[[Lecture 18, 4-7/Problems, 4-7|3 problems]], due April 21

## 1 Reminders

Recall the following two results which we proved in the previous lectures.

Theorem 1.0.1 Consider the stochastic six vertex model with inhomogeneous parameters $u_{1}, u_{2} \ldots$ along the horizontal direction, and $v_{1}, v_{2}, \ldots$ along the vertical direction. Assume that $\left|u_{i} v_{j}\right|<1$ for all $i, j$. Recall that the parameter $t \in[0,1)$ is assumed fixed once and for all. In the stochastic vertex model, the probabilities at $(x, y)$ are

$$
b_{1}\left(u_{x}, v_{y}\right):=\frac{1-u_{x} v_{y}}{1-t u_{x} v_{y}}, \quad b_{2}\left(u_{x}, v_{y}\right):=t b_{1}\left(u_{x}, v_{y}\right)=\frac{t\left(1-u_{x} v_{y}\right)}{1-t u_{x} v_{y}}
$$

Then the height function $h(x, y)$ in this model has the same distribution as $m_{0}\left(\lambda^{(x, y)}\right)=y-\ell\left(\lambda^{(x, y)}\right)$, where $\lambda^{(x, y)}$ is a random partition distributed according to the Hall-Littlewood measure

$$
\operatorname{Prob}(\lambda)=\prod_{i=1}^{x} \prod_{j=1}^{y} \frac{1-t u_{i} v_{j}}{1-u_{i} v_{j}} P_{\lambda}\left(v_{1}, \ldots, v_{y}\right) Q_{\lambda}\left(u_{1}, \ldots, u_{x}\right)
$$

Theorem 1.0.2 The Hall-Littlewood polynomials possess the following explicit formula:

$$
F_{\lambda}\left(u_{1}, \ldots, u_{N}\right)=\prod_{i \geq 0}(t ; t)_{m_{i}(\lambda)} P_{\lambda}\left(u_{1}, \ldots, u_{N}\right)
$$

and

$$
F_{\lambda}\left(u_{1}, \ldots, u_{N}\right)=(1-t)^{N} \sum_{\sigma \in S_{N}} \sigma\left(u_{1}^{\lambda_{1}} \ldots u_{N}^{\lambda_{N}} \prod_{1 \leq i<j \leq N} \frac{u_{i}-t u_{j}}{u_{i}-u_{j}}\right)
$$

where $\sigma$ acts by permutations of the $u_{j}$ 's.

## 2 Eigenoperators

- 


## 2 Eigenoperators

### 2.0.1 Recall the Schur case $\}$



Definition 2.0.2. First Macdonald operator $\}$


This operator is a member of a whole family of $N$ commuting operators, which form a "quantum integrable system"

Theorem 2.0.3. Eigenrelation $\}$

$$
\begin{aligned}
\text { Theorem } & D\left(t_{1} 0\right) P\left(x_{\lambda} \ldots x_{N} \mid t\right)= \\
& =\frac{1-t^{N-l(\lambda)}}{1-t} P_{\lambda}\left(x_{1}, \ldots, x_{N}\right) \\
l(\lambda) & =\not \subset \text { of nonzero parts in } \lambda
\end{aligned}
$$

The same eigenrelation holds for the other polynomials $F_{\lambda}, Q_{\lambda}, Q_{\lambda}^{*}$, since they are all proportional to each other, and the multiplicative constants are independent of the variables $x_{j}$.

Proof 2.0.3. Step 1. \{ \}

$$
\begin{aligned}
& \text { Patat (0 D } D(t) P_{\phi}=\frac{1+t^{0}}{1-t} P_{\phi} \\
& P_{\phi}=1 \text { as the only db } O \\
& \text { polynomial w, lex. } \\
& \text { highest term } x^{0}=1 \\
& \text { Indeed, } \sum_{i=1}^{N}\left(\prod_{\substack{j \neq i}} \frac{t x_{i}-x_{j}}{x_{i}-x_{j}}\right)=\frac{1-t^{N}}{1-t}
\end{aligned}
$$

hint: $\frac{1}{\partial \pi i} \oint \prod_{j=1}^{N} \frac{t z-x_{j}}{z-x_{j}}, \frac{d z}{(t-1) z}$

\& look at res. at $0, \infty$, and $x_{i}$

See also [[../Lecture 17, 4-5/Problems, 4-5\#1|Problem 1]] from the previous lecture.

Proof 2.0.3. Step 2. $\}$

$$
\begin{aligned}
& \text { (2) } F_{\lambda}\left(x_{1} \ldots, x_{N}\right) \\
& \text { Fix } \lambda, l(x)=l \\
& =(1-t)^{N} \sum_{\sigma \in S_{N}} \sigma\left(x_{1}^{\lambda_{1}}-x_{N}^{\lambda_{N}} \prod_{1 \leq i<j \leq N}^{x_{i}-t x_{j}} \frac{x_{i}-x_{j}}{1}\right) \theta \\
& x_{1}^{\lambda_{1}} \cdots x_{N}^{\lambda_{N}}=x_{1}^{\lambda_{1}} \ldots x_{l}^{\lambda_{l}} \\
& T_{0, x_{i}} \text { eliminates it vuten } \\
& i>e \\
& \text { \{ \} }
\end{aligned}
$$



In words, we have split the $F_{\lambda}$ polynomial into three pieces. In the first piece, the operator $T_{0, x_{i}}$ eliminates all the variables. The third piece is a constant thanks to the symmetrization identity ([[../Lecture 17, 4-5/Problems, 4-5\#2|Problem 2]] from the previous lecture). In the second piece, the action of $D(t, 0)$ removes one of the factors and then restores it back. So we see that the overall action of $D(t, 0)$ on $F_{\lambda}$ is diagonal and produces the same eigenvalue as if we applied $D(t, 0)$ in $N-\ell$ variables to a constant. This produces the eigenvalue $\left(1-t^{N-\ell}\right) /(1-t)$.

## 3 Contour integral formulas

### 3.1 Contour integral for $\mathrm{D}(0, t)$

## Lemma 3.1.1 \{ \}

$$
\begin{aligned}
& {\left[\sum_{i=1}^{N} \prod_{j \neq i} \frac{t_{x_{i}-x_{j}}}{x_{i}-x_{j}} T_{q_{1}, x_{i}}\right] f\left(x_{2}\right) \ldots f\left(x_{1}\right)} \\
& \text { / }
\end{aligned}
$$

Here the integration contour is around all $x_{i}$ and no other poles of the integrand.
For $q=0$, we get a contour integral for the action of $D(t, 0)$ on product functions.

### 3.2 Expectation

We repeat the technology used for Schur measures (see [[../Lecture 11, 3-10/2.1 Expectation via q-difference operators|this part]]), and compute the expectation of the eigenvalue through Cauchy identity.

We have:
\{ \}

$$
\begin{aligned}
& \mathbb{E}_{H L\left(x_{1}, x_{N} ; y_{1} \ldots y_{M}\right)} \frac{1-t^{N-l(\lambda)}}{1-t}= \\
&=\prod_{i j} \frac{1-t x_{i} y_{j}}{1-x_{i} y_{j}} \sum_{\lambda} \frac{1-t^{N-l(\lambda)}}{1-t} P_{\lambda}\left(x_{1}, x_{N}\right) Q_{\lambda}\left(y_{1} \ldots y_{M}\right) \\
&=\prod_{i j} \frac{1-t x_{i} y_{j}}{1-x_{i} y_{j}} \sum_{\lambda}\left[D^{(x)}(0, t) P_{\lambda}\left(x_{1}, x_{N}\right) Q_{\lambda}\left(y_{1} \ldots y_{M}\right)\right. \\
&=\prod_{i j} \frac{1-t x_{i} y_{j}}{1-x_{i} y_{j}} D^{(x)}(0, t) \sum_{\lambda} P_{\lambda}\left(x_{1}, x_{N}\right) Q_{\lambda}\left(y_{1}, y_{M}\right) \\
&=\frac{D(0, t) f\left(x_{1}\right)-\ldots f\left(x_{N}\right)}{f\left(x_{1}\right) \ldots f\left(x_{N}\right)}, \\
& \text { where } f(z)=\prod_{j=1}^{M} \frac{1-z y_{j}}{1-t z y_{j}}
\end{aligned}
$$

Now we apply contour integrals, and also recall that $h(M, N)$ has the same distribution as $N-\ell(\lambda)$, to write:

Proposition 3.2.1 \{ \}

$$
\mathbb{E} \frac{1-t^{h(N}(M, N)}{1-t}=\frac{1}{2 \pi_{i}} \oint_{x_{i}} \prod_{i=1}^{N} \frac{t z-x_{j}}{z-x_{j}} \cdot \frac{d z}{(t-1) z} \cdot \prod_{i=1}^{M} \frac{1-z y_{i}}{1-t z y_{i}}
$$

Here the integration contour is around all the $x_{i}$ 's.

We have proven this result above.

Note on notation Here I silently switched from $u_{i}, v_{j}$ variables to $x_{i}, y_{j}$ ones, and the lattice coordinates are $M, N$. This notation with $x_{i}, y_{j}$ will persist till the end of the lecture.

Proposition 3.2.2 Integration over a different contour corresponds to a simpler quantity that we take expectation of. Namely,
\{ \}

$$
\mathbb{E}^{h(M, N)}=\frac{1}{2 \pi_{i}} \oint_{0, x_{1}, x_{N}} \prod_{i=1}^{N} \frac{t z-x_{j}}{z-x_{j}} \cdot \frac{d z}{z} \cdot \prod_{i=1}^{M} \frac{1-z y_{i}}{1-t z y_{i}}
$$

There are two changes: different contour, and the lack of $(t-1)$ in the denominator.

## Proof 3.2.2

This follows from the fact that the residue of the integrand at zero is equal to $1 /(t-1)$, which is immediate since 0 is a simple pole.

### 3.3 Multiple t-moments

To address multiple $t$-moments, consider a slightly different operator \{ \}


Proposition 3.3.1 \{ \}

$$
\begin{gathered}
\frac{\tilde{\oplus} f\left(x_{1}\right) \ldots f\left(x_{N}\right)}{f\left(x_{1}\right) \ldots f\left(x_{N}\right)}=\frac{1}{2 \pi i} \oint_{j=1} \prod_{j=1}^{N} \frac{z-x_{j} / t}{z=x_{j}} \frac{d z}{z \cdot f(z)} \\
f(0)=1
\end{gathered}
$$

This follows from the expression for $D(t, 0)$.

Next:
\{ \}


Using the relation between the stochastic six vertex model and Hall-Littlewood measures, we have:

Corollary 3.3.2 \{ \}

$$
\begin{aligned}
& \text { For any } k \geqslant 0 \text {, } \\
& \qquad \mathbb{E} t^{k h(\mu, N)}=t^{k N} \frac{[\tilde{D}]^{k} f\left(x_{1}\right) \ldots f\left(x_{N}\right)}{f\left(x_{1}\right) \ldots f\left(x_{N}\right)}
\end{aligned}
$$

Now let us express this in integrals. Start with the case $k=2$.
\{ \}

How about $\tilde{D}^{2}$ ?

$$
\begin{aligned}
& \tilde{f}(\omega)=f(\omega) \cdot \prod_{j=1}^{N} \frac{z-w / t}{z-w} \quad \tilde{f}(0)=1 \\
& \Downarrow \\
& \frac{\tilde{D}^{2} f\left(x_{1}\right) \ldots f\left(x_{N}\right)}{f\left(x_{1}\right) \ldots f\left(x_{N}\right)}=\frac{1}{(2 \pi i)^{2}} \oint_{0, x_{1} \ldots x_{N}}^{w} \frac{d w}{0, x_{1} \ldots x_{N}} \frac{d z}{z} . \\
& \frac{z-w}{(z-w) t} \cdot \prod_{0}^{N} \frac{z-x_{j} / t}{z-x_{j}} \cdot \frac{w-x_{j} / t}{w-x_{j}} \cdot \frac{1}{f(z) f(w)}
\end{aligned}
$$

Here, however, we need to respect algebra, and not pick the residue at $w=z t$.

This is achieved by a careful selection of contours.
Take these (I call $z_{1}=z, z_{2}=w$ for easier matching with the next contour picture):
\{ \}


By continuing for larger $k$, we get the following $k$-fold contour integral formulas:

Theorem 3.3.3 \{ \}



Corollary 3.3.4 For the stochastic six vertex model:
\{ \}

$$
\begin{aligned}
& \pi t^{k h(M, N)}=\frac{t^{\frac{k(k-1)}{2}}}{(2 \pi i)^{k}} \oint \ldots \oint \frac{d z_{1} \cdots d z_{k}}{z_{1} \ldots z_{k}} \\
& \prod \prod_{\substack{ \\
1 \leqslant A \angle B \leq k} Z_{A}-z_{B}}^{z_{A}-t z_{B}} \prod_{i=1}^{k}\left[\prod_{j=1}^{N} \frac{t z_{i}-x_{j}}{z_{i}-x_{j}} \prod_{j=1}^{M} \frac{1-z_{i} y_{j}}{1-t z_{i} y_{j}}\right]
\end{aligned}
$$

Contours are the same as in [[\#Theorem $333 \mid$ the previous theorem]].

### 3.4 Moment problem

The expectations $\mathbb{E} t^{k h(M, N)}$ for all $k \geq 1$ determine the distribution of $h(M, N)$. Indeed, since $t^{h(M, N)} \in(0,1]$, this follows from the compact moment problem:

Theorem 3.4.1 Let $\nu$ be a probability measure supported on a compact segment $[a, b]$. Then its moments $\alpha_{k}=\int_{a}^{b} x^{k} \nu(d x)$ completely determine $\nu$.
This statement follows from Weierstrass theorem on approximation of continuous functions by polynomials.

## 4 Idea of the asymptotic analysis

- 


## 4.1 q-binomial theorem

Let us step back and consider possible generating functions associated with our $t$-moments of the height function.

Theorem 4.1.1. q-binomial theorem This is usually called the q-binomial theorem, however, our t plays the role of q .
If $|t|<1,|\zeta|<1$, then
\{ \}


Proof 4.1.1 This would follow from a more general identity as $N \rightarrow \infty$. \{ \}


We will write down a recurrence for the series coefficients $c_{N, n}$. \{ \}

$$
\begin{aligned}
& \left(\sum_{k} C_{N, k} \zeta^{k}\right)\left(1-3 t^{N-1}\right)=\left[\sum_{n} C_{N-1, n} 3^{n}\right] \\
& C_{N-1, n}=C_{N, n}-t^{N-1} C_{N, n-1} \\
& \}
\end{aligned}
$$



This shows the more general identity depending on $N$. Finally, the limit as $N \rightarrow \infty$ looks as
\{ \}

$$
\lim _{N \rightarrow \infty} \frac{(t, t)_{N+u-1}}{(t, t)_{N-1}(t ; t)_{n}}=\frac{1}{(t, t)_{n}}
$$

### 4.2 Asymptotic fluctuations via t-Laplace transforms

Let us apply the t-binomial theorem to t-moments:

Theorem 4.2.1 We have
\{ \}


$$
\begin{aligned}
& =\sqrt{\left[-\frac{1}{2},\right.} \\
& \longrightarrow \\
& \begin{array}{c}
t \text {-Laplace } \\
\text { trmus form }
\end{array}
\end{aligned}
$$

Proof 4.2.1 This is allowed: we can interchange expectation with summation because the random variable $t^{h(M, N)}$ is bounded.

Note that the right-hand side is called "t-Laplace transform", because
\{ \}



$$
\text { Unifion } x \in(-\infty, 0)
$$

The t-Laplace transform is helpful for asymptotic analysis:
\{ \}


Here $\eta$ should be the Tracy-Widom random variable, as in TASEP.
Take
\{ \}


Then:
\{ \}
$\theta \mathbb{E} \frac{}{\left(t^{(\eta-s) N^{1 / 3}} ; t\right)_{\infty}}$


$\frac{1}{(\ldots)_{\infty}} \rightarrow 0$


We believe that the probability in the right-hand side is expressed as (1 minus the) Fredholm determinant of the Airy kernel. This determinantal structure is quite special, and we don't see it yet before the limit.

To see the Fredholm determinant in the limit, it is actually possible to first obtain this structure before the limit. We will do this next time.

## Notes and references

There are several papers on the method of $t$ - (or $q$-) moments in the analysis of interacting particle systems. Here are the main references:

1. Alexei Borodin, Ivan Corwin. Macdonald processes. https://arxiv.org/abs/1111.4408; Proposition 3.2.1 onwards
2. Alexei Borodin, Ivan Corwin, Tomohiro Sasamoto. From duality to determinants for q-TASEP and ASEP. https://arxiv.org/abs/1207.5035. Section 3
3. Alexei Borodin, Ivan Corwin, Leonid Petrov, Tomohiro Sasamoto. Spectral theory for the q-Boson particle system. https://arxiv.org/abs/1308.3475. Appendix 7.2 has the proof of the contour shift argument.
4. Alexey Bufetov, Matteo Mucciconi, Leonid Petrov. Yang-Baxter random fields and stochastic vertex models. https://arxiv.org/abs/1905.06815. Section 9.1 discusses operator approach to getting $t$-moments of the stochastic six vertex model, and the corresponding Fredholm determinants.

## Problems

## 1

Show the following $t$-binomial identity:

$$
(-\zeta ; t)_{N}=\sum_{k=0}^{N} t^{k(k-1) / 2} \frac{(t ; t)_{N}}{(t ; t)_{N-k}(t ; t)_{k}} \zeta^{k}
$$

## 2

The binomial coefficient $\binom{n}{k}$ is, for example, the number of up-right lattice paths from $(0,0)$ to $(k, n-k)$. Find a similar interpretation for the t-binomial coefficient

$$
\frac{(t ; t)_{n}}{(t ; t)_{k}(t ; t)_{n-k}} .
$$

More precisely, find a statistic $f_{t}(\pi)$ on up-right paths such that the partition function

$$
\sum_{\pi \text { up-right paths }} f_{t}(\pi)
$$

is the t-binomial coefficient $\frac{(t, t)_{n}}{(t, t) k_{k}(t, t)_{n-k}}$.

## 3

Show that the Macdonald operator $D(t, q)$

\{ \}

preserves the ring of symmetric polynomials in $N$ variables $x_{1}, \ldots, x_{N}$.
[[Problems, 4-12|2 problems]], due May 6
1 Reminder

1 What we have and what we need
We have the $t$-Laplace transform for the height function of the (inhomogeneous) stochastic six vertex model:
\{ \}

t-Laplace transform,
useful for asymptotics

We expect the $t$-Laplace transform to converge to a Fredholm determinant (which expresses the Tracy-Widom distribution).
\{ \}

$$
\begin{aligned}
& \operatorname{det}(1+z A) \quad \text { Recall Fredholm det: } \\
& =1+\sum_{m=1}^{\infty} \frac{z^{m}}{m!} \int_{\Omega^{m}} \int_{\Omega^{m}} \operatorname{det}_{m \times m}\left[K\left(x_{i}, x_{j}\right)\right] d x_{1} \cdot d x_{m} \\
& (A f)(x)=\int_{\Omega} K(x, y) f(y) d y \\
& A: L^{2}(\Omega) \rightarrow L^{2}(\Omega), \quad \text { locily trace class }
\end{aligned}
$$

We would like to rewrite the $t$-Laplace transform as a Fredholm determinant. We have $k$-fold nested contour integral expressions for the $k$-th moment:
\{ \}
want this to

\{ \}


$$
f(z)=\prod_{i=1}^{x} \frac{1-z u_{i}}{1-t z u_{i}} \prod_{i=1}^{y} \frac{t z-v_{i}}{z-v_{i}}=g(z) / g(t z)
$$

$$
g(z)=\prod_{i=1}^{x}\left(1-z u_{i}^{c}\right) \prod_{i=1}^{y} \frac{1}{z-v_{i}}
$$

We would like to have a determinant under the non-nested $k$-fold integral. How about getting the determinant?
\{ \}

$$
\begin{aligned}
& \operatorname{det}\left[\frac{1}{x_{i}-y_{j}}\right]= \pm \frac{V(\vec{x}) v(\vec{y})}{\prod_{i j}\left(x_{i}-y_{j}\right)} \\
& \operatorname{det}\left[\frac{1}{x_{i}-t x_{j}}\right]= \pm \frac{v(\vec{x}) v(t \vec{x})}{\prod_{i}\left(x_{i}-t x_{j}\right)}
\end{aligned}
$$

## 2 Contour shift theorem

- 


### 2.1 Contour shifting

There are the following contours in our $k$-fold contour integrals \{\}


This is what were going to do to make the contours the same: \{ \}

Move $z_{k}, z_{k-1}, \ldots, z_{1}$ contours in this order to $\gamma$ around $O$ and $v_{\tau}$ and contains tr but not $t^{-1} u_{j}^{-1}$ 。
\{ \}

This picks $\Theta$ residues from

$$
\prod_{A<B} \frac{z_{A}-z_{B}}{z_{A}-t z_{B}}
$$

at $\quad z_{A}=t z_{B}$
for all $A<B$
Surprisingly, this is not a nightmare*
( $x$-becomes one for maltipoint formulas)
2.2 Warm up for $\mathrm{k}=2$

Let us perform the contour shifts for $k=2$.
\{ \}


Next, $z_{1}$ pricks $\theta$ Res at $z_{1}=t z_{2}$
\{ \}


In general we see that $f$ is evaluated at one of the integration variables, or at $t^{j}$ times the integration variable. This is called a string specialization

### 2.3 Sting specializations

\{ \}
\{ \}


$$
\begin{array}{r}
+\frac{(t, t)_{2}}{2 \pi i} \oint \operatorname{det}\left[\frac{1}{\omega_{1}-t^{\lambda j} w_{j}}\right]_{1 \times 1} f(\omega 0 \lambda) \\
\lambda=(2)
\end{array}
$$

$2.4 \mathrm{k}=3$, more warm up
Let us check how this works for $k=3$ :
\{ \}
$\frac{t^{3}}{(2 \pi i)^{3}} \oint_{\gamma_{1} \gamma_{2} \gamma_{3}} \frac{z_{1}-z_{2}}{z_{1}-t z_{2}}, \frac{z_{1}-z_{3}}{z_{1}-t z_{3}} \frac{z_{2}-z_{3}}{z_{2}-t z_{3}} \quad \frac{f\left(z_{1}\right) f\left(z_{2}\right) f\left(z_{3}\right)}{z_{1} z_{2} z_{3}}$ deform $z_{3}$ - nothing is picked deform $z_{2}$


$$
\text { poles } z_{1}=t z_{2}, z_{1}=t z_{3}
$$

$$
\begin{aligned}
& \text { t integral } \\
& (3 \text { terms })
\end{aligned} \quad \lambda=(111)
$$

pick ${ }_{\text {res }}$ at $z_{2}=t z_{3}$

$$
\searrow \quad \lambda=(3)
$$

$$
\underbrace{\text { Cole at }^{2} z_{1}=t^{2} z_{3}}_{\text {from }}
$$



Here we have contributions from all partitions $\lambda$ with $|\lambda|=\lambda_{1}+\ldots+\lambda_{\ell}=3$. The number $\ell=\ell(\lambda)$ of nonzero parts is precisely the number of the "free" integration variables.
-

### 2.5 Theorem formulation

Theorem 2.5.1. Contour shift theorem \{ \}

Thus．（Brodinu－Comin

$$
\begin{aligned}
& f(z)=g(z) / g(t z) \\
& f(z) f(t z-f(t ⿲ 二 丨 匕 刂) \\
& =(z) / g\left(t^{n} z\right)
\end{aligned}
$$

$\mathbb{E} t^{k h(x, y)}=(t, t)_{k}$

$$
\sum_{\lambda:}^{\left.\prod_{i}^{i} \frac{1}{m_{i}^{0}(\lambda)} \right\rvert\,}
$$

$$
|\lambda|=K
$$

$$
\frac{1}{(2 \pi i)} l(\lambda) \int_{\gamma} d e t\left[\frac{q_{\gamma}\left(\omega_{i}^{0}\right) / g\left(t^{\lambda_{i}} \omega_{i}\right)}{\omega_{i}-t^{\lambda} \omega_{j}}\right] l(\lambda) \times l(\lambda)
$$

here I already put g＇s coming from $f(w \circ \lambda)$

2．6 Theorem proof
2．5．1．Proof．The proof proceeds by a careful bookkeeping of residues and substitutions occurring during contour deformations．
\｛ \}

$$
\frac{t^{\frac{k(k-1)}{2}}}{(2 \pi i)^{k}} \oint_{\gamma} \frac{d z_{1}}{z_{1}} \cdots \oint_{\gamma_{k}} \frac{d z_{k}}{z_{k}} \cdot \prod_{1 \leqslant A \angle B \leqslant K} \frac{z_{A}-z_{B}}{z_{A}-t z_{B}} \cdot \frac{g\left(z_{1}\right)--g\left(z_{k}\right)}{g\left(t z_{-}\right)--g\left(t z_{k}\right)}
$$

Each time we pick either a residue，or an integral．
［［．．／img／Pasted image 20210426211752．png］］
We regard this as a function of the remaining free variables $z_{i_{\lambda_{1}}}, z_{j_{\lambda_{2}}}$ ，and so on．

Step 1. $\}$

$$
\sum_{|\lambda|=K} \frac{t^{\frac{k(k-1)}{2}}}{m_{1}!m_{2}!\cdots \cdots} \sum_{\lambda \sim \lambda} \oint_{\gamma} \oint_{\gamma} \frac{d z_{i} d \lambda_{1} j_{\lambda_{2}} \ldots}{(2 \pi i)^{l}}
$$



Step 2. \{ \}

Step 2. From sets I to $\tilde{E} \in S_{k}$
Fix $\lambda \&$ box at $\Sigma_{I \sim \lambda}$.
Relabel:


That is, instead of calling the free variables $z_{i_{\lambda_{1}}}, z_{j_{\lambda_{2}}}, \ldots$, which remembers the structure of the original integral, we will define a "canonical" way to encode residues.
\{ \}

$$
\begin{aligned}
& \text { Note: } \exists!\quad \zeta \in S_{k} \text { st. } \\
& \qquad \begin{aligned}
\left(z_{1}, z_{k}\right)= & \left(y_{b}(1) \ldots y_{z(k)}\right) \\
\text { Call } \omega_{j}= & y_{\lambda_{1}+\cdots+\lambda_{j-1}+1, j=1-l(\lambda)} \\
& \text { all free vewiabies }
\end{aligned} \\
& \}
\end{aligned}
$$

Let Res ${ }_{\lambda}$ be the canonical residue

$$
\begin{gathered}
\text { (*) } y_{\lambda_{1}}=t y_{\lambda_{1}-1}, y_{\lambda_{1}-1}=t y_{\lambda_{1}-2}, \ldots y_{2}=t y_{1} \\
\text { etc }
\end{gathered}
$$

(a function of $w_{1} \cdots w_{l}$ ).
Note: all poles for res. are simple
Also let Sired $\lambda_{\lambda}$ substitution of ( $(*)$

Therefore, for a fixed $\lambda$ the sum over $I$ :
\{\}

$$
\begin{aligned}
& \sum_{I \sim \lambda} \oint_{\gamma}-\oint_{\gamma} d z_{i_{\lambda_{1}}} d z_{\lambda_{\lambda_{2}}} \cdots \\
& \quad \operatorname{Res}_{I}\left[\prod_{1 \leqslant A \angle B \leq K} \frac{z_{A A}-z_{B}}{z_{A}-t z_{B}} \cdot \frac{f\left(z_{1}\right) \ldots f\left(z_{k}\right)}{z_{1} \ldots z_{k}}\right](-1)^{k \cdot l}
\end{aligned}
$$

equals
\{ \}

$$
\begin{aligned}
& \sum_{\sigma \in S_{K}} \oint_{\gamma} d \omega_{1} \ldots \oint_{\gamma} d \omega_{e} \operatorname{Res}\left[\prod_{A<B} \frac{y_{b(A)}-y_{b}(B)}{y_{b(A)}-\operatorname{tg}_{b(B)}}\right](-1)^{k-l} \\
& -S_{u} b_{\lambda}\left[\frac{f\left(y_{1}\right) \ldots f\left(y_{k}\right)}{y_{1} \ldots y_{k}}\right]
\end{aligned}
$$

If $b$ did mot come
from $I \sim \lambda$, the $z_{G(A)}-z_{z(B)}$ would
be wrong \& will cancel the demaninator at some point $\Rightarrow$ get $O$ residue

Step 3. Now we can separate the part of the formula which needs a residue, and part which is symmetrized.
\{ \}

$$
\prod_{A<B} \frac{y_{6(A)}-y_{B(B)}}{y_{B(A)}-t y_{B(B)}}=\prod_{A \neq B} \frac{y_{A}-y_{B}}{y_{A}-t y_{B}} \prod_{B<A} \frac{y_{6(A)}-t y_{G(B)}}{y_{G(A)}-y_{6(B)}}
$$

${ }^{\text {Res }}{ }_{\lambda}$ needed ouby here

For symmetrization we have \{ \}

$$
\sum_{b \in S_{k}} \pi_{B<A} \frac{y_{G(A)}-t y_{\sigma(B)}}{g_{\sigma(A)}-y_{6}(B)}=(t, t)_{k} /(1-t)^{k}
$$

See [[../Lecture 17, 4-5/Problems, 4-5\#2| this problem]].
Step 4. For the residue, one can explicitly compute it:
\{ \}
Step 4. Lemma lexerase 2
$(-1)^{k l} \operatorname{Res}_{\lambda}\left[\prod_{A \neq B} \frac{y_{A}-y_{B}}{y_{A}-t y_{B}}\right]=$


We're now checking that the powers of $t$ agree:
\{ \}

$$
\begin{aligned}
& \frac{t^{k(k-1) / 2} t^{-k^{2} / 2} \omega_{i}^{\lambda_{i}} t^{\lambda_{i}^{2} / 2}}{y_{1}-y_{k}}= \\
& =\frac{t^{-k / 2} t^{\lambda_{i}^{2} / 2}}{1 \cdot t \cdot \ldots \cdot t^{\lambda_{1}-1} \cdot 1 \cdot t \cdot \ldots \cdot t^{\lambda_{2}-1} \ldots}=1, \\
& \quad \begin{array}{r}
\text { (theorem is } \\
\text { done) }
\end{array}
\end{aligned}
$$

This completes the proof of the theorem.

## 3 Two Fredholm determinants

- 


### 3.1 Fredholm 1

From the contour shift theorem we get our first Fredholm determinantal expression for the $t$-Laplace transform.

Theorem 3.1.1 \{ \}

$$
\begin{aligned}
& \text { T. } \mathbb{E} \frac{1}{\left(3 t^{n(x) j} ; t\right)_{\infty}}=\operatorname{det}\left(1+k_{\xi}^{1}\right) \\
& (a, t)_{\infty}=(1-a)(1-a t)\left(1-a t^{2}\right) \ldots \quad|\rho|<\varepsilon-\text { cons. } \\
& K_{3}^{1}\left(n, w ; n^{\prime} w^{\prime}\right)=\frac{3^{n}}{w^{\prime}-t^{n} w} g(w) / g\left(t^{n} \omega\right) \\
& \begin{array}{r}
\text { Kernel in } L^{2}\left(\mathbb{Z}_{\geq 0} \times \gamma\right) \\
L_{\text {counting measure }}
\end{array}
\end{aligned}
$$

Proof 3.1.1 One can reorganize the sum over partitions into the sum over independent indices, and there are determinants inside the integrals which produces the Fredholm determinant:
\{ \}

3.2 Mellin-Barnes summation

There is another way to write the Fredholm determinant which is more convenient for asmyptotics. A helpful fact is the Mellin-Barnes summation:

Lemma 3.2.1 \{ \}

$$
\begin{aligned}
& \text { L. } \sum_{n=1}^{\infty} F(n) s^{n}=\frac{1}{2 n i} \int_{C} \Gamma(-s) \Gamma(1+s)(-3)^{s} F\left(s^{s}\right) d s \\
& \text { For nice } F \\
& \left(\text { mop poles of } \text { of } \text { isht }^{\text {etc }} \text { ) } \mid c\right. \\
& \begin{array}{c}
|\zeta|<1, \zeta \notin \mathbb{R} \geqslant 0 \\
(-\zeta)^{s} \text { branch } \\
\text { out at } \\
s \in \mathbb{R} \geqslant 0
\end{array}
\end{aligned}
$$

Proof 3.2.1 \{ \}

3.3 Fredholm 2 with kernel as a contour integral Theorem 3.3.1 \{ \}

$$
\begin{aligned}
& K_{3}^{1}\left(n, w ; n^{\prime}, w^{\prime}\right)=\frac{3^{n}}{w^{\prime}-t^{n} w} g(w) / g\left(t^{n} w\right)
\end{aligned}
$$

$$
\begin{aligned}
& K_{\zeta}^{2}\left(\omega, \omega^{\prime}\right)=\frac{1}{2 \pi u^{i}} \int_{C} \Gamma(-s) \Gamma(1+s)(-\zeta)^{s} \frac{g(w)}{g\left(t^{s} w\right)} \frac{d s}{w^{\prime}-t_{w}^{s}}
\end{aligned}
$$

Together with $\oint_{\gamma} d w$, this finally produces our favorite double contour integral structure!

Proof 3.3.1 \{ \}

$$
\begin{aligned}
& \text { Prod } \quad \operatorname{det}\left(1+k_{\frac{1}{1}}^{3}\right)=
\end{aligned}
$$

$$
\begin{aligned}
& \sum_{n \geq 0} k_{3}^{1}\left(n, w, n^{\prime} w^{\prime}\right)= \\
& n \geqslant 0 \\
& =\sum_{n \geqslant 0} \frac{子^{n}}{w^{\prime}-t^{n} w} g(w) / g\left(t^{n} \omega\right) \\
& =\frac{1}{2 \pi^{i}} \int_{C} \Gamma(-s) \Gamma(1+s)(-\zeta)^{s} \frac{g(w)}{g\left(t^{s} w\right)} \frac{d s}{w^{\prime}-t^{s} w}
\end{aligned}
$$

## Notes and references

There are several papers on the method of $t$ - (or $q$-) moments in the analysis of interacting particle systems. Here are the main references:

1. Alexei Borodin, Ivan Corwin. Macdonald processes. https://arxiv.org/abs/1111.4408; Proposition 3.2.1 onwards
2. Alexei Borodin, Ivan Corwin, Tomohiro Sasamoto. From duality to determinants for q-TASEP and ASEP. https://arxiv.org/abs/1207.5035. Section 3
3. Alexei Borodin, Ivan Corwin, Leonid Petrov, Tomohiro Sasamoto. Spectral theory for the q-Boson particle system. https://arxiv.org/abs/1308.3475. Appendix 7.2 has the proof of the contour shift argument.
4. Alexey Bufetov, Matteo Mucciconi, Leonid Petrov. Yang-Baxter random fields and stochastic vertex models. https://arxiv.org/abs/1905.06815. Section 9.1 discusses operator approach to getting $t$-moments of the stochastic six vertex model, and the corresponding Fredholm determinants.

## Problems

## 1

Compute the residue in [[2.6 Theorem proof\#Step 4|step 4]] of the proof.

## 2

Compute the residue of the Gamma function $\Gamma(x)=\int_{0}^{\infty} t^{x-1} e^{-t} d t$ at nonpositive integers $-n, n=0,1,2 \ldots$ The Gamma function is not defined there by this formula, but rather one could use the reflection formula for the Gamma function:

$$
\Gamma(x) \Gamma(1-x)=\frac{\pi}{\sin (\pi x)}, \quad x \notin \mathbb{Z}
$$

[[Problems, 4-14|1 problem]], due May 6

## 1 Asymptotics of Fredholm determinants

Throughout this section we perform a critical point analysis of the Fredholm determinant for the $t$-Laplace transform of the height function in the stochastic six vertex model.
-
1.1 Recall what we work with
\{ \}

What we have \& what we want

$$
g(z)=\prod_{i=1}^{x}\left(1-z u_{i}\right) \prod_{i=1}^{y} \frac{1}{z-v_{i}}
$$



Th.

$$
\begin{aligned}
& \mathbb{E}\left[\frac{1}{\left(\zeta t^{n(x, y)} ; t\right)_{\infty}}\right]=\operatorname{det}\left(1+k_{\zeta}^{1}\right) \\
& k_{3}^{1}\left(n, w j n^{\prime}, w^{\prime}\right)=\frac{3^{n}}{w^{\prime}-t^{n} w} g(w) / g\left(t^{n} w\right) \\
& \frac{|\xi|<\varepsilon-\text { con. }}{\text { kernel in } L^{2}\left(Z_{z 0} \times \gamma\right)}
\end{aligned}
$$

\{\}
L.

$$
\sum_{n=1}^{\infty} F(n) s^{n}=\frac{1}{2 \pi^{i}} \int_{C} \Gamma(-s) \Gamma(1+s)(-3)^{s} F(s) d s
$$

Mellin-Barnes sumeation

\{\}

Th. (Fredholm 2)

$$
\begin{aligned}
& \operatorname{det}\left(1+K_{3}^{1}\right)=\operatorname{det}\left(1+K_{3}^{2}\right) \\
& K_{3}^{2}-\text { kerwel on } \gamma \\
& K_{5}^{2}\left(w, w^{1}\right)=\frac{1}{2 n^{2}} \int_{C} \Gamma(-s) \Gamma(1+s)(-\zeta)^{s} \frac{g(w)}{g\left(t^{s} w\right)} \frac{d s}{w^{\prime}-t_{w}^{s}}
\end{aligned}
$$

\{\}

Proof $\operatorname{det}\left(1+K_{3}^{1}\right)=$

$$
\begin{aligned}
& \sum_{n \geqslant 0} K_{3}^{1}\left(n, w, n_{1}^{\prime}, w^{\prime}\right)= \\
& =\sum_{n=0} \frac{3^{n}}{w^{\prime}-t^{n} w} g(\omega) / g\left(t^{n} \omega\right) \\
& =\frac{1}{2 n^{i}} \int_{C} \Gamma(-s) \Gamma(1+s)(-\zeta)^{s} \frac{g(w)}{g\left(t^{s} w\right)} \frac{d s}{w^{s}-t^{s} w}
\end{aligned}
$$

1.2 Expansion we work with
\{ \}

$$
\begin{aligned}
& \left(3 t^{\text {hex) }}{ }_{5} t\right)_{\infty} \\
& \begin{array}{l}
\sum_{l \geqslant 0} \frac{1}{l!} \frac{1}{(2 n i)^{2}} \underbrace{\oint_{C}}_{c} \underbrace{\oint}_{\gamma} \oint_{i=1}^{\oint} \prod_{i}^{l} r\left(-s_{i}\right) r\left(s_{i}+1\right)(-3)^{s_{i}} \frac{g\left(w_{i}\right)}{g\left(t^{s_{i}} w_{i}\right)} \\
\operatorname{det}\left[\frac{1}{\omega_{0}-t^{s_{i}} w_{i}}\right]_{i, j=1}^{l} d \vec{\omega} d \vec{s}
\end{array} \\
& g(z)=\prod_{i=1}^{x}\left(1-z u_{i}\right) \prod_{i=1}^{y} \frac{1}{z-v_{i}}
\end{aligned}
$$

- 

1.3 Homogeneous parameters in the model
\{ \}

$$
\begin{gathered}
u_{1}^{a} \equiv 1, \quad v_{j} \equiv v, \quad \theta<v<1 \\
g(z)=(1-z)^{x}(z-v)^{-y} \quad 0<t<1 \\
\text { Let } \quad t_{1}^{S} \omega=z, \quad\left|t^{s}\right|=\left|t^{\delta+i p}\right|=\left|t^{\delta}\right|<1
\end{gathered}
$$

So $z$ belongs to $\left|t^{\delta}\right| \cdot|\omega|$-circle
\{ \}

\{ \}


So, let us summarize how the asymptotics analysis would look like.
\{ \}

Then integrands become

$$
\Gamma(-s) \Gamma(1+s)=\frac{-\pi}{\sin \pi s}
$$

$$
\begin{aligned}
& \prod_{i=1}^{l} r\left(-s_{i}\right) \Gamma\left(s_{i}+1\right)(-3)^{s_{i}} \frac{g\left(w_{i}\right)}{g\left(t^{s_{i}} w_{i}\right)} \operatorname{det}\left[\frac{1}{w_{j}-t^{s_{i}} w_{i}}\right]_{i, j=1}^{l} \\
& =\prod_{i=1}^{l}\left(-\frac{n}{\sin (n s i)}\right) \operatorname{det}\left[\frac{1}{w_{j}-z_{i}}\right] \prod_{i=1}^{l} \frac{(-3)^{s i} g\left(w_{i}\right)}{g\left(z_{i}\right)} \\
& t^{\left.t^{s i}=z_{i} / w_{i}\right)} \\
& s_{i}=\log \left(z_{i} / w_{i}\right) / \log (t)
\end{aligned}
$$

\{\}

1
Here the brauch of leg merters, let us fook at another are.

$$
\begin{aligned}
& \text { Evectadily } \quad \begin{aligned}
& z_{i} / w_{i} \simeq \frac{w_{c}+L^{-1 / 3} \tilde{z}}{w_{c}+L^{-1 / 3} \tilde{w}} \\
&=1+O\left(L^{-1 / 3}\right) \\
& \log \left(z / w_{i}\right)=O\left(L^{-1 / 3}\right)+2 \pi i \cdot k \\
& k \in \mathbb{Z}
\end{aligned}
\end{aligned}
$$

\{\}

$$
\begin{gathered}
\operatorname{Sin}\left(\frac{1}{\log t}\left[O\left(L^{-1 / 3}\right)+2 \pi i \cdot k\right]\right) \\
\frac{e^{i z}-e^{-i z}}{2 i}: \pm \frac{1}{2 i}\left[t^{2 \pi k}+t^{-2 \pi k}+\left(L^{-1 / 3}\right)\right] \\
\frac{1}{\sin (\ldots)} \quad k=0, \quad \text { arge } O\left(L^{+1 / 3}\right) \\
k \neq 0, \quad \text { constant. }
\end{gathered}
$$

Oury one of these cortributes - teorns out it's $k=0$.
\{\}
$\Rightarrow$ lonelusion:

$$
\text { we piek } \quad s=\frac{1}{\log t} \log (z / w)
$$

priuciple brauch
\{ \}

$$
\begin{aligned}
& \text { Next, } \frac{(-3)^{s} g(w)}{g(z)}= \\
& =t^{-\left[L H+p L^{1 / 3}\right] \cdot s} \cdot \frac{(1-w)^{s}=z / w}{(1-z)^{L x}(w-v)^{L y}} \\
& =\exp \left\{(\log z-\log w)\left[-L J-L+L^{L / 3}\right]+\right. \\
& \\
& +\operatorname{Lg}(\log (z-v)-\lg (w-v)) \\
& \\
& +L x[\log (1-w)-\log (1-z)]\}
\end{aligned}
$$

\{ \}

$$
\begin{aligned}
& =\exp \left\{L[S(w)-S(z)]+L^{1 / 3}[\lg z-\log \omega\}\right\} \\
& S(w)=H \lg w-y \log (\omega-v)+x \lg (1-w)
\end{aligned}
$$

\{ \}

\{ \}

$$
\begin{aligned}
& S(w) \text { dable crit pts: } \\
& S^{\prime}=s^{\prime \prime}=0 \text { eqn on } w, \text { th. }
\end{aligned}
$$

1.4 Summary of the strategy
\{ \}

$$
\begin{aligned}
& \mathbb{E} \frac{1}{\left(\zeta t^{h(x, y)}, t\right)_{\infty}}=\operatorname{det}\left(1+k_{2}^{3}\right) \\
& =\sum_{l \geq 0} \frac{1}{l!} \frac{1}{(2 \pi i)} \underbrace{2 l}_{c} \underbrace{\oint}_{\gamma} \underbrace{\oint}_{\gamma} \underbrace{\oint}_{i=1} \prod_{i=1}^{l} r\left(-s_{i}\right) r\left(s_{i}+1\right)(-3)^{s i} \underbrace{g\left(w_{i}\right)}_{g\left(t^{s i} w_{i}\right)} \\
& \operatorname{det}\left[\frac{1}{\omega_{j}-t^{s i} w_{i}}\right]_{i . j=1}^{l} d \vec{\omega} d \vec{s}
\end{aligned}
$$

\{\}

$$
=\sum_{l \geqslant 0} \frac{1}{l!} \frac{1}{(2 \pi i)^{2}} \oint_{\gamma^{\prime}} \oint_{\gamma} d \vec{z} \oint_{\gamma}-\oint_{d \vec{o}} d e t\left[\frac{1}{w_{i}^{j}-z_{j}^{\prime}}\right]_{1}^{l}
$$

$$
S(w)=H \lg w-y \log (w-v)+x \log (1-w)
$$

> Eituer bounded
> main terme
\{ \}


Goal: Expand the $\oint$ around
 to remove exp. growth.
-
1.5 Double critical point
\{ \}

$$
\begin{aligned}
& S(w)=H \lg w-y \log (w-v)+x \lg (1-w) \\
& S^{\prime}(w)=\frac{H L}{w}-\frac{x}{1-w}-\frac{y}{w-v} \\
& S^{\prime \prime}(w)=-\frac{J L}{w^{2}}-\frac{x}{(1-w)^{2}}+\frac{y}{(w-v)^{2}} \\
& S^{\prime}(w)=S^{\prime \prime}(w)=0 \Rightarrow \\
& w S^{\prime \prime}+S^{\prime}=0, \frac{v y}{(v-w)^{2}}=\frac{x}{(1-w)^{2}}
\end{aligned}
$$

\{ \}

$$
\begin{aligned}
& \text { So }\left\{\begin{array}{l}
w_{c}=\frac{v(x-y) \pm(1-v) \sqrt{v x y}}{x-v y} \\
\text { And } \\
\nVdash\left(\omega_{c}\right)
\end{array}\right.
\end{aligned}
$$

Note, it must be $H \geqslant 0$.
\& also $H(x, y)$ is indep of (t):

Simplifying for two $\omega_{c}$ :

$$
f(x, y)=\frac{(\sqrt{y} \pm \sqrt{\sqrt{x}})^{2}}{1-v}
$$

Which to prick?
\{ \}

\{ \}
$v:=1 / 2 ; \operatorname{Plot} 3 \mathrm{D}\left[\left\{-\frac{(\sqrt{v} \sqrt{x}+\sqrt{y})^{2}}{-1+v},-\frac{(-\sqrt{v} \sqrt{x}+\sqrt{y})^{2}}{-1+v}\right\},\{x, 0,2\},\{y, 0,2\}\right]$

we need the lower one

$$
F(x, y)=\frac{(\sqrt{y}-\sqrt{\sqrt{x}})^{2}}{1-v}
$$

$$
\mathcal{H}(x, y)=0
$$


1.6 Heuristics of the cone in the limit shape
\{ \}

Heuristics of the cone on S6V

first live is a Riv.
Indef. first live $\Rightarrow$
\{ \}


So the limit shape looks like this and it tangent to two planes: \{ \}

\{ \}

$$
\begin{aligned}
w_{c}(x, y) & =\frac{v(x-y)-(1-v) \sqrt{v x y}}{x-v y} \\
w_{c}(x, y) & =\infty \quad \text { at } \quad y=\frac{x}{v} \\
w_{c}(x, y) & =0 \quad \text { at } \quad y=v x \\
& >0 \quad \text { outside the cone } \\
& <0 \quad \text { inside }
\end{aligned}
$$

1.7 Moving the contours
\{ \}

$$
\begin{aligned}
& S(w)=H \log w-y \log (w-v)+x \operatorname{leg}(1-w) \\
& H(x, y)=\frac{(\sqrt{y}-\sqrt{\sqrt{x}})^{2}}{1-v} \leftarrow \text { car pheg in wow }
\end{aligned}
$$

$$
\begin{aligned}
& v<\frac{y}{x}<\frac{1}{v} \quad(\text { interesting zone) } \\
& \\
& -\infty<\omega_{c}(x, y)<0
\end{aligned}
$$

$$
\begin{aligned}
& \text { Plot of the zone } \\
& \text { Re } \delta(z)>\text { Re } S\left(\omega_{c}\right)
\end{aligned}
$$

\{ \}


$$
\begin{gathered}
(v=1 / 3, \quad x=1, \quad y=3 / 2) \\
w_{c}=-1.27
\end{gathered}
$$

We now need to move the contours so that the exponent in the integrals has negative real part. This is possible:
\{ \}

\{ \}

If in the $l$-th term of $\operatorname{det}\left(1+k_{5}^{2}\right)$ there is a var. $w_{i}$, Ei outside of abd of $w_{c}$
then this term
$\longrightarrow 0$

Note that in the "wrong zone" when $w_{c}>0$ we cannot move the contours as desired, due to poles:
\{ \}


### 1.8 Expansion around the critical point and asymptotics

 \{ \}$$
\begin{aligned}
& z=\omega_{c}+L^{-1 / 3} \tilde{z} / \sigma \\
& \omega=\omega_{c}+L^{-1 / 3} \quad \tilde{\omega} / 6 \\
& \sigma=-\sqrt[3]{\frac{S^{\prime \prime \prime}\left(\omega_{c}\right)}{2}} \quad S^{\prime \prime \prime}\left(\omega_{c}\right)<0 \\
& \text { (in general } \\
& \text { can be either way) } \\
& \text { \{ \} } \\
& \sum_{l=0} \frac{1}{l!} \frac{1}{(d \pi i)^{2 l}} \oint_{\gamma^{\prime}} \oint_{\gamma} d z \oint_{\gamma}-\oint_{d \overrightarrow{0}} d e t\left[\frac{1}{w_{i}-z_{j}}\right]_{1}^{l} \\
& \times \prod_{i=1}^{\ell}\left[-\frac{\pi}{\sin \left[\frac{\pi \log z_{i}-\pi \log \omega_{i}}{\log t}\right]}\right] \cdot \frac{e^{L\left(S\left(w_{i}\right)-S\left(z_{i}\right)\right)+r L^{1 / 3\left(\log z_{i}-\log \omega_{i}\right)}}}{z_{i} \log t}
\end{aligned}
$$

Let us now collect the asymptotic contributions of all the terms in the integrals: \{ \}

$$
\begin{gathered}
\frac{d w_{i} d z_{j}}{w_{i}-z_{j}}=\frac{d \tilde{w}_{i} \tilde{z}_{j}}{\tilde{w}_{i}-\tilde{z}_{j}} \cdot L^{-1 / 3} / 2 \\
p L^{1 / 3}\left[\log \left(z_{i}\right)-\lg \left(\omega_{i}\right)\right]=\frac{r}{\delta w_{c}}(\tilde{z}-\tilde{\omega})+O\left(L^{-1 / 3}\right) \\
L\left(\delta\left(w_{i}\right)-S\left(z_{i}\right)\right)= \\
=\frac{\tilde{w}^{3}-\tilde{z}^{3}}{6 z^{3}} \cdot S^{\prime \prime \prime}+O\left(L^{1 / 3}\right)
\end{gathered}
$$

\{\}

$$
=\frac{\tilde{z}^{3}-\tilde{w}^{3}}{3}+O\left(L^{-1 / 3}\right)
$$

Denom:

$$
z_{i}=W_{c}+O\left(L^{-1 / 3}\right)
$$

\{\}

$$
\begin{aligned}
\frac{-\pi}{\sin \left[\frac{\pi \log z_{i}-\pi \log \omega_{1}}{\log t}\right]} & \simeq \frac{-\pi}{\sin \left[\frac{\pi}{\log t}\left(L^{-1 / 3} / 2 \omega_{c}(\tilde{z}-\tilde{\omega})\right)\right]} \\
& \simeq \frac{-1 \cdot \log t \cdot}{(\tilde{z}-\hat{w})} \cdot 6 \omega_{c} L^{+1 / 3}
\end{aligned}
$$

We see that the scaling limit of the Fredholm determinant looks like:

$$
\begin{aligned}
& \text { \{ \} } \\
& \mathbb{E} \frac{1}{\left.\left.\left(-t^{14(x) y}\right)-\operatorname{craky}(x)+v_{0}^{2 k}\right)^{3}\right)_{\infty}}-\operatorname{det}\left(1+k_{2}^{3}\right)
\end{aligned}
$$

This is our final result for this lecture, and next time we will identify this formula
with the Fredholm determinant of the Airy kernel.

## Notes and references

There are several papers on the method of $t$ - (or $q$-) moments in the analysis of interacting particle systems. Here are the main references:

1. Alexei Borodin, Ivan Corwin. Macdonald processes. https://arxiv.org/abs/1111.4408; Proposition 3.2.1 onwards
2. Alexei Borodin, Ivan Corwin, Tomohiro Sasamoto. From duality to determinants for q-TASEP and ASEP. https://arxiv.org/abs/1207.5035. Section 3
3. Alexei Borodin, Ivan Corwin, Leonid Petrov, Tomohiro Sasamoto. Spectral theory for the q-Boson particle system. https://arxiv.org/abs/1308.3475. Appendix 7.2 has the proof of the contour shift argument.
4. Alexey Bufetov, Matteo Mucciconi, Leonid Petrov. Yang-Baxter random fields and stochastic vertex models. https://arxiv.org/abs/1905.06815. Section 9.1 discusses operator approach to getting $t$-moments of the stochastic six vertex model, and the corresponding Fredholm determinants.

## Problems

## 1

Show (heuristically) that the slope of the other side of the cone in the limit shape of the stochastic six vertex model is equal to $1 / v$ (see [ $[1.6$ Heuristics of the cone in the limit shape|this part]] about the lower boundary of slope $v$ ).
[[Problems, 4-19|1 problem]], due May 6
1 Summary of the results

1 Recall main points
Here is roughly the path we had towards the asymptotic results: \{ \}
$\begin{aligned} h(x, y) \leftrightarrow & H L \text { rueasure } \\ & (\text { (randomize } Y B E)\end{aligned}$
(2) $P_{\lambda}=\sum_{\delta} \quad \begin{array}{r}(\text { algebraic Bethe } \\ \text { ansontz })\end{array}$
\{ \}
(3) $D(0, t)$ eigenoperator, action $\oint$

(4) $t^{k h(x, y)+\ln (x, y)}$
(5) $\mathbb{E} \frac{1}{\left(\zeta t^{h}, t\right)_{\infty}}=\sum_{k=0}^{\infty} \frac{3^{k}}{(t, t)_{k}} E t^{k h}$
$\rightarrow$ via contour smift theorem
\{ \}
(6) $\mathbb{E} \frac{1}{\left(\zeta t^{h}, t\right)_{\infty}}=\operatorname{det}\left(1+K_{3}^{2}\right)$
$K_{S}^{2}$ - kerinel on $\gamma$


$$
\begin{aligned}
K_{s}^{2}\left(w, w^{\prime}\right) & =\frac{1}{2 n^{i}} \int_{C} \Gamma(-s) \Gamma(1+s)(-5)^{s} \frac{g(w)}{g\left(t^{s} w\right)} \frac{d s}{w^{\prime}-t_{w}^{s}} \\
& \prod_{0} \prod_{1}^{c} g(z)=\prod_{i=1}^{x}\left(1-z u_{i}^{i}\right) \prod_{i-1}^{y} \frac{1}{z-v_{i}}
\end{aligned}
$$

Then we performed the asymptotic analysis:
\{ \}

$$
\begin{array}{r}
u_{i} \equiv 1, \quad v_{j} \equiv v \quad, \quad \theta<v<1 \\
0<t<1
\end{array}
$$

$$
g(z)=(1-z)^{x}(z-v)^{-y}
$$

$$
z=t^{s} w, \quad \zeta=-t^{-L H+r L^{1 / 3}}
$$

\{ \}

$$
\begin{aligned}
& \operatorname{det}\left(1+K_{5}^{2}\right)= \\
& =\sum_{l \geqslant 0} \frac{1}{l!} \frac{1}{(2 \pi i)^{2 l}} \oint_{\gamma^{\prime}}-\oint_{\gamma} d \vec{z} \oint_{\gamma}-\oint_{d} d \overrightarrow{0} d e t\left[\frac{1}{w_{i}-z_{j}^{\prime}}\right]_{1}^{l}
\end{aligned}
$$

> Eituer bounted
> main term
or $O\left(L^{1 / 3}\right)$

$$
S(w)=H \log w-y \log (w-v)+x \log (1-w)
$$

\{\}

$$
s^{\prime}=s^{\prime \prime}=0
$$

eau in $\omega, H$ both
\{ \}
Double critical point we of $S(\cdot)$

$$
\begin{array}{r}
w_{c}=\frac{v(x-y)-(1-v) \sqrt{v x y}}{x-v y}<0 \\
\neq J\left((x, y)=\frac{w_{c}\left(y-v x+w_{c}(x-y)\right)}{\left(1-w_{c}\right)\left(v-w_{c}\right)}\right. \\
\left(v<\frac{y}{x}<\frac{1}{v}\right) \quad=\frac{(\sqrt{y}-\sqrt{v x})^{2}}{1-v}
\end{array}
$$

\{ \}

Let $\sigma^{\prime}=-\sqrt[3]{\frac{s^{\prime \prime}\left(w_{c}\right)}{2}}>0, w_{c}<0$
We showed (by moring coutours \& Scating)

$$
\mathbb{E} \frac{1}{\left(-t^{h(x, y)-L \mathcal{H}(x, y)+\Gamma L^{1 / 3}}, t_{\infty}\right.}=\operatorname{det}\left(1+K_{2}^{3}\right)
$$

(\}

$$
\begin{aligned}
& =\left(1+0\left(L^{-1 / 3}\right)\right) \cdot \sum_{l=0}^{\infty} \frac{1}{l!} \frac{1}{(2-i)^{2}} \iint \ldots \int \cdot \int \\
& \quad d \vec{z} d \vec{w} \cdot \operatorname{det}\left[\frac{1}{w_{t}-t j}\right]_{1}^{l}
\end{aligned}
$$

(x)

$$
\cdot \prod_{i=1}^{l} e^{z_{i}^{3} / 3-w_{i}^{3} / 3+r / 3 w_{c}\left(z_{i}-w_{i}\right)} \frac{1}{z_{i}-\omega_{i}}
$$

## 2 Identification with the Tracy-Widom distribudion

- 


### 2.1 Limiting result

We have shown that the asymptitcs

$$
\mathbb{E} \frac{1}{\left(-t^{h(L x, L y)+L \mathcal{H}(x, y)+r L^{1 / 3}} ; t\right)_{\infty}}
$$

gives $\left(1+O\left(L^{-1 / 3}\right)\right)$ times the Fredholm determinant \{ \}

$$
(*)=\operatorname{det}\left(1+{\underset{S}{r / 2 w_{c}}}^{k^{\prime}}\right)
$$

$$
W T S=F_{2}\left(-\sqrt{\omega_{c}} \sigma\right)
$$

$$
\bigcap_{p}\left(u, u^{\prime}\right)
$$

$$
\binom{r \varepsilon<a l l}{\sigma w_{c}<0}
$$

$$
=-\frac{1}{2 n i} \int e^{\omega^{3} / 3-u^{3} / 3+r(\omega-u)}
$$

$$
\frac{d w}{(w-u)\left(w-u^{\prime}\right)}
$$




(An y-like
Contours $)$

Indeed, to see that this is the desired Fredholm determinant, we take the integrations and put them inside the determinants, using the so-called Andreief's identity (there is in fact $\ell$ ! in the right-hand side):
\{ \}

$$
\begin{gathered}
\left.\operatorname{Becoance}_{\int_{l}-\int d_{1}-\cdots d x e} \operatorname{det}\left[f_{i}\left(x_{j}\right)\right] \operatorname{det}\left[g_{i}(x)\right)\right] \\
=\operatorname{det}\left[\int f_{i}(x) g_{j}(x) d x\right]_{1}^{l} \\
\text { (Andreief's identity) } \\
\text { (Exercise) }
\end{gathered}
$$

See also [[Problems, 4-19\#1|Problem 1]].

### 2.2 Airy kernel recall

Recall the Airy ${ }_{2}$ kernel
\{\}

\{ \}

$$
\begin{array}{r}
A_{i}(x)=\frac{1}{2 \pi} \int_{-\infty}^{+\infty} e^{i t^{3} / 3+i t x} d t \\
\xrightarrow{?} \quad{ }^{n}(\text { reg. }) \int_{-\infty}^{+\infty} \cos \left(\frac{t^{3}}{3}+t x\right) d t
\end{array}
$$

Remark. In the "Gaussian" case, the normal distribution pdf can also be written as a Fredholm determinant:
\{ \}

2.3 Identification

We can identify two Fredholm determinants:
Theorem 2.3.1 \{ \}

$$
\text { Theorem. } \operatorname{det}(1-A)_{(-r, \infty)}=\operatorname{det}\left(1+\widetilde{K}_{r}\right)_{\neq \mathbb{R}}
$$

Proof 2.3.1 \{ \}

Proof. An stance of

$$
\operatorname{det}(1-A B)=\operatorname{det}(1-B A)
$$

(if both $A B, B A$ are trace class)

$$
\begin{aligned}
& \tilde{k}_{r}\left(u, u^{\prime}\right)=-\frac{1}{2 n_{i}} \int_{0}^{w_{0} \not \omega^{3 / 3-u^{3} / 3+\sim(w-u)}} \frac{d w}{(w-u)\left(w-u^{\prime}\right)} \\
& \frac{1}{w-n}=\int_{0}^{\infty} d \lambda \cdot e^{-\lambda(w-u)} \quad \operatorname{Re}(w-u)>0
\end{aligned}
$$

\{\}

$$
\begin{aligned}
& \tilde{K}_{r}=-A B \quad ; \quad A: L^{2}(\zeta) \rightarrow L^{2}(0, \infty) \\
& \text { B: } L^{2}(0, \infty) \rightarrow L^{2}(\Sigma) \\
& A(u, \lambda)=e^{-u^{3} / 3+u(\lambda-r)} \\
& B\left(\lambda, u^{\prime}\right)=\frac{1}{2 \pi u^{i}} \int \frac{d z}{z-u^{\prime}} e^{z^{3} / 3-z(\lambda-r)} \\
& \text { o'\& } \\
& B A\left(\lambda, \lambda^{\prime}\right)=\int_{\rangle_{u}} B(\lambda, u) A\left(u, \lambda^{\prime}\right)
\end{aligned}
$$

\{ \}

Compute BA:
$(B A)\left(\lambda_{3} \lambda^{\prime}\right)=\frac{1}{(a r r)^{2}} \int_{\lll} d z \int d u \frac{e^{z^{3} / 3-z(\lambda+r)-\frac{u^{3}}{3}}}{z-u}$.

$$
e^{-u^{3} / 3+u(\lambda-r)}
$$

$$
=A\left(\lambda-r, \lambda^{\prime}-r\right) \quad \text { Qu } \quad \lambda, \lambda^{\prime} \epsilon L^{2}(0, \infty)
$$

In the end we get the following result:
\{ \}


$\Downarrow \angle \rightarrow \infty$

$$
\operatorname{det}(1-A)\left(-\frac{\Gamma}{\omega_{c} \sigma},+\infty\right)
$$

$$
=F_{2}\left(-r / w_{e} \sigma\right)
$$

$$
\begin{aligned}
& \text { Tracy-widonn } \\
& \text { distribution } \\
& \text { coif }
\end{aligned}
$$

Summarizing, we have proved the following theorem:
\{ \}


## 3 Large deviations

- 


### 3.1 Simple random walk

Let us now talk briefly about large deviations in the stochastic six vertex model. We start from the simple random walk, as a warm up.

For it, we have the following law of large numbers and central limit theorem:

$$
\begin{aligned}
& S_{n}=x_{1}+\cdots+x_{n}, \quad x_{i}=0,1 \text { wp. } \frac{1}{2} \\
& \frac{S_{n}}{n} \rightarrow \frac{n}{2} \\
& \mathbb{P}\left(\frac{S_{n}-n / 2}{\frac{1}{2} \sqrt{n}}<x\right) \rightarrow \Phi(x)=\int_{-\infty}^{x} \frac{1}{\sqrt{2 \pi}} e^{-z^{2} / 2} d z
\end{aligned}
$$

Large deviations are statements of the following form:
\{ \}

$$
\begin{aligned}
\mathbb{P}\left(S_{n}-n / 2\right. & >\alpha n) \asymp e^{-n I_{+}(\alpha)} \\
\mathbb{P}\left(S_{n}-n / 2<-\alpha n\right) & \frown e^{-n I_{-}(\alpha)} \\
\alpha>0 \quad & \begin{array}{r}
\text { polynomial prefactors } \\
\text { ignored }
\end{array}
\end{aligned}
$$

More rigorously,
\{ \}

$$
\lim _{n} \frac{1}{n} \log \mathbb{P}=-I_{+}(\alpha)
$$

Sometimes, for more complicated models, even lime inf or him sup type estimates are good enough. For the random walk we can compute everything explicitly: \{ \}

$$
\begin{aligned}
& I_{ \pm}(\alpha) \text { are increasing functions } \\
& \mathbb{R}\left(S_{n}>\frac{n}{2}+\alpha n\right) \asymp \mathbb{P}\left(S_{n}=n\left(\alpha+\frac{1}{2}\right)\right) \\
& =\frac{1}{2^{n}}\binom{n}{n \alpha+n / 2} \quad\left(\Rightarrow I_{+}=I_{-}\right) \\
& =(\operatorname{Stinling}) \quad N!\sim e^{N \log N-N} \\
& =e^{-n\left(\left(\alpha+\frac{1}{2}\right) \log \left(\alpha+\frac{1}{2}\right)+\left(\alpha-\frac{1}{2}\right) \log \left(\alpha-\frac{1}{2}\right)\right)} \\
& \xrightarrow{I_{-} \overbrace{\alpha}} I_{+}
\end{aligned}
$$

### 3.2 Stochastic six vertex model

For interacing particle systems, the large deviations are different on two sides of the law of large numbers.
Consider TASEP:
\{ \}


We have $c=t / 4$.
\{ \}


Then large deviations take the form:
\{ \}

Let us now look at large deviations for the stochastic six vertex model, using our formulas.

In the "too slow" regime, the convergence should be as
\{ \}

$$
\begin{aligned}
& \text { too slow- covid } \\
& \mathbb{P}(h(L x, L y)>L \cdot H(x, y)-\alpha L) \quad \rightarrow 1 \\
& \zeta \\
& \mathbb{E} \frac{1}{\left(t^{h(x+y)-L z t+\alpha L} ; t\right)_{\infty}} \\
& =\operatorname{det}\left(1+K_{5}^{2}\right) \\
& =1+\frac{\int K_{3}^{2}(x, x) d x+\frac{1}{2} \iint(1+7}{e^{-L}}+\ldots
\end{aligned}
$$

In the Fredholm determinant, we have 1 as the first term, and the rest would be a correction of exponentially small size.

Remark. In the "too fast" regime, the Fredholm determinant should converge to zero, which means that all integrals of all orders would nontrivially contribute. This is much harder to analyze.
\{ \}

$$
\begin{aligned}
& \text { In } \exp \text { in Fredholm: } \\
& e^{L(S(\omega)-S(z))}
\end{aligned}
$$

$$
S(\omega)=(f l-\alpha) \lg \omega-y \log (\omega-v)+x \log (1-\omega)
$$


\{ \}


$$
\text { Example, } x=y=1, v=1 / 4
$$

$$
w_{c}=-\frac{1}{2}
$$

$$
\begin{aligned}
& \lambda_{c}(\alpha)=\frac{4+15 \alpha+3 \sqrt{3} \sqrt{\alpha(8+3 \alpha)}}{24 \alpha-8} \\
& 2 \text { single crit pats. } \alpha>0 \text { : Real }
\end{aligned}
$$

When $\alpha<0$ ("too fast" regime), the two roots are complex conjugate. For the "too slow" regime,
\{ \}


However, in the "too fast" case,
\{ \}

and it is not clear what would be the end contribution.

## Notes and references

There are several papers on the method of $t$ - (or $q$-) moments in the analysis of interacting particle systems. Here are the main references:

1. Alexei Borodin, Ivan Corwin. Macdonald processes. https://arxiv.org/abs/1111.4408; Proposition 3.2.1 onwards
2. Alexei Borodin, Ivan Corwin, Tomohiro Sasamoto. From duality to determinants for q-TASEP and ASEP. https://arxiv.org/abs/1207.5035. Section 3
3. Alexei Borodin, Ivan Corwin, Leonid Petrov, Tomohiro Sasamoto. Spectral theory for the q-Boson particle system. https://arxiv.org/abs/1308.3475. Appendix 7.2 has the proof of the contour shift argument.
4. Alexey Bufetov, Matteo Mucciconi, Leonid Petrov. Yang-Baxter random fields and stochastic vertex models. https://arxiv.org/abs/1905.06815. Sectimon 9.1 discusses operator approach to getting $t$-moments of the stochastic six vertex model, and the corresponding Fredholm determinants.

Large deviations of particle systems have a long history, starting from TASEP in the 1990s: 1. Seppalainen, "Coupling the Totally Asymmetric Simple Exclusion Process with a Moving Interface" http://mathmprf.org/journal/articles/id830/. 2. Johansson, "Shape Fluctuations and Random Matrices" https://arxiv.org/abs/math/9903134. 3. For an approach explained in the lecture, but for ASEP, see also https://arxiv.org/abs/1708.05806, section 3.

## Problems

## 1

Show the Andreief's identity:

$$
\int_{A} d x_{1} \ldots \int_{A} d x_{N} \operatorname{det}\left[f_{i}\left(x_{j}\right)\right]_{N \times N} \operatorname{det}\left[g_{i}\left(x_{j}\right)\right]_{N \times N}=N!\operatorname{det}\left[\int_{A} f_{i}(x) g_{j}(x) d x\right]_{N \times N}
$$

In this lecture we give an overview of coloured vertex models. There is a motivation coming from quantum groups / R matrices, and after briefly discussing it I introduced a number of concrete examples.
\{ \}


## 1 Some quantum group ramblings

- 


## 1 On quantum groups

\{ \}

R matrices from representation th. why the combinatorics of coloured model works, an overview wo many details
\{\}

Quantum group

- Hope algebra $H \quad B \quad: \begin{gathered}a(B)=A \otimes B+ \\ +B \otimes D\end{gathered}$ (coalgebra w. wore properties)
comertip! $\Delta: H \rightarrow H \otimes H$
(multiplication $H \otimes H \rightarrow M$ )
following diagram commutes:


Here $\Delta$ is the comultiplication of the bialgebra, $\nabla$ its multiplication, $\eta$ its unit and $\varepsilon$ its counit. In the sumless Sweedier notation, this property can also be expressed as
\{\}

$$
S_{2}
$$



$$
\begin{gathered}
{[a, b]=a b-b a} \\
{[e, f]=h \quad[h, f]=-2 f} \\
{[h, e]=2 e}
\end{gathered}
$$

\{ \}

$$
\begin{aligned}
& U_{q}\left(s l_{2}\right) \text { basis of } 3 \text { el's } \\
& {\left[\underset{i}{e}, f_{i}\right]=\frac{k_{i}-k_{i}^{-1}}{q-q^{-1}}} \\
& u_{q}\left(\hat{s e}_{n}\right)_{e} e_{i}, f_{j}, k_{i}^{k_{i}} \\
& {[e, f]=e f-f e \quad(q=1, \quad[e, f]=h \quad[h, f]=-2 f} \\
& [h, e]=2 e)
\end{aligned}
$$

\{ \}
quentun: $\left[e, f_{i}\right]=\frac{k_{i}-k_{i}^{-1}}{q-q^{-1}}$
etc.


$$
\Delta(k)=k \otimes k
$$

$$
A(E)=k \otimes E+E \otimes 1
$$

$$
k \leftrightarrow q^{H}
$$


\{ \}

isoworghism
\{ \}

Yary - Baxter equation

- uhen we have 3 spaces

\{\}

$$
\begin{aligned}
\left(R_{v \omega}^{\otimes \mid}\right)\left(1 \otimes R_{u w}\right) & \left(R_{u v} \mid\right)> \\
& =\left(1 \otimes R_{u v}\right)\left(R_{u w} \otimes \mid\right)\left(\mid \otimes R_{w w}\right)
\end{aligned}
$$

\{\}

$$
\begin{aligned}
& \text { So, } R_{12} R_{13} R_{23}=R_{23} R_{13} R_{12} \\
& \text { Universal } R \text { matrix: } \\
& \text { Ron } H \otimes \nVdash \text { set } \\
& \mathcal{H} \otimes l \xrightarrow{n \otimes \rho} \operatorname{bud}(v \otimes W) \\
& \downarrow 6 \quad \downarrow R_{v w} \\
& \mathrm{H} \otimes \mathrm{H} \xrightarrow{\varrho \otimes \pi} \text { End }(\omega \otimes V) \\
& R_{n w}=\operatorname{rog}(R)
\end{aligned}
$$

The 6 V model comes from the quantum $s l_{2}$ universal R matrix, specialized at two two-dimensional representations. Possible generalizations come from changing the representations, or changing the underlying quantum group.

2 Higher spin models
-
2.1 Six vertex model from sld
\{ \}


$$
b_{1}=\frac{1-u}{1-t u}
$$

$t$ fixed
u spectral parameter
\{ \}
\& eperator in $V \otimes V$

$$
\left.\begin{array}{c}
V=\mathbb{C}^{2}=\text { span }\left\{e_{0}, e_{1}\right\} \\
R_{u}=\left[\begin{array}{cccc}
0 & 01 & 10 & 11 \\
1 & 0 & 0 & 0 \\
0 & b_{1} & 1-b_{1} & 0 \\
0 & 1-t b_{1} & t b_{1} & 0 \\
0 & 0 & 0 & 1
\end{array}\right]_{11}^{\infty} \\
00 \\
k_{u}\left(b_{1}=\frac{1-u}{1-u t}\right. \\
\left.i_{1}, j_{2} ; i_{2}, j_{2}^{2}\right)
\end{array} 0_{i}\right]_{i_{1}}^{i_{2}} .
$$

\{\}

2.2 Higher spin sl2 examples
\{ \}

Migher spin


Say $W=\mathbb{C}^{2}$
$V=$ "Verma modile" $\sim \begin{gathered}\text { spinn } \\ \text { param.. } s\end{gathered}$ $=\operatorname{span}\left(e_{0}, e_{1}, e_{2}, \ldots\right)$
\{ \}


$$
s u=-\tilde{u}, \quad s \rightarrow 0 \quad-\quad H L(\text { stock })
$$



One can also take both representations to be arbitrary higher spin, this leads to the following model:
\{ \}

Or $R$ in $V \otimes W$, both generic

$$
\begin{align*}
\mathrm{L}_{u, \mathbf{s}}^{(J)}\left(i_{1}, j_{1} ; i_{2}, j_{2}\right)= & \mathbf{1}_{i_{1}+j_{1}=i_{2}+j_{2}} \frac{(-1)^{i_{1}} q^{\frac{1}{2} i_{1}\left(i_{1}+2 j_{1}-1\right)} u^{i_{1}} \mathrm{~s}^{j_{1}+j_{2}-i_{2}}\left(u \mathrm{~s}^{-1} ; q\right)_{j_{2}-i_{1}}}{(q ; q)_{i_{2}}(\mathrm{~s} u ; q)_{i_{2}+j_{2}}\left(q^{J+1-j_{1}} ; q\right)_{j_{1}-j_{2}}} \\
& \left.\times \bar{\phi}_{3}\binom{q^{-i_{2}} ; q^{-i_{1}}, \mathrm{~s} u q^{J}, q \mathrm{~s} / u}{\mathrm{~s}^{2}, q^{1+j_{2}-i_{1}}, q^{J+1-i_{2}-j_{2}}} q, q\right) \tag{5.6}
\end{align*}
$$

3 Higher rank / coloured model

3 Coloured models
\{ \}


$$
\frac{1}{2}
$$



$$
u_{q}\left(\widehat{s}_{n+1}\right)
$$

Basic rep is

$$
V=\operatorname{spcan}\left(e^{0}, e^{1}, \ldots, e^{n}\right)
$$

$$
0 \leftrightarrow \text { empty }
$$

$i \longleftrightarrow \operatorname{colos} i^{i}$

\{ \}

\{ \}

Def.


$t_{b_{1}}=\frac{t(1-u)}{1-t u}$

$$
1-t b_{1}=\frac{1-t}{1-t u}
$$



$$
b_{1}=\frac{1-u}{1-t u}
$$

$$
1-b_{1}=\frac{(1-t) u}{1-t_{u}}
$$

Starkastic
coloured 6V model (Boradin-wheeler 2018)
\{ \}


Rainbow initial/boundary conclition
\{ \}


## Notes and references

1. Coloured stochastic vertex models are discussed in many recent works, starting from Borodin-Wheeler, https://arxiv.org/abs/1808.01866
2. Connections of $R$ matrices to quantum groups is a rich subject which we only briefly mentioned. For example, an accessible introduction (by Ivan Loseu) may be found here: https://gauss.math.yale.edu/~il282/RT/RT13.pdf, available also at this link: [[../img/RT13.pdf]]
3. Higher spin stochastic vertex models are discussed, for example, in papers https://arxiv.org/abs/1601.05770, https://arxiv.org/abs/1905.06815

## Problems

No problems at this lecture
[[Problems, $4-26 \mid 2$ problems]], due May 6
This lecture discusses colour-position symmetry, and a Hecke algebra approach to vertex models

## 1 Coloured stochastic six vertex model

- 

1.1 Recall the definition of the model
\{ \}

Def.



$$
t_{b_{1}}=\frac{t(1-u)}{1-t u}
$$

$$
1-t b_{1}=\frac{1-t}{1-t u}
$$




$$
b_{1}=\frac{1-u}{1-t u}
$$

$$
1-b_{1}=\frac{(1-t)_{u}}{1-t_{u}}
$$

Stachastic
coloured 6V model
(Borodin-Wheeler 2018)
\{\}


Rainbow initial / boundary conelition

### 1.2 Degeneration to ASEP and TASEP

\{ \}

\{ \}


Coloured TASEP
\{ \}

Coloured TASFP $\Leftrightarrow$ mery usnel TASEPS,
coler I color 4 col. ${ }^{\circ}$ but wupled


$$
\eta^{(1)} \leq \eta^{(2)} \leq \eta^{(3)} \leq-\quad \eta^{(j)}=\text { coufig. }
$$

empty col. $\Leftrightarrow$ wo patticle at leved
\{\}


Note: one can define a discrete time ASEP, which is more general:
\{ \}


## 2 Colour-position symmetry

- 


### 2.1 Formulation of symmetry

\{ \}

$$
\begin{aligned}
& \rightarrow \text { Angel, Amir, Valko } \\
& \text { TASEP (2008) } \\
& \rightarrow \text { Borodin - wheeler (2018) } \\
& P_{\lambda}\left(x_{1}, x_{N}\right)=\text { via wonsymum. functions }
\end{aligned}
$$

$$
\begin{aligned}
& \vec{k}=\lambda \\
& \text { via keck algebras } \\
& \rightarrow \text { Galashin } 2020 \\
& \text { Borodin - orin - ureter } \\
& \text { - extensions, applications }
\end{aligned}
$$

Define two probabilities, the uncoloured one $P_{6 V}$ and the coloured one $P_{c o l}$
Definition 2.1.1 \{ \}

$$
\begin{aligned}
& \text { (no colors) }
\end{aligned}
$$

Definition 2.1.2 \{ \}

$$
P_{\text {cot }}(I, J)=
$$



Theorem 2.1.3 \{ \}

$$
\begin{array}{r}
\text { Theorem } \mathbb{M}_{G V}(I, J)=\underbrace{}_{\text {col }}(I, J) \\
(\text { prove sext time via } \\
\text { Heckle algebras) }
\end{array}
$$

Let us discuss some corollaries:

Corollary 2.1.4 \{ \}

$$
\begin{aligned}
& \text { for a single-paint observable } \\
& P_{G V}(h(M, N) \geqslant l)=\sum_{|J|=l} \mathbb{P}_{G V}(\phi, J) \quad \sum_{i l}^{(M, N)} \\
& \text { colored height. } \\
& =\sum_{|J|=t} \mathbb{P}_{c o l}(\phi, J)=\mathbb{P}_{a d}\left(h_{\geqslant 1}(\mu, N) \geqslant l\right)
\end{aligned}
$$

(this is just a consistency check)
Corollary 2.1.5 \{ \}

Cor (non-obvious)

$$
\mathbb{P}_{\text {col }}(\phi,\{k\})=\mathbb{P}_{G v}(\phi,\{k\})
$$


2.2 Application to TASEP second class particle

Theorem 2.2.1 \{ \}

$$
\begin{aligned}
& \forall x \in \mathbb{Z}, \\
& \text { Theorem } \mathbb{P}_{\substack{\text { calsep }}}(\text { color } 1 \text { to the right of } x) \\
& =\mathbb{P}_{\text {TASSEl }}(\text { particle at location } x+1) \\
& \text { follows from } b_{1}, b_{2} \rightarrow 0, t=0 \\
& \text { limit of the previous theorem } \\
& \text { for } I=\phi, \quad J=\{1\} \text {. }
\end{aligned}
$$

TASEP with a single second-class particle $\}$


How one could compute the limit for the uncoloured TASEP?
Via Schur measures / processes:
\{ \}

Structure:

$$
X_{N}(t)+N \stackrel{d}{=} \lambda_{N}, \lambda \sim S_{c l e u r} \text { measure }
$$

Also,

$$
\left\{X_{1}(t)+1, \ldots, X_{N}(t)+N\right\} \longleftrightarrow \text { Selw proven }
$$



Schur Process is determinental
$\Rightarrow$ Prob. the nt there is a part. at $x+1$ car be wreyputed.

However, from the hydrodynamics / density arguments it follows that:
\{ \}

\{ \}

$$
\begin{aligned}
& \begin{aligned}
\Rightarrow \lim _{t \rightarrow \infty} \mathbb{P}\left(\frac{L(t)}{t}<x\right)=\frac{1+x}{2} \\
\text { vioform out }
\end{aligned} \\
& \downarrow \quad[-1,1] \\
& \text { seconal class part- } \\
& \text { is "uniform on }[-t, t] \text { " } \\
& \text { (w. Gaussian fluctuations, } \\
& \text { coming from bernoulli } \\
& \text { product measures). }
\end{aligned}
$$

3 Heck algebras
3.1 Definition and involution

We begin the discussion of the Hecke algebra from the Coxeter presentation of the symmetric group:
\{ \}

$$
\begin{aligned}
& \text { (Iwatori-) Hecre algebra (q) } \\
& W=S_{N}=\text { symm group on }\{1, \ldots, N\} \\
& \text { geverated loy } S_{i}=(i, i+1) \\
& i=1 \ldots N-1 \\
& \text { \{ \} } \\
& \left\{\begin{array}{l}
S_{i}^{2}=e \\
S_{i} S_{i+1} S_{i}=S_{i+1} S_{i} S_{i+1} \quad(*) \quad \begin{array}{r}
\text { more general: } \\
\text { Coxeter grous) }
\end{array} \\
S_{i} S_{j}=S_{j} S_{i} \quad|i-j| \geqslant 2
\end{array}\right. \\
& \text { Indeed, look at "braids": }
\end{aligned}
$$

Definition 3.1.1 \{ \}

Can we put $R$ matrix into this context? Def. Ht - basis of $\left\{T_{\omega}: \omega \in S_{N}\right\} \quad \omega \in S_{N}$

$$
\begin{aligned}
& \left\{\begin{array}{l}
T_{s_{i}}^{2}=(1-g) T_{s_{i}}+9 \\
T_{s_{i}} T_{s_{i+1}} T_{s_{i}}=T_{s_{i+1}} T_{s_{i}} T_{s_{i+1}} \\
T_{s_{i}} T_{s j}=T_{s j} T_{s_{i}} \quad|i-j| \geqslant 2
\end{array}\right. \\
& \left(q=1 \text {, back to } S_{N}\right)
\end{aligned}
$$

Equivalently, here is the multiplication table of the Hecke algebra:
\{\}

Hecve algebra $H(\omega)=\operatorname{span}\left\{T_{\omega}: \omega \in W\right\}$

$$
\begin{aligned}
& \text { s.t }
\end{aligned}
$$

$$
\begin{aligned}
& \text { ( } s=s_{i} \text { for some } i \text { ) } \\
& \rightarrow l(\omega) \text { - leugth of } \omega \in W=S_{N}
\end{aligned}
$$

consistat rules, define an algebra
\{\}

Indeed, if $\omega=s_{1} \ldots s_{r}$ a reduced expression,

$$
T_{w}=T_{s_{1}}-T_{s_{r}} \text {. Then } T_{w} \text { satisfy }(*)
$$

Case 1 -easy
Case 2, $\omega=s \omega^{\prime}, \quad l\left(\omega^{\prime}\right)+1=l(\omega)$

$$
T_{s w}=T_{s} T_{s} T_{w}=\left[(1-q) T_{s}+q\right] T_{w}=(1-q) T_{w}+q T_{s w}
$$

So It can be def.a.s generators \& relations

$$
\begin{aligned}
& T_{s_{i}} T_{s_{i+1}} T_{s_{j}}=T_{s_{i+1}} T_{S_{i}} T_{S_{i+1}} \\
& T_{s_{i}} T_{s_{j}}=T_{s_{j}} T_{s_{i}} \quad|j-j| \geqslant 2 \\
& T_{S}^{2}=(1-g) T_{S}+q \\
& \quad\left(q=1, \text { back to } S_{N}\right)
\end{aligned}
$$

$\Rightarrow 7$ is a $q$-deform of the symmetric algebra of $S_{N}$.

Here is the main result about the Hecke algebra we need:
Theorem 3.1.2 \{ \}

$$
\begin{array}{ll}
\text { Let } I: H(w) \rightarrow \mathcal{H}(w) \\
I\left(T_{w}\right)=T_{w^{-1}} \\
\text { Then } I\left(T^{(1)} T^{(2)} \ldots T^{(r)}\right)=I\left(T^{(r)}\right) \ldots I\left(T^{(1)}\right) \\
\& & I^{2}(T)=T
\end{array}
$$

The proof is left as an exercise: [[Problems, $4-26 \# 2 \mid$ see here]].
3.2 Probability measures and random walks on Hecke algebras
\{ \}

$$
\mathcal{H}_{\text {prob }}(w)=\left\{h \in \mathcal{H}(w), \begin{array}{c}
h=\sum_{w \in S_{N}} c_{w} T_{w} \\
\sum c_{w}=1 \\
c_{w} \geqslant 0
\end{array}\right\}
$$

Random step:

$$
h=\sum_{w \in W} c_{w} T_{w} \in H_{p .06}(w)
$$

$$
\begin{aligned}
h & =\sum_{w \in w} c_{w} T_{w} \\
& \downarrow h=\sum_{w} d_{w} T_{w}
\end{aligned}
$$

Linear operator

$$
\left\{C_{w}\right\} \mapsto\left\{d_{w}\right\}
$$



$$
h=e \cdot \xrightarrow{T_{1}} T_{(213)}^{T_{1}} \xrightarrow{T_{2}} T_{2} T_{1} \xrightarrow{(3 \mid 2)} \xrightarrow{(1-q) T_{2} T_{1}}+q_{1}
$$

$$
\begin{aligned}
& T_{1} \\
& T_{2} T_{1}=T_{2} T_{1} T_{2}=\delta_{(321)}
\end{aligned}
$$

## Notes and references

1. References on Matt Walker's presentation about the Mpemba effect:

- https://journals.aps.org/prx/abstract/10.1103/PhysRevX.9.021060
- https://meetings.aps.org/Meeting/MAR21/Session/F18.3

2. Coloured stochastic vertex models are discussed in many recent works, starting from Borodin-Wheeler, https://arxiv.org/abs/1808.01866
3. The connection to Hecke algegras is discussed in https://arxiv.org/abs/2003.02730

## Problems

## 1

For an elementary permutation $s=(i, i+1)$, compute the inverse of the Hecke element $T_{s}^{-1}$.

2
Prove the main property of the involution $I$ :
\{ \}


Hint: use induction on $\ell(w)$, the length of the permutation.
See [[3.1 Definition and involution\#Theorem $312 \mid$ Theorem 3.1.2]].
[[Lecture 24, 4-28/Problems, 4-28|2 problems]], due May 6
1 Hecke algebra and involution

1 Heck algebra and involution
Recall the main algebraic object and the main involution result \{ \}
\{ \}

$$
T_{w}=T_{s_{i_{1}}--} T_{s_{i}} \text { if } \quad w=s_{i_{1}}-s_{i_{r}} \in S_{N}
$$

is a reduced word
(shortest product of transp representing
$\omega$ )

Example 1.0.1. Reduced vs non-reduced $\}$


Theorem 1．0．2．Involution \｛ \}

$$
\begin{aligned}
& H=\operatorname{sp\omega }\left(T_{\omega}: \omega \in S_{N}\right) \\
& \text { I: 孔 } \rightarrow \text { 孔 } \\
& I\left(T_{\omega}\right)=T_{\omega^{-1}} \\
& \text { Then } \\
& \begin{array}{c}
I\left(T^{1} T^{2} \ldots T^{p}\right)=I\left(T^{r}\right) \ldots I\left(T^{1}\right) \\
\forall T^{1} \ldots T^{r} \in J-L
\end{array} \\
& I^{2}=i d
\end{aligned}
$$

## 2 Coloured ASEP via Hecke algebra

## 2．1 ASEP on Hecke algebra

Definition 2．1．1．Coloured ASEP on 1，．．，N $\}$

Asep on $\{1, \ldots, N\}$
(identity initial condition)
-(1) $2 .=(N) \rightarrow$ initial condition

$$
\begin{aligned}
& j>i \quad i j \xrightarrow{j} \quad j i \\
& (q \text { same as } t) \\
& j_{i} i \xrightarrow{\text { sate } q} i{ }^{\circ} \\
& \text { as a part. syst. }
\end{aligned}
$$

Realization of ASEP as a random walk on the Heck algebra $\}$
Let us pick each of $T_{\text {si }}$ at rate 1, ind pruelently (according to Poisson procures)
at time $t$, we pick elements

$$
\begin{aligned}
& d t \cdot T_{s_{i}}+(1-d t) T_{e} \quad \text { (one at a time) } \\
& \text { \& multiply the existing word by it }
\end{aligned}
$$

Condition on the event that the current configuration is $w$, so the current Hecke element is $T_{w}$. And multiply it by $T_{s_{i}} d t+(1-d t)$. There are two cases \{ \}
Tu. 2 cases.
(1) $l(s ; w)=l(w)+1$
(bi) $\operatorname{bin}_{i+1} 6_{i}<b_{i+1}$

$$
\left(d t \cdot T_{s_{i}}+(1-d t) T_{e}\right) T_{w}=T_{s_{i} u} d t+\ldots
$$

swap $b_{i}, b_{i+1}$ at rate 1
\{ \}

$$
\begin{aligned}
& \text { (2) } l\left(s_{i} w\right)=l(w)-1, \quad b_{i+1}>b_{i} \\
& \omega=s_{i} \omega^{\prime}, \omega^{\prime} \text {-rechuced } R(\text { Exercise) } \\
& \begin{array}{r}
\left(d t \cdot T_{s_{i}}+(1-d t) T_{e}\right) T_{s_{i} \omega^{\prime}}=\left\{\begin{array}{l}
\omega^{\prime}=s_{i} \omega \\
=
\end{array} T_{s_{i} \omega^{\prime}} \cdot(1-q) d t+T_{\omega^{\prime}} \cdot q d t\right.
\end{array} \\
& +(1-d t) T_{s i w^{\prime}} \\
& \frac{\uparrow}{\begin{array}{r}
\text { swap at } \\
\text { rate }
\end{array}}
\end{aligned}
$$

See [[Problems, 4-28\#1| Problem 1]].
Therefore, we have proven:
Proposition 2.1.2 \{ \}

So, $\frac{\text { ASEP (coloured) }}{\text { is modeled }}$ ley $\{1, \ldots, N\}$
Poissou-Like clocks attached to keck generators $T_{s i}$
$(\Rightarrow$ TASEP for $q=0$ )
2.2 Remark on discrete time ASEP
\{ \}
Discrete time ASEP
(pick pair of prosticles $\quad \stackrel{0}{0}$, it at random \& exchange dep.on their colors)
\{\}

Let $Y_{s, x}=x T_{s}+(1-x)$
Consider tue proves on Ht
which consists in picking $Y_{s, x}$ at random (came $x$, over all $s=s_{1}, \ldots s_{1}$.)

We arrive at the following swap rules:
\{ \}


Remark 2.2.1 The elements $Y_{s_{i}, x}$ may be also taken with different parameters $Y_{s_{i}, x_{i}}$. Then the parameters $x_{i}$ are attached to bonds in the ASEP. This particle system is not (yet?) integrable, in the sense that we don't know of reasonable formulas which allow for an asymptotic analysis. However, the color-position symmetry is still present even in this system.

Remark 2.2.2 In matching coloured systems with Heck algebras, sometimes we may identify color $j$ with element $j$, and sometimes identify color $j$ with element $N+1-j$. This is not an essential difference, and the importance here is that the swapping rules are different and depend on the order of the elements.

## 3 Stochastic six vertex model via Heck algebra

- 


### 3.1 R matrix as a Hecke element

In the stochastic six vertex model, R matrices (which are stochastic operators in $V \otimes V, \operatorname{dim} V=n+1$ for the $n$-colored model) behave "like" elements of the Hecke algebra.
Motivating discussion:

$V \otimes V$


$$
\begin{gathered}
R_{i} R_{j}=R_{j} R_{i} \\
|i-j| \geqslant 2 \\
R_{i} R_{i+1} R_{i}= \\
=R_{i+1} R_{i} R_{i+1} \\
\text { YBE }
\end{gathered}
$$

It is very natural to ask whether they satisfy some quadratic relation similar to the one satisfied by the $T_{s}$ 's. In fact, the $Y_{s, x}$ also satisfy a certain quadratic relation.

We will give an example for 2 colors, when $R$ is an $9 \times 9$ matrix.
Recall the weights in the stochastic six vertex model.
\{ \}


Let the $R$ matrix be the one with a swap in the target space $V \otimes V$ (which interchanges the tensor factors). That is, the following matrix is made up of the weights $w\left(i_{1}, j_{1} ; j_{2}, i_{2}\right)$ :
\{ \}

$$
2 \text { for } n=2 \text {. (starch. } 6 V \text { with } 2 \text { colors) }
$$

$\left(\begin{array}{ccccccccc|c}1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 00 \\ 0 & \frac{(-1+t) u}{-1+t u} & 0 & \frac{-1+u}{-1+t u} & 0 & 0 & 0 & 0 & 0 & 01 \\ 0 & 0 & \frac{(-1+t) u}{-1+t u} & 0 & 0 & 0 & \frac{-1+u}{-1+t u} & 0 & 0 & 02 \\ 0 & \frac{t(-1+u)}{-1+t u} & 0 & \frac{-1+t}{-1+t u} & 0 & 0 & 0 & 0 & 0 & 10 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 11 \\ 0 & 0 & 0 & 0 & 0 & \frac{(-1+t) u}{-1+t u} & 0 & \frac{-1+u}{-1+t u} & 0 & 12 \\ 0 & 0 & \frac{t(-1+u)}{-1+t u} & 0 & 0 & 0 & \frac{-1+t}{-1+t u} & 0 & 0 & 20 \\ 0 & 0 & 0 & 0 & 0 & \frac{t(-1+u)}{-1+t u} & 0 & \frac{-1+t}{-1+t u} & 0 & 21 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1\end{array}\right) 22$

One can check:
Proposition 3.1.1 The twisted R matrix satisfies the following quadratic relation
\{ \}

$$
R^{2}=\frac{(1+u)(1-t)}{1-t u} R+\frac{t-u}{1-t u} I_{d}
$$

(where $I d$ is the identity matrix)
This is shown for 2 colors. See [[Problems, 4-28\#2|Problem 2]] for a hint in the general case.
Therefore,

Proposition 3.1.2 We may identify the R matrix with the Heck element $R=Y_{s, x}=x T_{s}+(1-x)$, for a suitable $x$.

So, we can build the stochastic coloured six vertex model as a result of an application of a product of the $Y$ elements to the identity permutation.

Define
\{ \}

$$
W_{a, b}=Y_{(b-1, b), x} Y_{(b-2, b-1), x} \cdots Y_{(a, a+1), x}
$$

Then
Lemma 3.1.3 \{ \}
Lemma.

$$
\begin{aligned}
& W_{a-k}, b-k \cdots W_{a-1}, b-W_{a}, b \\
& \qquad \text { is a }(k+1) \text { by }(b-a+1) \text { box }
\end{aligned}
$$

This application produces a vertex model in the $(k+1) \times(b-a+1)$ box.
Examples:
\{ \}



### 3.2 Color-position symmetry

We have the following pictures for the coloured vertex model:
\{ \}


$$
[\infty]
$$

$$
\begin{aligned}
(Y \ldots Y) & T_{e} \\
& =\sum_{\zeta} P_{\text {rib }}(\sigma) T_{\sigma} \\
I\left(Y_{s, x}\right) & =Y_{s, x}
\end{aligned}
$$

$$
\begin{array}{rlll}
\operatorname{Prob}(b)=\text { coett. ley } & T_{b} & \text { in } \\
& W_{a-k, b-k^{\prime \prime \prime}} & W_{a, b} & T_{e}
\end{array}
$$

$$
\begin{aligned}
& \text { incoming color } \\
& \text { permutation } \\
& \text { (can track the } \\
& \text { below colors, } \\
& \text { too!) }
\end{aligned}
$$

$\qquad$
We can apply involution, and reorder the application of the operators $Y_{s, x}$. We get the following result:

Theorem 3.2.1, in the form with crosses For each $\sigma \in S_{N}$ :
\{ \}


Theorem 3.2.1, in the grid form $\}$
Thu.

(The color-position

Parodin-Pufetov 2019 Galashiv 2020 Pufetor 2020

### 3.3 Example for a second class particle

Let us apply [[3.2 Color-position symmetry\#Theorem 321 in the grid form|the previous theorem]] to the second class particle

Remark 3.3.1 Most computations are done in the $3 \times 4$ grid, but the statements and the corresponding proofs are fully general.

Example 3.3.2 \{ \}


### 3.4 Matching of probabilities

Here we use [[3.2 Color-position symmetry\#Theorem 321 in the grid form|the previous theorem]] to prove a part of Borodin-Wheeler's colour matching result. Recall the result

Distribution matching The probability in the coloured model \{ \}

$$
\begin{array}{r}
\text { Fer the color }- \text { position symmetry o } \\
\text { recall }
\end{array}
$$



Is equal to the following probability in the uncoloured model:
\{ \}

Proof of the distribution matching for empty I Turns out that the Hecke approach allows to easily prove the above equality of probabilities, but for the particular case $I=\varnothing$. Note that this was the case we used for the second class particles.
\{ \}

\{ \}
Probability equals tautologically to

$$
\left\{1_{\rho j,} N\right\} \supset\left\{\tilde{S}_{J_{1}+M}^{-1} \ldots \sigma_{J e+M}^{-1}\right\}
$$

Now we can apply the involution, and get \{ \}


## 4 Coloured height functions

- 

4 Matching of joint distributions of coloured height functions

Let us now begin discussing height functions in coloured models
In the usual uncoloured case, recall that we have
\{ \}


Let us add colours, and define

Definition 4.0.1 Let $H_{k}(M, N)$ be the height function for colours $\geq k$, that is, the number of paths of colour $\leq k$ that exit through the right boundary.

Lemma 4.0.2 + proof We have an easy matching between (single-point) distributions of coloured and uncoloured height functions:
\{ \}


Let us prove that this extends to multiple colours.

Theorem 4.0.3 \{ \}


Theoren.


$$
\left\{H_{M+1}^{\mathrm{Col}}(M, N), \ldots, H_{M+N}^{\mathrm{Col}}(M, N)\right\}
$$

$$
\begin{array}{r}
\stackrel{d}{=}\left\{\mathcal{H}^{\text {uncol }}(M, N), H^{\text {uncol }}(M, N-1), \ldots\right. \\
\left.\cdots, \mathcal{H}^{\text {uncol }}(M, 1)\right\}
\end{array}
$$

Proof 4.0.3 \{ \}

We have
\{ \}

$$
\begin{aligned}
& \text { joiut distrib.of }
\end{aligned}
$$

$$
\begin{aligned}
& H_{k}^{c o l}(4,3)=\sum_{i=1}^{3} \mathbb{H}_{i} \geqslant k \\
&=\sum_{i=1}^{3} \mathbb{L}_{i \in} \in\left\{b_{i} \in\{k, k+1, \ldots\}\right. \\
&\left.b_{k},-\sigma_{k+1}, \ldots, \sigma_{M+N}^{-1}\right\}
\end{aligned}
$$

Now apply the involution:
\{ \}

$$
\begin{aligned}
& H_{7}^{c o l}(4,3) \xlongequal{d} \sum_{i=1}^{3} \|_{i=b_{7}^{-1}}=H^{\text {uncol. }}(4,1) \\
& H_{6}^{\text {col }}(4,3) \xlongequal{d} \sum_{i=1}^{3} \|_{i=b_{7}^{-1} \text { orb }}^{6}=H^{\text {uncol. }}(4,2)
\end{aligned}
$$

This discusion readily extends to the general case.

## Notes and references

1. Borodin-Wheeler, original distributional matching result. https://arxiv.org/abs/1808.01866
2. Borodin-Bufetov, color-position symmetry. https://arxiv.org/abs/1905.04692
3. Bufetov, Hecke algebra paper. https://arxiv.org/abs/2003.02730
4. Work on another symmetry, the shift invariance. https://arxiv.org/abs/1912.02957
5. Galashin, more discussion of symmetries via Hecke algebras. https://arxiv.org/abs/2003.06330

Problems
1
Show that if $s$ is an elementary transposition $(i, i+1)$ and $w \in S_{N}$, such that $\ell(s w)=\ell(w)-1$, then we can represent $w=s w^{\prime}$ for some $w^{\prime} \in S_{N}$.

2
Show that the twisted $R$ matrix composed of the stochastic coloured vertex weights satisfies the quardatic relation
\{ \}

$$
R^{2}=\frac{(1+u)(1-t)}{1-t u} R+\frac{t-u}{1-t u} I_{d}
$$

for an arbitrary number of colours $n$.
Hint: represent the operator $R$ in the tensor product $V \otimes V$ as \{ \}

In general, $R$ is on op. in $V \otimes V$

$$
\begin{array}{ll} 
& \operatorname{div} V=n+1 \quad, n=\not F \text { of colors } \\
\& & R=\sum \alpha_{i k j l} E_{i k} \otimes E_{j l} \quad i k, j l=0 \ldots n
\end{array}
$$

$$
\text { for explicit } \alpha_{i j k e}
$$

Then square it using relations for $E_{i j}$ )

$$
E_{i j}=\left(\begin{array}{cc}
0 & \phi \\
0 & 0 \\
0 & 1 \\
0 & 0
\end{array}\right)
$$

Then squaring this expression is straightforward using the matrix units $E_{i j}$.
[[Problems, 5-3|4 problems]], due May 6

## 1 Joint distribution at a single point, recall

- 


## 1 Joint distribution in the coloured model at a single point

Recall that we're working with the coloured stochastic six vertex model:
\{ \}




$$
1-b_{1}=\frac{(1-t) u}{1-t u}
$$

In the previous lecture we proved the following theorem, based on the involution in the Hecke algebra:

Theorem 1.0.1 \{ \}


Here the probability of the same permutation $\sigma \in S_{N}$ is the same for the vertex model on both sides.

This theorem implies the following interpretation of the joint distribution of the coloured height function values at a single point $(M, N)$. Recall the definition:

Definition 1.0.2 The coloured height function is defined as
\{ \}

$$
\begin{aligned}
H_{k}(M, N)= & \# \text { paths of color } \geqslant k \\
& \text { exiting through the right } \\
& \text { huonddary }\{M\} \times[0, N] \\
& \rightarrow \\
& \rightarrow
\end{aligned}
$$

Corollary 1.0.3 There is the following equality in distribution between the coloured and the uncoloured models:
\{ \}

$$
\begin{aligned}
& \left\{H_{M+1}^{\operatorname{col}}(M, N), \ldots, H_{M+N}^{\operatorname{Col}}(M, N)\right\} \\
& \quad d \\
& =\left\{H^{\operatorname{uncol}}(M, N), H^{\operatorname{uncol}}(M, N-1), \ldots\right. \\
& \left.\cdots, H^{\operatorname{uncol}}(M, 1)\right\}
\end{aligned}
$$

Pictorially:
\{ \}

$$
\begin{gathered}
M+N \\
\vdots \\
M+1 \\
M — 21
\end{gathered}
$$

and
\{ \}


## 2 Computing the single-point observables

Here we prove a formula for the joint distribution of the coloured height functions at a single point, which is a slight extension of our analysis of the uncoloured stochastic six vertex model by means of difference operators.
-

### 2.1 Mapping to Hall-Littlewood processes

Recall that the distribution of the height function of the usual uncoloured stochastic six vertex model is mapped to the Hall-Littlewood measure:
\{ \}


In fact, one can extend this to joint distributions along down-right paths: \{ \}


We don't need all down-right paths, but rather a single down line:
\{\}


Here's the key lemma:
Lemma 2.1.1 \{ \}

\{ \}

$$
\begin{aligned}
& \text { If } \lambda_{1} x, \mu \text { have the } \\
& \text { joint distribution } \\
& \frac{1}{Z} P_{\mu}\left(u_{1} \ldots u_{k}\right) Q_{\mu / e}(v) P_{\lambda / e}(u) Q_{\lambda}\left(v_{1} \ldots v_{m}\right)
\end{aligned}
$$

\& we apply a raubionized cross notion then the end joint distr. of $\lambda, v, \mu$ is

$$
\frac{1}{\widetilde{z}} P_{\mu}\left(u_{1} \ldots u_{k}\right) P_{v / \mu}(u) Q_{v / \lambda}(v) Q_{\lambda}\left(v_{1} \ldots v_{m}\right)
$$

This is left as [[Problems, 5-3\#2|Problem 2]].
In particular, the joint distribution of $\mu, \nu$ after summing over $\lambda$ is given by the Hall-Littlewood process
\{ \}

$$
\text { HL process } \frac{1}{z} P_{\mu}\left(u_{1},-u_{k}\right) P_{v / \mu}(n) Q_{\nu}\left(v_{1}, v_{\mu}, v\right)
$$

This discussion leads to:
Corollary 2.1.2 \{ \}

Follows Joint distil of

$$
\lambda^{(1)}-\lambda^{(N)} \text { is }
$$


of the HL process

$$
\begin{aligned}
\frac{1}{z} P_{\lambda^{(1)}}\left(u_{1}\right) P_{\lambda^{(2)} / \lambda^{\prime 1)}}\left(u_{2}\right)- & -P_{\lambda^{(N)} / \lambda^{(N-1)}}\left(u^{(v)}\right) \\
& Q_{\lambda^{(N)}}\left(v_{1} \ldots v_{M}\right)
\end{aligned}
$$

Corollary 2.1.3 \{ \}

Jorut distr. of height f. at

$$
\begin{aligned}
& (M, 1) \ldots(M, N) \text { is } \\
& \left\{K-l\left(\lambda^{(K)}\right)\right\}_{K=1 \ldots N} \\
& \text { for the HL prows (*) }
\end{aligned}
$$

Remark about the Schur case In the case $t=0$, the joint distribution of the height function along down-right paths is given by the Schur process, instead of the Hall-Littlewood process.
The Schur process is determinantal, which means that the joint distribution
may be expressed as a certain Fredholm determinant. In this case, multipoint asymptotic analysis is accessible (in the Kardar-Parisi-Zhang regime).

For the case $t \neq 0$, similar multipoint asymptotic analysis is much harder (and we omit this). However, some other asymptotic may be performed; they lead to correlated Gaussian fields.
2.2 Contour integral via difference operators

We begin by recalling the eigenoperators for the Hall-Littlewood polynomials. \{ \}

$$
\begin{aligned}
D(t, 0) & =\sum_{i=1}^{N} \prod_{j \neq i} \frac{t_{x_{i}-x_{j}}^{x_{i}-x_{j}} T_{0, x_{i}}}{\left.D\left(t_{j},\right)\right) P_{\lambda}\left(x_{1}, x_{N} \mid t\right)=} \\
& =\frac{1-t^{N-\ell(\lambda)}}{1-t} P_{\lambda}\left(x_{1, \ldots} x_{N}\right)
\end{aligned}
$$

$$
\begin{aligned}
& \tilde{D}=t^{-N}[1+(t-1) D(t, 0)] \\
& \tilde{D} P_{\lambda}=t^{-l(\lambda)} P_{\lambda}
\end{aligned}
$$

The action of $\widetilde{D}$ on product functions takes a contour integral form \{ \}

$$
\begin{aligned}
& \frac{\tilde{D} f\left(x_{1}\right)-f\left(x_{N}\right)}{f\left(x_{1}\right) \ldots f\left(x_{N}\right)}=\frac{1}{2 \pi_{i}} \oint_{j} \prod_{j=1}^{N} \frac{z-x_{j} / t}{z-x_{j}} \frac{d z}{z \cdot f(z)} \\
& f(0)=1
\end{aligned}
$$

Our objective now is to compute Hall-Littlewood process expectations \{ \}

$$
\mathbb{E} t^{-k_{1} l\left(\lambda^{(1)}\right)-k_{2} l\left(\lambda^{(2)}\right)-\ldots-k_{N} l\left(\lambda^{(\omega)}\right)}
$$

or, equivalently,

$$
\begin{aligned}
& \sqrt[N]{L} t^{-\sum_{i=1}^{k} l\left(\lambda^{\left(n_{i}\right)}\right)} \\
& \text { for some } \quad N \geqslant n_{1} \geqslant \ldots \geqslant n_{k} \geqslant 1
\end{aligned}
$$

Lemma 2.2.1 Let $\widetilde{D}_{n}$ be the operator applied in the variables $u_{1}, \ldots, u_{n}$. Then, expectations
\{ \}

$$
\begin{aligned}
& \text { Et } t^{-\sum_{i=1}^{k} l\left(\lambda^{\left(n_{i}\right)}\right)} \\
& \text { for some } \quad N \geqslant n_{1} \geqslant \ldots \geqslant n_{k} \geqslant 1
\end{aligned}
$$

are computed by applying
\{ \}

$$
\begin{aligned}
& \tilde{D}_{n_{k}} \ldots \tilde{D}_{n_{2}} \tilde{D}_{n_{1}} \quad \text { in this order } \\
& \text { to } \\
& Z=\sum_{\lambda^{(1)} \lambda^{(\nu)}} P_{\lambda^{(1)}}\left(u_{1}\right) \times
\end{aligned}
$$

$$
\begin{aligned}
& \text { (and then dividing bey } Z \text { ) }
\end{aligned}
$$

Proof 2.2.1 We display the proof in the particular case, but the general case is analogous.
\{ \}

$$
\begin{aligned}
& \text { Proof. (excuaple } N=3, n_{1}=3, n_{2}=2 \text { ) } \\
& \lambda^{(3)}=\nu \\
& Z=\sum_{\lambda \mu \nu} P_{\lambda}\left(u_{1}\right) P_{\mu \lambda}\left(u_{2}\right) P_{\nu / \mu}\left(u_{3}\right) \cdot \begin{array}{l}
\lambda^{(2)}=\mu \\
\lambda^{(1)}=\lambda
\end{array} \\
& =\sum_{\mu, \nu} P_{\mu}\left(u_{1}, u_{2}\right) P_{v} / \mu\left(u_{3}\right) Q_{\nu}(\vec{v}) \\
& =\sum_{v} P_{v}\left(u_{1}, u_{2}, u_{3}\right) Q_{v}(\vec{v})
\end{aligned}
$$

These forms of the partition function $Z$ are useful for applications of various operators. In particular, we have

Apply $\tilde{D}_{2} \tilde{D}_{3}$

$$
\begin{aligned}
& \tilde{D}_{2} \tilde{D}_{3} z \\
= & \tilde{D}_{2} \sum_{v} \tilde{D}_{3} P_{v}\left(u_{1}, u_{2}, u_{3}\right) Q_{v}(\vec{v}) \\
= & \tilde{D}_{2} \sum_{v} t^{-l(\nu)} P_{v}\left(u_{1}, u_{2}, u_{3}\right) Q_{v}(\vec{v}) \\
= & \tilde{D}_{2} \sum_{v, \mu} t^{-l(\nu)} P_{\mu}\left(u_{1}, u_{2}\right) P_{v / \mu}\left(u_{3}\right) Q_{v}(\vec{v})
\end{aligned}
$$

\{\}

$$
\begin{aligned}
& =\sum_{v, \mu} t^{-(l v)} \tilde{D}_{\nu} P_{\mu}\left(u_{1}, u_{2}\right) \quad P_{\nu / \mu}\left(u_{3}\right) Q_{\nu}(\vec{v}) \\
& =\sum_{v, \mu} t^{-(\mid v)-l(\mu \mu)} P_{\mu}\left(u_{1}, u_{2}\right) \quad P_{\nu / \mu}\left(u_{3}\right) Q_{\nu}(\vec{v}) \\
& \text { desired expectation }
\end{aligned}
$$

Therefore, the operator application leads to the following result. (We omit the contour integral manipulations that are the same as we did in the single-point case.)

Theorem 2.2.2 \{ \}
The rem

$$
(k-\text { fold nested } \oint)
$$

$$
\begin{aligned}
& h\left(M, n_{i}\right) \\
& E t^{\sum_{i=1}^{k}\left(n_{i}-l\left(\lambda^{\left(n_{i}\right)}\right)\right)}=\frac{t^{\frac{k(k-1)}{2}}}{(2 \pi i)^{k}} \oint \ldots \oint \frac{d z_{1}-d z_{k}}{z_{1} \ldots z_{k}} \\
& \prod_{1 \angle A \angle B \leq k} \frac{z_{A}-z_{B}}{z_{A}-t z_{B}} \prod_{i=1}^{k}\left[\prod_{j=1}^{n_{i}} \frac{t z_{i}-u_{j}}{z_{i}-u_{j}} \prod_{j=1}^{m} \frac{1-z_{i} V_{j}}{1-t z_{i} V_{j}}\right]
\end{aligned}
$$



Note:
\{ \}

Note:

$$
\begin{aligned}
& \prod_{i=1}^{k} \prod_{j=1}^{n_{i}} \text { is the only difference } \\
& \quad \text { from tue single - point } \\
& \quad \text { case where we hack } \\
& \mathbb{E} t^{\left(N-e\left(\lambda^{(N)}\right)\right) \cdot k} \longleftrightarrow \prod_{i=1}^{k} \prod_{j=1}^{N}
\end{aligned}
$$

Corollary 2.2.3 \{ \}


$$
\begin{aligned}
& =\frac{t^{k(k-1)}}{(2 \pi i)^{k}} \oint \ldots \oint \frac{d z_{1}-d z_{k}}{z_{1} \ldots z_{k}} \prod_{1 \leqslant A<B \leqslant k}^{\frac{z_{A}-z_{B}}{z_{A}-t z_{B}}} \\
& \times \prod_{i=1}^{k}\left[\prod_{j=1}^{N+1-n_{i}} \frac{t z_{i}-u_{j}}{z_{i}-u_{j}} \prod_{j=1}^{M} \frac{1-z_{i} v_{j}}{1-t z_{i} v_{j}}\right]
\end{aligned}
$$

## 3 Multipoint observables in the coloured model

Here we switch the notation to that of the Bufetov-Korotkikh paper, instead of rewriting their results for our notation. This leads to equivalent results. In particular, the "quantum parameter" $t$ is replaced everywhere by $q$.
-

### 3.1 Action of the Hecke algebra on functions

 \{ \}Dunk - Demazure - Lusztig representation of the Heme algebra

$$
\begin{gathered}
T_{i} \leftrightarrow T_{(i, i+1)} \\
\left(T_{i}-q\right)\left(T_{i}+1\right)=0 \\
\left(\Leftrightarrow T_{i}^{2}-q T_{i}+T_{i}-q=0\right. \\
\left.T_{i}^{2}=(q-1) T_{i}+q\right)
\end{gathered}
$$

$$
\overline{T_{i}, i=1 \ldots N-1}
$$

braid type rel.

$$
T_{i} T_{j}=T_{j} T_{i},(i-j \geqslant 2
$$

$$
T_{i} T_{i+1} T_{i}=
$$

$$
\stackrel{=}{i} T_{i t 1} T_{i} T_{i t 1}
$$

we had 1-q but
fris is just $T_{i} \rightarrow-T_{i}$
so, not essential

Proposition 3.1.1 \{\}

Proposition The following is a represent. of the Hecke algebra in the space of (for example) rational functions in $\vec{\omega}$

$$
\begin{aligned}
& T_{0} f\left(\omega_{1}, \ldots, \omega_{N}\right)= \\
= & q f\left(\omega_{1} ;-\omega_{N}\right)+\frac{w_{i+1}-q w_{i}}{w_{i+1}-w_{i}}[\underbrace{\sigma_{i}-I_{d}}_{i}] f\left(w_{1} \ldots \omega_{N}\right) \\
& { }_{\text {permutation }} \\
& w_{i} \leftrightarrow w_{i+1}
\end{aligned}
$$

This is left as an exercise, see [[Problems, 5-3\#1|Problem 1]]. \{ \}

$$
\begin{aligned}
& \text { For } \quad \rightarrow \in S_{N}, \text { denote } T_{\pi} \text { as usual as } \\
& T_{\pi}=T_{i_{1}} . T_{i_{k}} \text {, } \\
& \text { Where } \pi=S_{i_{1}} \ldots S_{i_{k}} \text { is a } \\
& \\
& \text { Shortest expression }
\end{aligned}
$$

### 3.2 General formula for the joint distributions

Let us now specify which data is required for multipoint observables of the coloured stochastic six vertex model.
\{ \}


Figure 2. Left: an example of the data for a SC6V model, namely, a skew domain, rapidities of rows and columns and a monotone coloring. Here the coloring is ( $1,2,2,3,4,5,5,5$ ). Right: an example of a path configuration satisfying the boundary condition on the left picture, as well as the values of the height function $h_{\geq 4}^{(\alpha, \beta)}$ with respect to this configuration.
incoming colts must increase but may repeat

The domain is between two down-right paths.
The incoming colour configuration must be increasing, but colours can repeat.
Definition 3.2.1 The initial data (down-right path Q and incoming colours) is encoded as follows:
\{ \}


Definition 3.2.2 $\}$

$$
P_{i}=\operatorname{conton} s \text { line }
$$



Definition 3.2.3 The outgoing data is encoded by arbitrary points (along the down-right path P ):
\{ \}
Let $\alpha_{1} \leq \ldots \leq \alpha_{k}$

$$
\beta_{1} \geqslant \cdots \Gamma_{k}
$$

encode

$$
\left(\alpha_{i}, \beta_{i}\right) \text { on } P_{1} \text { the }
$$ outside boundary



Also we specify, which coloured height functions are taken at these points. This is done by picking colours $0 \leq c_{1} \leq c_{2} \leq \ldots \leq c_{k}$. Moreover, the assignment of colours to the points is arbitrary, so we need also to specify a permutation $\pi \in S_{k}$.

Definition 3.2.4. Observable Consider the following random variable
\{ \}

$$
\eta=H_{>C_{\pi^{-1}(1)}\left(\alpha_{1}, \beta_{1}\right)+\ldots+H_{>c_{\pi^{-1}(k)}\left(\alpha_{k}, \beta_{k}\right)} \text { coloured height fo's }}^{\substack{ \\\text { his er }}}
$$

Theorem 3.2.5 (Bufetov-Korotkikh) \{ \}

$$
\begin{aligned}
& \pi q^{\eta}=q^{\frac{k(k-1)}{2}-l(\pi)} \oint_{\Gamma} d w_{1} \ldots \oint_{r_{1}} d w_{k} \prod_{r_{k}} \frac{w_{b}-w_{a}}{w_{b}-q w_{a}} \\
& \times\left[T_{\pi}\left(\prod_{a=1}^{k} \prod_{i=1}^{i<\delta\left(c_{a}\right)} \frac{1-x_{i} w_{a}}{1-q x_{i} w_{a}} \prod_{j>\gamma\left(c_{a}\right)}^{M} \frac{1-y_{j} w_{a}}{1-q y_{j} w_{a}}\right)\right] \prod_{a=1}^{k}\left(\prod_{i=1}^{i<\beta_{a}} \frac{1-q x_{i} w_{a}}{1-x_{i} w_{a}} \prod_{j>\alpha_{a}}^{M} \frac{1-q y_{j} w_{a}}{1-y_{j} w_{a}} \frac{\frac{1}{2 \pi \mathrm{i} w_{a}}}{)}\right.
\end{aligned}
$$

Corollary 3.2.6 There are two easy cases:
\{ \}
(1) If $c_{1}=\cdots=c_{k}=1$,

Cotor-blind model, $\quad T=e, \quad T_{\pi}=\operatorname{Iel}$
(2) all points $\left(\alpha_{i}, \beta_{i}\right)$ are equal,

$$
\text { also can tare } T_{刀}=I_{d}
$$

See also [[Problems, 5-3\#4| Problem 4]].

### 3.3 On the proof of the general multipoint formula

Let us say a little about the proof of the Bufetov-Korotkikh formula. The proof is by induction, in a direct "verification style". The inductive step is based on local relations.

## Local relation for the uncoloured model

\{ \}



$\pi$
uncoloold model

Lemma 3.3.1 \{ \}


The proof is left as an exercise, see [[Problems, 5-3\#4| Problem 4]].

## Local relation for the coloured model

\{ \}

Let $u, v$ be row and column parameters. For a pair of colors $i<j$ set

$\longrightarrow=\frac{u-v}{u-q v}$
$\longrightarrow=\frac{v(1-q)}{u-v}$
$\longrightarrow=1$
$\longrightarrow=\frac{q(u-v)}{u-q v}$
$\longrightarrow=\frac{u(1-q)}{u-q v}$
\{ \}


$$
\begin{aligned}
& \mathbb{E}\left[\mathcal{Q} \geq c\left([p]^{\prime}\right) \left\lvert\, \begin{array}{c}
\text { incoming } \\
\text { coloss }
\end{array}\right.\right] \\
& =\frac{q^{r} u-q v}{u-q v} \mathcal{Q}_{\geq c}\left(\left[p_{t}\right]^{r}\right) \\
& +\frac{v-q u}{u-q v} \sum_{i=0}^{r-1} q^{i} Q_{\geq c}\left(\left[p_{t}\right]^{i}\left[p_{k}\right]\left[p_{*}\right]^{r-i-1}\right) \\
& +\frac{u-v}{u-q v} \sum_{i=0}^{r-1} q^{i} \mathcal{Q}_{\geq \mathrm{c}}\left(\left[p_{p}\right]^{i}\left[p_{+}\right]\left[p_{v}\right]^{r-i-1}\right)
\end{aligned}
$$

## Notes and references

1. Reference on the diffusion limit for the push-block process: https://arxiv.org/abs/1206.3817
2. Bufetov-Korotkikh paper with the proof of the multipoint formula for the coloured stochastic model: https://arxiv.org/abs/2011.11426

## Problems

## 1

Prove that the Demazure-Lusztig operators give rise to a representation of the Hecke algebra of $S_{N}$ in the space of rational functions in $w_{1}, \ldots, w_{N}$, in which the quadratic relation has the form

$$
\left(T_{i}-q\right)\left(T_{i}+1\right)=0, \quad i=1, \ldots, N
$$

\{ \}

$$
\begin{aligned}
& \prod_{i} f\left(w_{1}, \ldots, w_{N}\right)= \\
= & f\left(w_{1,}, w_{N}\right)+\frac{w_{i+1}-4 w_{i}}{w_{i+1}-w_{i}}[\underbrace{\sigma_{i}}_{i}-I_{d}] f\left(w_{1} \ldots w_{N}\right) \\
& w_{i} \leftrightarrow w_{i+1}
\end{aligned}
$$

## 2

Prove [[2.1 Mapping to Hall-Littlewood processes\#Lemma 21 1|Lemma 2.1.1]].

## 3

Write down the two "easy" particular cases of the general q-moment formula (see [[3.2 General formula for the joint distributions\#Corollary $326 \mid$ Corollary 3.2.6]]). Match these formulas together, and deduce from this the flip invariance: \{ \}


4
Prove the local relation ([[3.3 On the proof of the general multipoint formula\#Lemma 331 |Lemma 3.3.1]]).

No problems after this lecture

## 1 On various Gaussian asymptotics

- 


### 1.1 Finitely many particles

When there are finitely many particles in a system, we expect (correlated) Gaussian type behavior, or something similar and explicit (like a spectrum of a Gaussian random matrix). In the TASEP / Push-block case, the system is a deterministic transform of a collection of independent random walks.
\{ \}

TASEP / push-block

related to
random matrices
(Don's talk)

-

### 1.2 Gaussian Free Field

\{ \}


In the case of large systems, determinantal structure (and sometimes more general structure) gives rise to Gaussian Free Field asymptotics. This is a 2d analogue of a Brownian bridge.
\{ \}

$$
\begin{aligned}
& h(L x, L y)-\mathbb{E} h(L x, L y) \sim \text { Gaussian Free } \\
& \text { Field } \phi(x, y) \\
& \phi(x, y) \text { - random generalized funct. on } \\
& \phi(x, y)=\Phi\left(\Omega^{-1}(z)\right) \\
& \mathbb{E}(\Phi(z) \Phi(\omega))=-\frac{1}{2 \pi} \log \left|\frac{z-\omega}{z-\bar{\omega}}\right| \\
& \Phi(z) \text { as a rand. var. } \\
& \text { does wot save sense }
\end{aligned}
$$

### 1.3 Symmetric systems

In symmetric systems, we also expect more classical Gaussian fluctuations, since they model "short-time" behaviour.

Consider the symmetric versions of ASEP and stochastic 6 V model (ie., with $t=1$ ).
\{ \}

SSEP, symuetric excl.process
$\Leftrightarrow$ ASEP with $t=1$


Then heigint function satisfies CLT

$$
\begin{aligned}
& h(t, 0) \stackrel{t \rightarrow \infty}{\sim} \sqrt{t} J t_{0}+t^{1 / 4} \text { 。Ganssian } \\
& \text { S6V, } \quad t=1
\end{aligned}
$$

\{ \}
uncoloured GV model

$$
\begin{aligned}
& \rightarrow b_{b_{1}} \\
& \mathbb{E} t^{k \cdot h(M, N)}=\oint .-\phi
\end{aligned}
$$

not well -adapted to setting $t=1$.
One takes limit as $t \rightarrow 1, b_{1}, b_{2} \rightarrow 1$ :

$$
\begin{array}{rlr}
b_{1}=e^{-\beta_{1} / L}, \quad b_{2} & =t b_{1}=e^{-\beta_{2} / L}, & \beta_{1}, \beta_{2}>0 \\
\beta_{1} \neq \beta_{2} \\
\Leftrightarrow t=t^{1 / L}, \log \tau & =\beta_{1}-\beta_{2}, & \\
s=\beta_{1} / \beta_{2} & \simeq \frac{1-b_{1}}{1-t b_{1}} &
\end{array}
$$

\{ \}

Then. $\frac{1}{L} h(L x, L y) \rightarrow H(x, y)$, where

$$
t^{H(x, y)}=1+\frac{1}{\operatorname{ari}} \oint_{-1} \tau^{\left[\frac{y z}{1+z}-\frac{x s z}{1+s z}\right]} d z / z
$$

(limit $\tau \rightarrow 0$ gives back the 6 V him-sh.)

Moreover (ex.) we have that

$$
\begin{aligned}
& f(x, y)=\tau^{\partial H(x, y)} \quad \text { satisfies } \\
& \partial_{x y} f+\beta_{1} \partial y f+\beta_{2} \partial_{x} f=0 \quad\binom{\text { telegraph }}{\text { equation" }}
\end{aligned}
$$

\{ \}

This was the LLN. The CLT looks like:

$$
\begin{array}{r}
\frac{h(L x, L y)-\mathbb{E} h(L x, L y)}{\sqrt{L}} \rightarrow \begin{array}{c}
\text { centered } \\
\text { Gaussian } \\
\text { field } \\
\phi(x, y)
\end{array}
\end{array}
$$

(Def: $\forall(x, y i)$ distinct,

$$
\left(\phi\left(x_{1}, y_{1}\right), \cdots, \phi\left(x_{k}, y_{k}\right)\right)
$$

is a centered Gaussian vector

$$
\text { w. cover. } \left.\mathbb{E} \phi(x, y) \phi\left(x^{\prime}, y^{\prime}\right)=\ldots\right)
$$

\{ \}

Moreover,

$$
\begin{aligned}
& L \operatorname{Cov}\left(t^{h(L x, L y)}, t^{h\left(L x^{\prime}, L y^{\prime}\right)}\right) \longrightarrow \\
& \quad \longrightarrow(\text { au explicit expression })
\end{aligned}
$$

These LLN/CLT are del to Borodin- Goring
$(2018)$

That is, we did not put $t=1$ in the beginning, as that system would be harder to analyze. Rather, a limit $t \rightarrow 1$ simultaneously with $b_{1} \rightarrow 1$ produces a Gaussian behaviour, which can be seen in particular using our $t$-moments.

A random sample of the model with $b_{1}, b_{2}$ close to 1 looks like this:
\{ \}


## 2 Gaussian asymptotics via t-moments

### 2.1 Useful contour transformation

\{ \}
Step 1. Slightly cringe formula
for the moments.

$$
\begin{aligned}
& \sum h\left(m, n_{i}\right) \\
& =\frac{t^{k \frac{(k-1)}{2}}}{(2 \pi i)^{k}} \oint \ldots \oint \frac{d z_{1} \cdots d z_{k}}{z_{1} \ldots z_{k}}
\end{aligned}
$$

$$
\left.\prod_{\substack{ \\\angle A \angle B \leq K}}^{\frac{z_{A}-z_{B}}{z_{A}-t z_{B}} \prod_{i=1}^{k}\left[\prod_{j=1}^{n_{i}} \frac{t z_{i}-x_{j}}{z_{i}-x_{j}} \prod_{j=1}^{m} \frac{1-z_{i} y_{j}}{1-t z_{i} y_{j}}\right]}\right]
$$

We have the contours with parts around 0 :
\{ \}


Lemma 2.1.1 \{ \}

Lemma

$$
\begin{aligned}
& \quad \mathbb{E} \prod_{i=1}^{k}\left(t^{h\left(M, n_{i}\right)}-t^{i-1}\right) \\
& =\frac{t^{k \frac{(k-1)}{2}}}{(2 \pi i)^{k}} \oint \cdots \frac{d z_{1}-d z_{k}}{z_{1} \ldots z_{k}} \\
& \prod_{K A<B \leq K}^{z_{A}-z_{B}} \\
& z_{A}-t z_{B}
\end{aligned} \prod_{i=1}^{k}\left[\prod_{j=1}^{n_{i}} \frac{t z_{i}-x_{j}}{z_{i}-x_{j}} \prod_{j=1}^{M} \frac{1-z_{i} y_{j}}{1-t z_{i} y_{j}}\right] .
$$

Same $\oint$ lent over the contour around only $x_{j}$ and wot $D$.

Proof 2.1.1 \{ \}
Let $f_{n}(z)=\prod_{j=1}^{n} \frac{t z_{i}-x_{j}}{z_{i}-x_{j}} \prod_{j=1}^{M} \frac{1-z_{i} y_{j}}{1-t z_{i} y_{j}}$
$f_{n}(0)=1$
\{ \}
stant from

\{ \}
then we have $\sum$ over $2^{k}$ termas $I \subseteq\{1 \ldots k\}$ ubere $z_{i}, i \notin I$, susegrate arround $O$,

$$
\begin{aligned}
& I=\left\{i_{1}<\ldots<i e\right\} \\
& I^{c}=\left\{p_{1}<\ldots<p_{k}-\right\}
\end{aligned}
$$

Each $z_{p_{i}}$ is shrunk to 0 ,
in order $z_{p_{k} 0_{0}} \ldots, z_{\rho_{1}}$
\& we get faltop $t^{-\sum\left(k-p_{j}\right)}$
\{\}
v $k$-fold with o's
Iuitial $\oint . \beta$ is $l$-fold w/o O's

$$
\begin{aligned}
& \sum_{l=0}^{k} \sum_{|I|=l} t^{k l-\sum_{\alpha} i_{a}} \oint \oint \frac{d w_{1} \ldots w_{w}}{w_{1} \ldots w e(2 \pi i)^{l}}
\end{aligned}
$$

\{ \}

Now let $X_{j} \longleftrightarrow t^{h\left(M, m_{j}\right)}$
\& then the $\sum_{I}$ is a combination

$$
\begin{gathered}
\text { of } \sqrt{ } \text { poly's in } \\
X_{1}, \ldots, X_{k}
\end{gathered}
$$

\{ \}
$\Rightarrow$ result follows from the idbutity

$$
\begin{aligned}
& \sum_{l=0}^{k} \sum_{I I=l}\left(t^{-1}\right)^{(k-l)(k-l+1) / 2} \\
& \left.0\left(X_{i}-\left(t^{-1}\right)^{i}\right)\left(X_{i_{2}}-\left(t^{-1}\right)^{i_{2}-1}\right)_{\cdots}\right)\left(X_{i l}-\left(t^{-1}\right)^{i l-l+1}\right) \\
& =X_{1} X_{2} \ldots X_{k}
\end{aligned}
$$

This identity can be proven by induction \{ \}
both sides are linear in $X_{1}$, 30 enough for 2 prints,

$$
\left.x_{1}=\infty \text { and } x_{1}=t^{-1}\right)
$$

2.2 Limit shape
\{ \}
Recall $b_{1}=\frac{1-x y}{1-t x y}=e^{-\beta_{1} / L} \rightarrow 1$,

Let $x_{j} \equiv 1$ and $y_{j} \equiv s$ fixed
\{ \}

$$
\begin{aligned}
& \mathbb{E} \prod_{i=1}^{k}\left(t^{h\left(M, n_{i}\right)}-t^{i}\right) t^{h}=\tau^{1 / L h} \\
& \quad=\frac{t^{k(k-1)}}{(2 \pi i)^{k}} \oint^{h} \ldots \frac{d z_{1} \cdots d z_{k}}{z_{1} \cdots z_{k}} \\
& \prod_{i \leq A<B \leq k}^{z_{A}-z_{B}} \prod_{i=1}^{k}\left(\frac{t z_{i}-1}{z_{i}-1}\right)^{n_{i}}\left(\frac{1-z_{i} s}{1-t z_{i} s}\right)^{M}
\end{aligned}
$$

\& all contours are around 1 .

\{ \}

$$
\begin{aligned}
& \text { Now take } k=1: \quad t^{h}=\tau^{1 / h \cdot h} \rightarrow \tau^{H-} \\
& \mathbb{E}\left(t^{h(l x, l y)}-1\right) \\
& \simeq \tau^{z(x, y)}-1
\end{aligned}
$$

In the $\oint$ :

$$
\begin{aligned}
& \frac{i}{2 \pi i} \oint_{\binom{\text {around d }}{1}} \frac{d z}{z}\left(\frac{t z-1}{z-1}\right)^{L x}\left(\frac{1-z s}{1-t z s}\right)^{L y} \\
& L x \log \frac{1-t z}{1-z}+\operatorname{Ly} \log \frac{1-z s}{1-t z s}=x \frac{z \beta_{1}}{1-z}-y \frac{s z \beta_{1}}{1-s z} \\
& \text { (get something similar to the chiming, } \\
& \text { the issue might be up to reflect. wot } \\
& \text { the main diagonal, of rearming } \\
& \text { of the parameters). }
\end{aligned}
$$

For the covariance, the scaling should be \{ \}


### 2.3 Covariance

Now let us get a formula for the covariance. This is not yet proving the Gaussian asymptotics, as for the Gaussian asymptotics one would have to show that all multipoint expectations / observables are expressed through the covariance in a certain specific form. We will omit the corresponding argument, and refer to the paper on stochastic telegraph equation for details.

First, we make the contours avoid the poles at $z_{i}=z_{j}$. This is possible for $t$ sufficiently close to 1 :
\{ \}


The extra transformation is possible because:
\{ \}

Indue, the extra residues vanish. $z_{2}=E^{-1}$, and we had

$$
\frac{t z_{1}-1}{z_{1}-1} \cdot \frac{t z_{2}-1}{z_{2}-1} \longrightarrow \frac{t z_{1}-1}{z_{1}-1} \cdot \frac{z_{1}-1}{t_{z_{1}-1}}
$$

$$
\text { So residue at } z_{1}=1 \text { vamiblges, } \oint=0 \text {. }
$$

With the other contours, we can now take asymptotics:
\{ \}

$$
\begin{aligned}
& L \operatorname{Cov}(\underbrace{t^{h(L x, L y)}}_{t^{h} \simeq \tau^{J t}}) \underbrace{t^{h\left(L x^{\prime}, L y^{\prime}\right)}}_{t^{h^{\prime}} \simeq \tau^{3 t^{\prime}}})=\cdots \\
& L\left[\mathbb { E } ( t ^ { h } - 1 ) \left(t^{\left.h^{\prime}-t\right)-\mathbb{E}\left(t^{h}-1\right) \mathbb{E}\left(t^{\left.h^{\prime}-1\right)}\right]}\right.\right. \\
& =L(\oint \oint-\oint)
\end{aligned}
$$

\{ \}

$$
\begin{aligned}
& =\frac{L}{(2 \pi i)^{2}} \oint_{\text {l }} \oint^{\frac{z_{1}(t-1)}{z_{1}-t z_{2}}} \underbrace{\left(\frac{z_{1}-z_{2}}{z_{1}-t z_{2}}-1\right.})\left(\frac{t z_{1}-1}{z_{1}-1}\right)^{L x}\left(\frac{t z_{2}-1}{z_{2-1}}\right)^{L x^{\prime}} . \\
& \text { nested } \cdot\left(\frac{1-z_{1} s}{1-t z_{1} s}\right)^{L y}\left(\frac{1-z_{2} s}{1-t z_{2} s}\right)^{L y^{\prime}} \frac{d z_{1} d z_{1}}{z_{1} z_{2}}
\end{aligned}
$$

$$
\begin{aligned}
\longrightarrow & \frac{\log \tau}{(2 \pi i)^{2}} \oint \oint \frac{z_{1}}{z_{1}-z_{2}} \cdot \frac{d z_{1} d z_{2}}{z_{1} z_{2}} \\
& \exp \left[x \frac{z_{1} \beta_{1}}{1-z_{1}}-y \frac{s z_{1} \beta_{1}}{1-s z_{1}}+x^{\prime} \frac{z_{2} \beta_{1}}{1-z_{2}}-y^{\prime} \frac{s z_{2} \beta_{1}}{1-s z_{2}}\right]
\end{aligned}
$$

contonss around 1 and $z$ inside $t z_{2}$.
\{ \}

$$
\begin{aligned}
& \mathbb{E}\left(t^{h}-1\right)\left(t^{h^{\prime}}-t\right)-\mathbb{E}\left(t^{h}-1\right) \mathbb{E}\left(t^{h^{\prime}}-1\right) \\
& \mathbb{E}\left(t^{h+h^{\prime}}-t^{h+1}-y^{\prime \prime}+t\right)-\mathbb{E} t^{h} \mathbb{E} t^{h^{\prime}}+\mathbb{E} t^{h}+\mathbb{E} t^{h^{\prime}}-1 \\
= & \operatorname{Cov}\left(t^{h}, t^{h^{\prime}}\right)+t-1+\mathbb{E} t^{h}-t \mathbb{E} t^{h} \\
= & \operatorname{Cov}\left(t^{h}, t^{h^{\prime}}\right)+(1-t)\left(\mathbb{E} t^{h}-1\right)
\end{aligned}
$$

$\Rightarrow$ get explicit $\operatorname{Cov}\left(t^{h}, t^{h^{\prime}}\right)$.
\{ \}

$$
\begin{aligned}
& \operatorname{LCov}\left(t^{h}, t^{n^{\prime}}\right) \simeq \\
& \frac{\log \tau}{(2 \pi i)^{2}} \oint_{\substack{\text { noted } \\
\text { send 1 }}} \oint_{z_{1}-z_{2}} \cdot \frac{d z_{1} d z_{2}}{z_{1} z_{2}} \\
& \exp \left[x \frac{z_{1} \beta_{1}}{1-z_{1}}-y \frac{s z_{\beta} \beta_{1}}{1-s z_{1}}+x^{\prime} \frac{z_{2} \beta_{1}}{1-z_{2}}-y^{\prime} \frac{s z_{2} \beta_{1}}{1-s z_{2}}\right] \\
& +\frac{\log \tau}{2 \pi i} \oint_{\text {grand 1 }} \frac{d z}{z} \exp \left[x \frac{z \beta_{1}}{1-z}-y \frac{s z \beta_{1}}{1-s z}\right] \\
& \simeq \operatorname{Cov}\left(\tau^{J t(x, y)}, \tau^{H\left(x^{\prime}, y^{\prime}\right)}\right)
\end{aligned}
$$

This shows an explicit covariance.

### 2.4 Getting the six vertex model cone

Let us take the limit $\tau \rightarrow 0$. This limit in fact recovers the stochastic six vertex's cone limit shape. That follows from (formal, non-rigorous) interchange of the limits. If

$$
t=\tau^{1 / L}
$$

then for a fixed $t$ we must have

$$
t=\tau^{1 / L}=\left(t^{L}\right)^{1 / L},
$$

so $\tau=t^{L}$ should be very small. It is nice that this formal interchange of the limits works, and we recover the fixed- $t$ limit shape.
This asymptotic analysis is performed via steepest descent:
\{ \}

$$
\begin{aligned}
& \operatorname{TE}\left(\tau^{J t(x, y)}-1\right)=\frac{t}{2 \pi i} \oint_{\substack{\text { around } \\
1}} \frac{d z}{z} \exp \left[x \frac{z \beta_{1}}{1-z}-y \frac{s z \beta_{1}}{1-s z}\right] \\
& t=\tau^{1 / L}, \log \tau=\beta_{1}-\beta_{2}, \\
& S=\beta_{1} / \beta_{2} \simeq \frac{1-b_{1}}{1-t b_{1}} \\
& S(z)=x \frac{z}{1-z}-y \frac{s z}{1-s z} ; e_{\beta_{1} \rightarrow-\infty}^{\beta_{1} s} e^{\beta_{1}} \simeq \tau \\
& S^{\prime}(z)=0, \quad z_{c} \text { (the correct one) is }
\end{aligned}
$$

\{\}

$$
\begin{aligned}
& z_{c}=\frac{\sqrt{y s / x}-1}{\sqrt{y s / x}-s} s<\frac{y}{x}<s^{-1} \\
& \tau^{H}-1 \simeq \frac{1}{z_{c}} \tau^{S\left(z_{c}\right)} \\
& \Rightarrow \quad \psi \simeq S\left(z_{c}\right)=\frac{(\sqrt{x}-\sqrt{s y})^{2}}{1-s},
\end{aligned}
$$

the SGV limit shape!

## 3 Shift invariance and Brownian bridges

- 


### 3.1 Shift invariance

Let us mention one more Gaussian instance of asymptotics in the coloured vertex model.

First, a shift-invariance result:
\{ \}

(Borodin-Gorin-Wheeler 2019)
This should in principle follow from our flip-invariance:
\{ \}

3.2 Connection to invariance for Brownian motion local times

In a certain limit, the stochastic coloured model leads to a Gaussian model, the additive coloured stochastic heat equation. Its correlations are expressed through expectations of certain Brownian bridge local times. The Brownian bridge local times are themselves shift invariant.


Intersection local time is tee leal time at $O$ of $\underbrace{B_{1}-B_{2}}_{\text {starts at }}$ $a$ and ends at -b
if we shift
endpoints of $B_{2}$
by $\Delta$ up sech that $a, b$ do not become
negative,
\{ \}
we have the same distr. of the local time. (classical pros. result)


## Notes and references

1. Gorin-Shkolnikov on diffusion limit of TASEP: https://arxiv.org/abs/1206.3817
2. Gaussian Free Field in random tilings: https://arxiv.org/abs/1206.5123
3. Borodin-Gorin's paper on the Gaussian limit of the stochastic six vertex model: https://arxiv.org/abs/1803.09137
4. Shift-invariance, with connection to local times of Brownian bridge: https://arxiv.org/abs/1912.02957
5. Galashin on how shift-invariance follows from Heck algebra: https://arxiv.org/abs/2003.06330

## Problems

No problems after this lecture

UHTETPUPYEMGIE
CUCTEMBI GACTULS
(cyrrastule, the kbattobole)
https://lpetrov.cc/
QTASEP@GMAIL.COM
https://publish.obsidian.md/particle-systems/
https://publish.obsidian.md/particle-systems/Ru/Ru-main
3ameiku + zaga4u
no「иCTuKA
(1) 3 Anиeb Zoom (He gre nyסn. goctyua)
(2) EMAJL PACCbInKa ( $\triangle O G A B b T E C b$ ) (Hanouccera ruls 0

$$
\begin{array}{r}
\text { nekywex }+ \text { zoom link) }
\end{array}
$$

+ Oquchble 4ACbI TH-CP
(3) $3 A \triangle A 4 u$

1. Tипичhble Kaptuhku
1.1. "Фuзическая" Moaenb
https://habr.com/ru/post/173905/
(2013)
https://physics.aps.org/articles/v6/7


липкии
тетрис"

1.2. "ИMTEГРИРYEMAя" MO

- TASEP Totally Asymmetric
simple Exclusia Proess
https://wt.iam.uni-bonn.de/ferrari/research/jsanimationtasep


Объяснить, как «решается» TASEP и его многочисленные «родственники»: явные формулы и асимптотическое поведение в разных режимах

Предполагается некоторое знакомство с:

- Анализом / теорией меры (очень желательно)
- Теорией вероятностей (тоже очень желательно)
- Плюс, если видели симметрические функции (не обязательно)


OXFORD MATHEMATICAL, MONOGRAPHS
Symmetric Functions and Hall Polynomials

SECOND EDITION
I. G. MACDONALD


* расска*ите
- cede

2. Mapkobckue yenu, gисиретвое и renpepolbroe bpeus
2.) $x$ - komatrose up-60 R1, 品

Def $T(x, y) \quad, x, y \in \notin$

- rampuya mepexoga.

$$
\begin{aligned}
& T(x, y) \geq 0 \\
& \sum_{y \in X} T(x, y)=1 \\
& \forall x
\end{aligned}
$$

Def.

$$
\begin{aligned}
& X_{0}, X_{1}, X_{2}, \ldots \in \notin \rightarrow \text { ogropogras } \\
& \text { yeur Mapkta } \sim T \\
& P\left(X_{i}=y \mid X_{i-1}=x, X_{i-2}=x_{i-2}, \ldots, X_{0}=x_{0}\right) \\
& =P\left(X_{i}=y \mid X_{i-1}=x\right)=T(x, y) \\
& M, y_{3}\left\{X_{n}\right\} \longleftrightarrow T_{n} \operatorname{Law}\left(X_{0}\right)
\end{aligned}
$$

Prop hepexeg 39 n wazob

$$
\begin{gathered}
P\left(X_{n+j}=y \mid X_{j}=x\right) \\
=\left(T^{n}\right)(x, y)
\end{gathered}
$$

Proof. $n=2$ alanguruep

$$
\begin{aligned}
& P\left(X_{j+2}=y \mid X_{j}=x\right) \\
&= \sum_{z \in X} P\left(X_{j+2}^{A}=y \mid X_{j+1}=z\right) P\left(X_{j+1}^{B}=z \mid X_{j}=x\right) \\
&= \sum_{z} T(x, z) T(z, j)=T^{2}(x, y) \\
& P(A \mid B C) P(B \mid C)=\frac{P(A B C)}{P(B C)} \frac{P(B C)}{P(C)} \\
&= P(A B \mid C)
\end{aligned} \sum_{z} \Rightarrow P(A \mid C),
$$


2.2. Buriomuanotioe / ryaccoteobcuoe pacn pegere tue


$$
\begin{aligned}
& X_{0}=0 \quad \text { 万ипом. p-e } \\
& P\left(X_{n}=k\right)={ }^{k} C_{n}^{k} p^{k}(1-p)^{n-k} \\
& (k \leqslant N, n \leqslant N) \quad\binom{n}{k} \\
& T=\left(\begin{array}{ccccc}
1-p & p & & 0 & \\
& 1-p & p & 0 & \\
& 0 & 1-p & p & \\
& & & & 1
\end{array}\right) \\
& P\left(X_{n}=k\right)-\text { элешенть৷ } T^{n}
\end{aligned}
$$

Prop. $p \rightarrow 0, n \longrightarrow \infty, \quad n p \rightarrow t>0, t \in \mathbb{R}$ Torga $P\left(X_{n}=k\right) \rightarrow e^{-t} \cdot \frac{t^{k}}{k!}, k=0,1, \ldots$
Praf.

$$
\begin{aligned}
& \binom{n}{k} p^{k}(1-p)^{n-k}= \\
& =\frac{n!p^{k}}{(n-k)!} \cdot(1-p)^{n-k} \\
& p^{k} \cdot\left[n^{k}+\ldots\right] \rightarrow t^{k} \\
& (1-p)^{-t}
\end{aligned}
$$

$$
\tilde{X}_{t}=\lim _{p \rightarrow 0, n p \rightarrow t} X_{n} \sim P_{0 i s s}(t)
$$

Mogerr yeme Mapkoba gues $\tilde{X}_{t}$
тре́syet renpeab) brozo bpeweru

$n \rightarrow \infty$
$\Rightarrow$ nobropeen wonbitey mrowo pay

$$
\begin{aligned}
& P\left(\tilde{X}_{t}=i \mid \tilde{X}_{s}=i\right) \quad s<t \\
& =\lim _{p \rightarrow 0} P\left(X_{[t / p]}=i \mid X_{[s / p]}=i\right) \\
& =\lim _{p \rightarrow 0}(1-p)^{\left|\frac{t}{p}\right|-\left\lfloor\frac{s}{p}\right]}=e^{s-t}=e^{-(t-s)} \\
& P(\text { xgei }>t \quad b \quad i)=e^{-t}
\end{aligned}
$$

$$
\pm
$$

Bperes oxetgarme ckayka - Inceror.

$$
P(Z>t)=e^{-\lambda t}, t>0 / / \lambda>0 \quad \operatorname{Exp}(1)
$$

$$
\mathbb{E}(Z)=1 / \lambda
$$

CTpgurypi mb c menp. brewethem \& gucup. up-bom (Markov jump
 processes)

$$
\left[\begin{array}{lll}
-\alpha & \alpha & 0 \\
0 & -\beta & \beta \\
\xi & \gamma & -\gamma-\xi
\end{array}\right]
$$

$T \leadsto \tilde{T}-$ матриуа urienculbrociéé

- $\tilde{T}(x, y) \geqslant 0 \quad x \neq y$
- $\tilde{T}(x, x) \leqslant 0$
- $\sum_{y \in \mathcal{X}} \tilde{T}(x, y)=0$

$$
-\mathbb{P}(\text { wait }>s)=e^{-s(\xi+\gamma)}, s>0
$$

$\tilde{X}_{t}$ 水ёт $\operatorname{Exp}(\xi+\gamma)$ b $x_{3}$, norom upuzaer b $x_{1} / x_{2}$
$c$ bep $\frac{\beta}{\xi+\gamma} / \frac{\gamma}{3+\gamma}$

$$
\begin{align*}
& P\left(\tilde{X}_{t}=y \mid \tilde{X}_{s}=x\right)=e^{(t-s) \tilde{T}}(x, y) \quad s<t \\
& e^{t\left[\begin{array}{cc}
-1 & 1 \\
0 & 0
\end{array}\right]}=\left[\begin{array}{cc}
e^{-t} & 1-e^{-t} \\
0 & 1
\end{array}\right]  \tag{7}\\
& \text { bином } \rightarrow \text { hyace. } T^{\lfloor t / p\rfloor} \longrightarrow e^{t \widetilde{T}} \\
& \begin{array}{l}
p \rightarrow 0 \\
n=\left\lfloor\frac{t}{p}\right\rfloor
\end{array} \\
& \frac{1}{p}(T-I d) \longrightarrow \tilde{J} \\
& \underset{T}{\sim}=\left[\begin{array}{cccc}
-1 & 1 & & 0 \\
& -1 & 1 & \\
& -1 & 1 \\
& & \ddots
\end{array}\right]
\end{align*}
$$

23. Пуассоновскии npоуесс / IR $^{\text {R }}$

$\Phi_{\text {ak } \text { T. }}$ Cnyrasinas worqury payms $S \subset \mathbb{R}$


PakT. oтpezok, cogepxaynu $O$,

$$
\text { uneer pacupegererue } \neq \operatorname{Exp}(1)
$$

(Ha comon gere, yerea glayx)
2.4. OS ofyux Hyacc. upoyeccax

$$
(\Omega, j, \mu) \quad\left(\mathbb{R}^{2}, \beta\left(\mathbb{B}^{2}\right), d x^{2}\right)
$$



$$
\begin{aligned}
& 1 \\
& \text { cuyt }
\end{aligned}
$$

$z \subset \Omega$, gucap. nor. workey. log unox ectio

$$
N_{A}=\#(Z \cap A) \in \mathbb{Z}_{30}, \forall A \in J
$$

1) $\forall A_{1}, \ldots A_{n}, A_{i} \cap A_{j}=\varnothing \quad i \neq j$ bewrebith $N_{A_{i}}$ - reagebucumer
2) 

$$
\begin{aligned}
& N_{A} \sim P_{\text {oiss }}(\mu(A)) \\
& P\left(N_{A}=k\right)=e^{-\mu(A)} \frac{(\mu(A))^{k}}{k!}, k \geqslant 0
\end{aligned}
$$

Pakt Myack-npoyece Ha $\mathbb{R}$ - Dro nyace yroyecc $b$ orgen cubicre

Tect untryuyru:



Myacc. apsyece
gетериинантииии npoyece.


PakT. hyacc. uproyecc $=$ upeger
pabtromproix rejabuen ent $X$ to4eqralx korqurypays (T.e. hyace - „Hzusonee ceyreastors vorig.")

Nampruep
$2 N$ res-pabr. tayed

3. Cистемы 4actuy / tasep
3.1 TASEP: $\exists$, Maproboctb

Ha $G=(V, E)$ oxpanu4. cenerue kaugrebrentitani, $\infty$ um noversorioú sez nerest

np-bo wor. $\{0,1\}^{V}$
uproyece $X_{t}$ a reupp bp.

- $\forall$ yactuyaby xgët Exp $\left(^{t}\right) b_{\text {peens }}$
- korga sysumpuck bberume, ract.
unorzaer cayt. Wo

$$
G=\mathbb{Z}
$$

$$
G=
$$



$$
P(z>t+s \mid z>s)=P(z>t)
$$

oteyscrbe vocregéncturs

Thus. $\forall$ reay. porg. $\in\{0,1\}^{\mathbb{Z}}$, TASEP agyecwber
Proof (Harris 1974)


$$
X_{t}=\text { qutrayus of } X_{0} \text { \& }
$$

or nyace. yproyecca 48
$\ldots \Delta \mathbb{R} \cup \mathbb{R} \omega \mathbb{R} \omega \ldots$.

$$
=\bigsqcup_{i \in \mathbb{Z}}\left(\mathbb{R}, d_{x}\right)
$$

Лекция 2. 17 MAPTA


* C $_{\triangle A b A и ̆ t e ~}^{\text {3a }}$ 3A4и


1. TASEP, cyuseclobanue (8 prounow pay)


$$
X_{t}=\text { qutracyus of } X_{0} \text { \& }
$$ or myacc. yroyecca HR

$\ldots \boldsymbol{R} \cup \mathbb{R} \omega \mathbb{R} \omega \ldots$
2. TASEP / hepkonslyug (directed last parsage percolation)
2.1 Pyrkyus bbicotbl


$$
\begin{aligned}
& t \in \mathbb{R} \geqslant 0 \\
& x \in \mathbb{Z}
\end{aligned}
$$

2.2. Pact u3 grua (corner grouth)


Th. POCT $\Leftrightarrow$ TASEP us unotrous guavobun

Yup. Oxapanieprzdato bce pacup. upubage ugue k Mapnadocvory spoyecey we $\{0,1\} \mathbb{Z}$.
2.3. LPP ( directed last passage percolation)
«Направленная перколяция последнего прохода»
$y$

$\omega_{x y} \geqslant 0$ iid mosole

Def. $L_{x, y}=$ bpeuss, koya worpoeen kretry (cu-b.)
honnuro za bpeave $t$,

$$
B_{t}=\left\{(x, y): \quad L_{x y} \leq t\right\}
$$


Pf, $(1)=(2)$ : T.k. (2) yzabl. pekyperen (1)
(1) :

logive burnerar max $\left(L_{x-1, y}, L_{x, y-1}\right)$

Kaynumep.

2.4. Mpegenbras Popra

$$
\begin{aligned}
& \frac{1}{N} L_{\lfloor N \times J, L N y \perp} \longrightarrow \varphi(x, y) \quad ?_{a} \\
& 4 \\
& \frac{1}{t} B_{t} \longrightarrow B=\{(x, y): \varphi(x, y) \leqslant 1\}
\end{aligned}
$$

Ecur ecir
mecuys upegenol,
solepum, wo eero
nyligeukas popua
hpucep.

2.5. Cynepaggutubrocio

Plop.


Pf. wax no gorbwery vsi-by breaŕ racru. D

$\vec{v} \in \mathbb{Z}_{\geqslant 0}^{2} ;$


$$
(m \leqslant n)
$$

$L_{0 \rightarrow n \vec{v}} \geqslant L_{0 \rightarrow m \vec{v}}+L_{m \vec{v} \rightarrow n \vec{v}}$
$\prod_{0}$ relecpares $L_{0 \rightarrow n \vec{v}} \geqslant L_{0 \rightarrow m} \vec{v}+L_{0 \rightarrow(n-r u)} \vec{v}$

$$
\mathbb{E} L_{0 \rightarrow n \vec{u}}=f(n)
$$

$$
f(n) \geqslant f(m)+f(m-n)
$$

Yup. $\lim _{n \rightarrow \infty} \frac{1}{n} f(n)$ eypectoye $T$
$u$ paber $\sup _{n} \frac{f(n)}{n}, \leqslant+\infty$

Ex. $f(n)=\log (n!), \quad f(n) / n \longrightarrow+\infty$
$\|$
Prop.
$\frac{1}{n} \mathbb{E} L_{0 \rightarrow n \vec{v}} \rightarrow \varphi(\vec{v})$
gus beex $\vec{v} \in \mathbb{Z}_{\geq 0}^{2}$
(ogropogrocio)
$\begin{array}{ll}\text { Kpoue row, } \quad \varphi(c \vec{v})=c \varphi(\vec{v}), & \left(\begin{array}{l}\vec{v} \in \mathbb{R}^{2} \\ c \in \mathbb{Z} \\ c \vec{v} \in \mathbb{Z}^{2}\end{array}\right)\end{array}$
(u oraxerce renpepratror $\Rightarrow$ ra $\mathbb{R}^{2} \geqslant 0$ )

Это gaët orber $B=\{(x, y): \varphi(x, y) \leqslant 1\}$
ho se $\mathrm{c} x-0$
evyrastion20
k uplefey

2.6. Cynepags. $\frac{\text { כproguyeckfue teop. }}{\left(k_{u}+\right.}$
 "Cyraggutubkag"

Des. $(\Omega, f, p)$

$$
P(\Omega)=1
$$

$T: \Omega \rightarrow \Omega \quad$ coxp. meng

$$
P\left(T^{-1}(A)\right)=P(A) \quad \forall A \in \mathcal{F}
$$


Ex.


Def. Эproguruecue: $\quad T A=A \Rightarrow P(A)=0$

$$
\Uparrow
$$

$\forall f, \quad f(T x)=f(x) \Rightarrow f=$ const (urbays.pp-yus) $\quad P-n, b$.

Ex. $f\left(T\left(x_{1}, x_{2}, \ldots\right)\right)=f\left(x_{1}, x_{2} \ldots\right)$ lugers zpr. cglura Verranima)

$$
\begin{gathered}
f\left(x_{k}, x_{k+1}, \ldots\right) \\
f=\operatorname{limen}^{(n)} \quad f^{(n)} \text { sab. or } x_{1} \ldots x_{n}
\end{gathered}
$$

$\underline{I} \quad g_{n_{V_{0}}} \in L^{1}$, ugnepasg; $T$-coxpinepy.

$$
g_{m+n}(x) \geqslant g_{n}(x)+g_{m}\left(T_{x}^{n}\right)
$$

$\Downarrow$

$$
\lim _{n \rightarrow \infty} \frac{g_{n}(x)}{n}=g(x)^{\geq 0} \quad \leq+\infty
$$

aypuecibyer, $n g$-uthbaps. $p$-yuus
(Dok-bo za pankave kypca)

Cregcrburs, Эpr. T. bupkroga (1)

$$
-354 \text { (2) }(y \star p ?)
$$

(1)

$$
\begin{aligned}
& g_{n}(x)=f(x)+f(T x)+\cdots+f\left(T^{n-1} x\right) \\
& g_{m+n}(x)=g_{n}(x)+g_{m}\left(T^{n} x\right)
\end{aligned}
$$

$\Downarrow$

$$
\frac{1}{n} \sum_{i=0}^{\Downarrow-1} f\left(T^{i} x\right) \rightarrow \widetilde{f}(x)
$$

$$
T \text {-ıpг } \Rightarrow \tilde{f}=\text { const },=\int_{\Omega} f(x) P(d x)
$$

(2)

2.7. Hpuroxeruce $k$ LPP.
(octrobtas uges)
$\vec{v} \in \mathbb{Z}_{\geqslant 0}^{2}, \quad T-$ cglair $\quad \operatorname{Ha}(-\vec{v})$


$$
\begin{aligned}
& \Omega=\left\{\left(w_{x y}\right)_{x, y \in z_{\geqslant 0}}\right\} \\
& P-\text { upogant-wера }
\end{aligned}
$$

$$
=L_{0 \rightarrow n \vec{u}}\left(T^{m}\left\{v_{x y}\right\}\right)
$$

$$
L_{0 \rightarrow(n+m) \vec{v}} \geqslant L_{0 \rightarrow m \vec{v}}+L_{m \vec{v} \rightarrow(n+m) \vec{v}}
$$

$\Rightarrow \varphi(\vec{v})=\lim _{n \rightarrow \infty} \frac{1}{h} L_{0 \rightarrow n} \vec{v} \quad$ agayeabyer

Veccuyrasirts (const)
T.k. $T$ - дproguyen b enry rezabuc. epagon (kax ugluz bepryum)

$$
\left[\begin{array}{c}
\text { geram - Seppa"Läinen, Lectures on Corner } \\
\text { Cerowth }
\end{array}\right]
$$

$$
\begin{aligned}
\varphi(\vec{v}), \quad \vec{v} \in \mathbb{Z}_{\geq 0}^{2} \quad \Rightarrow & \vec{v} \in \mathbb{Q}_{\geq 0}^{z} \\
& (\text { ogroupogroctio })
\end{aligned}
$$

Ceosciba $\quad \varphi(x, y) \quad(x, y) \in \mathbb{R} \geqslant 0$
(1) cumм. no $x, y$, ogropogria,
reysinganoyas no $x, y$
(2) $\varphi=+\infty$ им $<+\infty$ на всёл $\mathbb{R}_{\geqslant 0}^{2}$ thycro $\varphi<\infty$ :
(3) Cyuepagg. $\varphi(p+q) \geqslant \varphi(p)+\varphi(q)$
(4) Bortyia $\quad \gamma \varphi(p)+(1-\gamma) \varphi(q) \leqslant \varphi(\gamma p+(1-\gamma) q)$ $0 \leqslant \gamma \leqslant 1$
(5) tenpenobrita

Pf. (1) $V$
(2)

(3) Cynepagg. kax rupegen eynepagg. $L$.

$$
厶_{B}^{A} \quad L_{D \rightarrow A} \geqslant L_{O \rightarrow B}+L_{B \rightarrow A}
$$

(4)

$$
\begin{aligned}
& \gamma \varphi(p)+(1-\gamma) \varphi(q)= \\
& =\varphi(\gamma p)+\varphi((1-\gamma) q) \\
& \leq(\text { egnepcgg. }) \quad \varphi(\gamma p+(1-\gamma) q)
\end{aligned}
$$

(5) Bortyoocro $\Rightarrow$ reupepolbrocro

2.8. Abнble upeg. формит

- Jkenor. $W_{x, y}$ Exp $(\lambda)$
- guckp. atearoz
$\operatorname{Geom}(q)$
$\ldots$.... вCë !


$$
\underbrace{(x, y)=y\left[\frac{\left(\sqrt{\frac{x y}{y}}+1\right)^{2}}{1-q}-1\right]}
$$



THSEP: 2 mogxega
$\longrightarrow$ rugpogurcanuka
$\longrightarrow$ unterpupgenoct $\quad(90-e)$

3. Tugpoguramuka
(Rost 1981)
3.1. Jbpuerika
(1) nokan p-e $\mathrm{Fa} \mathbb{Z}$ He meruetas ygur guramure TASEP

(2) nokar p-e ta $\mathbb{Z}$ tpancer umbapuaritios
(3) aguyecolyer rokan. unortocto ("Iproguyro")
$\forall$ (Liggett 1976)
Oro pacupegerenve $\frac{\operatorname{Ber}(\rho)}{/}, \rho \in[0,1]$

$$
\begin{aligned}
& p(0)=\rho \\
& p(0)=1-\rho
\end{aligned}
$$

Hezalencumo nobckm $x \in \mathbb{Z}$

Thovk b Ber $(\rho)$ :

$$
j(\rho)=\rho(1-\rho)
$$

$\frac{\text { yp-e bropupca }}{\text { (reloljuoe) }} \rho_{t}+[\rho(1-\rho)]_{x}=0$

$(\leftrightarrow$ rapadore $)$ sbugetus planenulum.

24 mapia. Nekyus 3.

1. Tugroguseanuka TASEP
1.1. Nokambrel pabHobecue

(1) llokarbroe p-e ra $\{0,1\} \mathbb{Z}$ Crayuorrapito uph garemure TASEP
(2) lokarbros p-e ra $\{0,1\}^{\mathbb{Z}}$ Tpanca. urbapuaritho
(3) Cyugectyer nokanotial mornesto

$$
\begin{aligned}
& \rho=\lim _{N \rightarrow \infty} \frac{\nmid \text { yavery } b(-N / 2, N / 2)}{N} \\
& \text { (oproguintocio) } \\
& \text { thoy } \\
& \Rightarrow \text { Na.p-e ects } \quad \operatorname{Ber}(\rho) \quad(\text { Ligett } 1976)
\end{aligned}
$$

Tloton $\operatorname{Ber}(\rho)$

$$
g(\rho)=\rho(1-\rho)
$$


$\oint \mathbb{P}$ (uparixia $0 \rightarrow 1$ za bpanes $d t) / d t$
7p. Teoprana
$\Phi_{\text {akt }}$ veap woroka $\rho_{t}+(j(\rho))_{x}=0$
$\Downarrow$
broprepe

$$
\rho_{t}+[\rho(1-\rho)]_{x}=0 \quad \rho(0, x)
$$

Demenve $\rho(t, x)=\frac{1}{2}-\frac{x}{2 t} \quad \leftrightarrow$
1.2. Tlpobepka koppertroctu: craywortaprocso $\operatorname{Ber}(\rho)$

Leame Poiss $(\rho) \mathbb{Z}_{\geqslant n} \rho \in(0,1)$

$\Rightarrow t$

n pans vereynux ravin \& 4acry oT n gom Hezalon cumb)

Pf. $m=n+1$, no krygkyme (Burke progerty)

(dt) Buixeg

$$
\rho / 1-\rho
$$



$$
(\overrightarrow{0})=
$$

$$
\begin{aligned}
& \int_{(1-\rho)}^{0} d t \\
& \rho(1-\rho) \cdot \rho d t=0
\end{aligned}
$$

$$
\begin{aligned}
& \rho^{2} \rho_{0}^{\rho(1-\rho)(1-d t)}= \rho(1-d t) \\
&-\rho^{2}(1-d t) \\
&+\rho^{2} \\
&= \rho(1-d t)+\rho^{2} d t \\
& \cdots=\rho+0(d t) \\
&(1-\rho)(1-d t)=1-\rho+0(d t)
\end{aligned}
$$

Bn.lbos: buxegace cobur p-e - np-e

$$
\operatorname{Rer}(\rho) \times \operatorname{poiss}(\rho)
$$

Teopema. Ber(g) coxpanserss.

$P_{n}$ na $\mathbb{Z}_{\geqslant-n} \times \mathbb{R}_{\geqslant 0}$ corracobaribl
$\Rightarrow 3 \lim _{n \rightarrow \infty} \begin{gathered}\text { (korumiopal o } \\ \text { upagarkeveren uepul) }\end{gathered}$
$\Rightarrow$ Cray. grutamura wa Ber $(\rho)$ nuect taros byg.
2. Dorazartubrbo T. Murreta
2.) Aranorus: T. De Фuretru
(Eompyne wat-ho up. ранама - экк-д. тоини мpherepan:
mpoent tonn ambe)
$\mu$ tha $\{0,1\}^{R_{\geqslant 1}}$
Def. heperamoboyneas:


$$
\begin{gathered}
=\left(X_{b}(1), \ldots, X_{b}(n)\right) \\
\forall n, \forall B \in S_{n}
\end{gathered}
$$

$$
\gamma \mu_{2}+(1-\gamma) \mu_{2}, 0 \leqslant \gamma \leqslant 1
$$

$\{$ hepect. ug(b) $\}$ - bun. uroxecolo $A$

$$
f_{\text {ext }}=\{u \in A, \text { roo ecu }
$$

$$
\mu=\gamma \mu_{1}+(1-\gamma) \mu_{2} ; \mu_{1}, \mu_{2} \in A
$$

To $\left.\quad \mu_{1}=\mu_{2}=\mu . \quad\right\}$

T. De Qurcetru:

$$
\text { (1) } \operatorname{Aext}=\{\operatorname{Ber}(\rho): 10,[0,1]\}
$$

2) $\forall \mu \in A, \exists!\eta к а ~[0,1]$ 40

$$
\mu=\int_{0}^{1} \operatorname{Ber}(\rho) \eta(d \rho)
$$

2.2. Kannurer (Coupling) u nopegok Ha repax

$$
\Omega=\{0,1\} \mathbb{Z}
$$

Def.

$\Rightarrow$ "Coupled process" Ha $\{0,1\}^{\mathbb{Z}} \times\{0,1\}^{\mathbb{Z}}$ (corracobartionén upayeel)

Def.

$$
\begin{aligned}
& \delta, \eta \in\{0,1\}^{\mathbb{Z}} \\
& \xi \leq \eta \quad \forall x, x \in 3 \Rightarrow x \in \eta \\
& \xi=00
\end{aligned}
$$

Lema. $(\xi(t), \eta(t))$ - corrac. uproyece

$$
\begin{aligned}
& \xi(0) \leq y(0) \quad \Rightarrow \quad \xi(t) \leq y(t) \quad \forall t \\
&\left(\begin{array}{l}
= \\
\geqslant \\
< \\
>
\end{array}\right)-T 0 k e
\end{aligned}
$$

Def. $\mu_{1}, \mu_{2}$ bep. wegs) ra $\{0,1\}^{\mathbb{Z}}=\Omega$
$\mu_{1} \leqslant \mu_{2} \quad$ ecm unyecrbyer
vepre $v$ the $\Omega x \Omega$,

$$
\left.v\right|_{1}=\mu_{1},\left.\quad \nu\right|_{2}=\mu_{2}
$$

u $v(\xi \leqslant \eta)=1$.

$$
(\text { toxe gor }=,<, \geqslant \geqslant)
$$

2.3. Dterynnerue. Ybetroú TASEP, zagara o ractuye 2 knacca

(i) (i)

Urerce. 1

$$
\begin{align*}
& \rightarrow j  \tag{4}\\
& i>j \\
& i \leqslant j \text { ver upakka }
\end{align*}
$$



$$
-t \leq L(t) \leq t
$$

2.4. Cobücrba corn. upryecce


Lemus. $\mathcal{L}$ ra $\Omega \times \Omega$ - экстр, cias,
grue corn uproyecca

$$
\begin{gathered}
\Rightarrow D(\eta-\xi), \quad v(\eta \leqslant \xi), \quad v(\eta \geqslant \xi) \\
=0 \text { или } 1
\end{gathered}
$$

Pf, $\{\xi=\eta\}, \ldots$ - uklogs wogutho wou Tyers $B$-uribap., $0<\underbrace{\nu(B)<1}_{=p} \underbrace{}_{i}$ TASEP

$$
\nu(\cdot)=p \nu(\cdot \mid B)+(1-p) \nu\left(-\mid B^{c}\right)
$$

$$
\begin{gathered}
\Rightarrow v(1 \mid B), \nu\left(\cdot \mid B^{C}\right)-\text { TOXe cray. vep } b \mid \\
T \cdot k^{\prime} . B-\text { cray. }
\end{gathered}
$$

$\Rightarrow p=0$ un 1 b ary thag
2.5. 「rabная теореиа (o nonrom nopegre) (ancurs sy>0-4ytr getares)

Th. $\mu_{1}, \mu_{2}$ на $\Omega=\left\{0,13^{\mathbb{Z}}\right.$

- cray, TP-uke., экcTP.

Mepual gus TASEP.
Torga $\mu_{1} \geqslant \mu_{2}$ um $\mu_{1} \leqslant \mu_{2}$.
kux. T.e. $\{\mu\}$ - bwore ywopregovemo, yto ramekaer the 1-napame ipurtroro

Pf 1) Xorum $V$ ra $\Omega \times \Omega$ cian, jop.ureb, Jocy.
e upoekyus un $\mu_{1}, \mu_{2}$.
Harrié c $\nu_{0}=\mu_{1} \times \mu_{2} ; \quad \nu_{t}=$ TASEP ciapirynougú $C$ Vo.
bce $\nu_{t}$ uverot upoexymi $\mu_{1}, \mu_{2}$
uTr- - unb.
$\Omega \times \Omega-$ vounakt $\quad\left(\simeq[-1,1]^{2}\right)$ m-leo bexp wep na $\Omega \times \Omega$ - Toke

$$
\begin{aligned}
\Rightarrow & \exists t_{n} \rightarrow \infty, \\
& \frac{1}{t_{n}} \int_{0}^{t_{n}} \nu_{t} d t \rightarrow v, n \rightarrow \infty
\end{aligned}
$$

$\Rightarrow \nu-$ CTay.

- тр uthl kak yreger $\nu_{t}$
- иneet upoengur $\mu 1, \mu_{2}$

2) Otardie rousth ancosp $\nu$
nyers $\quad A=\{b$ - way. Tp. nucb. rea $\Omega \times \Omega$ a nsoecyuem $\left.\mu_{1}, \mu_{2}\right\}$
$A$ - voun., boin., teryerde (cu.1)

$$
\Rightarrow A_{\text {ext }} \neq \varnothing \quad(\text { Kperrir-Muroware } \text { ) }
$$

Выпуклый компакт К в локально выпуклом пространстве L совпадает с замыканием выпуклой оболочки множества своих крайних точек Lext
329. Пусть $K$ - выпуклое множество в линейном протранстве $L$. Подмножество $A \subset K$ называется крайним, сли всякий отрезок, лежащий в $K$, середина которого гринадлежит $A$, целиком лежит в $A$. Доказать, что персечение любого семейства крайних подмножеств либо уусто, либо является крайним нодмножеством.
330. Пусть $K$ - выпуклый комшакт. Доказать, что созокупность его замкнутых крайних подмножеств (см. задачу 329), упорядоченная по включению, имеет минпмальныї элемент.
331. Пусть $K$ - замкнутое выпуклое ограниченное подмножество в ЛВП $L, \quad A$ - минимальный элемент ссмейства замкнутых крайних нодмножеств в $K$ (см. задачу 330). Доказать, что $A$ состоит из одной точки.
332. Доказать, что выпуклый компакт $K$ в ЛВП $L$ имеет хотя бы одну крайнюю точку.

333*. Доказать теорему Крейна - Мильмана: всякиі̆ выпуклый компакт $K$ в ЛВП $L$ совпадает с замыканием выпуклой оболочки своих крайних точек.

Tycro $v \in$ Aext.
Torga $v$ - okcip o kracce bex

$$
\text { cray. tp-nreb mep ra } \Omega \times \Omega
$$

Decier, $\nu=p \alpha+(1-p) \beta$
Bozovër usperyuso, $\mu_{1}, \mu_{2}$ ancf.

$$
\Rightarrow \quad \alpha, \beta \in A \Rightarrow \alpha=\beta=\nu .
$$

3) Torga ${ }_{p}(S \leqslant \eta \geqslant q)$

Dociroureo govagar, uso $=1$.
Dociaroureo govagar, uso $\forall x \in \mathbb{Z}$,

$$
\nu\left[\begin{array}{ll}
0 & 0 \\
0 & 0
\end{array}\right]=\nu\left[\begin{array}{ll}
0 & 0 \\
x & x+1
\end{array}\right]=0
$$

T.K. ecun
 ogruakobirie bce 0 unu
$\Rightarrow \exists$ tetyyebas beg-Tb, UTO Hru komghr. cosepyrices pogon za korcerpoe bpaus

$$
\text { T.e. } V\left[\begin{array}{ll}
0 & 0 \\
0 & 0
\end{array}\right]>0
$$

x $x+1$

Darome: Coporo - can Liggett 1976

$$
L .2 .4,2.5,3.1
$$



$$
b \quad\left[\begin{array}{ll}
0 & 0 \\
0 & 0
\end{array}\right],
$$

no y thac cray. $p-e$ t.e "y*e cumniocb".
2.6. Nocregnui ulas g-ba
$\operatorname{Ber}(\rho)$ - Tp.ures eray.

$$
\begin{aligned}
& \text { Takxe } \operatorname{Ber}(\rho)-\operatorname{skc} \beta \text {. } \\
& (\mu=p \alpha+(1-\rho) \beta \Rightarrow
\end{aligned}
$$

voyram $\alpha \ll 1 / p \mu$

$$
\begin{aligned}
& d \alpha=\frac{1}{p} f d \mu \\
& f-T p \cdot u \mu b \Rightarrow \text { const) }
\end{aligned}
$$

Wyers $\mu$ - tp. und way. f okerp. wa $\Omega$
$\forall \rho, \quad \operatorname{Ber}(\rho) \geqslant \mu$ weac $\operatorname{Ber}(\rho) \leqslant \mu$.

$$
\rho_{0}:=\sup (\rho: \mu \geq \operatorname{Ber}(\rho))
$$

Hyers $f$ - moror. rup. op. He $l$
$\rho \longmapsto \int_{\Omega} f d \operatorname{Ber}(\rho)$ monot ap war $\rho$
T.k. $\quad \int f d \mu=\int f d \operatorname{Ber}\left(\rho_{0}\right) \quad \forall f$
$\Rightarrow$ (Jru op oup-uepy) $\mu=\operatorname{Ber}\left(\rho_{0}\right)$

Teop. Nurretta osocrobblbaer rauto nugpogurcumatich $b$ TASEP, gue ocranomio ruyaribl Ip,4,17. u euè nopasotat6 $\longrightarrow$ uppesgém th urvenpurpgemocte.

(KPZ) Kardar - Parisi-Zhang enge cero "glingrigupyroyas ugpogurametra" upegckagbibenoyas gronuoty agum

$$
x_{N}\left(T_{N}\right)-N X=N^{1 / 3} \xi
$$

$$
\begin{aligned}
h(t, x): \quad h_{t} & =\frac{1}{2} h_{x x}+\left(h_{x}\right)^{2}+W \\
z & =e^{h^{\prime}}, z_{t}
\end{aligned}=\frac{1}{2} z_{x x}+W z \quad \text { sexaur }
$$

3. Интегрируешоcio e TASEP'e

- minoso cuocorab
- bee clozezarua e ger.uproy. \& unoro det.
(1)

$$
\operatorname{det}_{N \times N}\left(a_{i j}(N)\right)
$$

$$
\begin{aligned}
& x_{i}(0)=-i \\
& x_{N}(t)+N \stackrel{d}{=} \lambda_{N} \\
& \lambda_{N} \rightarrow u_{z} \quad \lambda=\left(\lambda_{1} \prod_{2} \ldots \geqslant \lambda_{N} \geqslant 0\right)
\end{aligned}
$$

$$
\text { C Johanssovs } 2000
$$

Borodim-Ferrari 2008)
RSK $\longrightarrow$ gerepm. то\%. upoyeect
$\longrightarrow$ acumnrotuka vepez $\oint$

$$
\begin{aligned}
& N, t \rightarrow \infty
\end{aligned}
$$

(2) Or yus ungráa TASEP © $N$ Yact.

$$
\begin{aligned}
P\left(y_{N}-y_{1}\right. & \left.\xrightarrow{t} x_{N}-x_{1}\right) \\
& =\operatorname{det}_{N \times N}\left[F_{i-j}\left(x_{N+1-i}-y_{N+1-j}, t\right)\right] \\
F_{n}(x, t) & =\frac{(-1)^{n}}{2 \pi_{i}} \oint_{0,1} \frac{d-w}{\omega} \frac{(1-\omega)^{-n}}{\omega^{x-n}} e^{t(\omega-1)}
\end{aligned}
$$

$$
(\text { Schutz, } 1997)
$$

Arzay bete (1930-e)
( $\simeq$ pasbiaer gus TASEP, 4-TASEP, ASEP,..)
$\rightarrow$ Mrowo bbiukco, yngoyeruí det
$\rightarrow$ KPZ fixed peint (2017)
Matetski-Quastel - Remenik

Лекуия 4
31 мартА

1. Ot TASEP К 山естивер山инной mодeли
2. 


$\rightarrow$
ASEP


Функynи Llypa

Mhterpupyembe cucteubl YACTUK

Ha ypobre 6B vogem ygosro nowruto renotofible pop-uyubl gus acuentotukn.
1.2. GB mogent



$$
\begin{aligned}
& \mathbb{P}(\omega)=\frac{1}{z} a_{1}^{\#(+)} a_{2}^{\#(+)} b_{1}^{\#(+)} c_{1}^{\#(+)} b_{2}^{\#(t)} c_{2}^{\#(+)} \\
& Z=\sum a_{1}^{\#(+)} a_{2}^{\#(+)} b_{1}^{\#(+)} c_{1}^{\#(+-)} b_{2}^{\#(t)} c_{2}^{\#(+)}
\end{aligned}
$$

becur
gouycrimblu kongmz. c gartiblum zp yca.
1.3. Metopus


- 1930e, Hayaunz
- 1960e, arzay bere (Lieb, Yaug-Yang,...)
- 1980e, aızesp arzay bete

01992,2014 - crox. $6 b$ moge to

- 1992 - 1995 - 3nakove pegy royneces watpuyb)
- 2015 - upaper u 2D rég

Published: 25 March 2015

# Square ice in graphene nanocapillaries 

G. Algara-Siller, O. Lehtinen, F. C. Wang, R. R. Nair, U. Kaiser $\boxtimes, ~ H . ~ A . ~ W u ~ \boxtimes, ~ A . ~ K . ~ G e i m ~ \& ~ I . ~ V . ~$ Grigorieva $\square$

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7140 Accesses $\mid \mathbf{3 9 0}$ Citations $\mid 280$ Altmetric $\mid$ Metrics


#### Abstract

Bulk water exists in many forms, including liquid, vapour and numerous crystalline and amorphous phases of ice, with hexagonal ice being responsible for the fascinating variety of snowflakes ${ }^{1,2}$. Much less noticeable but equally ubiquitous is water adsorbed at interfaces and confined in microscopic pores. Such low-dimensional water determines aspects of various phenomena in materials science, geology, biology, tribology and nanotechnology ${ }^{3,4,5,6,7,8}$. Theory suggests many possible phases for adsorbed and confined water ${ }^{9,10,11,12,13,14,15,16,17}$, but it has proved challenging to assess its crystal structure experimentally ${ }^{17,18,19,20,21,22,23}$. Here we report high-resolution electron microscopy imaging of water locked between two graphene sheets, an archetypal example of hydrophobic confinement. The observations show that the nanoconfined water at room temperature forms 'square ice'-a phase having symmetry qualitatively different from the conventional tetrahedral geometry of hydrogen bonding between water molecules. Square ice has a high packing density with a lattice constant of $2.83 \AA$ and can assemble in bilayer and trilayer




a

d
(Praн.ycrabue gomentros verlku)

(1)

$$
a_{1}=a_{2}=b_{1}=b_{2}=c_{1}=c_{2}=1 \rightarrow 4 \text { cro "3ranorepe- }
$$ gyrounut ces wappry"

$$
Z=\prod_{k=0}^{n-1}(3 k+1)!/(n+k)!
$$

(Zeilberger - Kuperberg)
(2)

$$
a_{1}=a_{2}=b_{1}=b_{2}=1, \quad c_{1}=c_{2}=\sqrt{2}
$$

3anoyeruels gometromka nee
Aytenckoto spurnuanta
(3) $a_{2}=0, a_{1}=b_{1}=b_{2}=c_{1}=c_{2}=1$


$$
\begin{aligned}
& Z=\prod_{i=1}^{a} \prod_{j=1}^{b} \prod_{k=1}^{c} \frac{i+j+k-1}{i+j+k-2} \\
& (\text { MakMaron } 1900-c)
\end{aligned}
$$

2. Ctoxactuqeckat 6 м moдenb
2.). Onределение


$$
0 \leq b_{1}, b_{2} \leq 1
$$



$$
z^{\text {stoch }}=1
$$


2.2. Burpoxgenus

$$
\begin{aligned}
\rightarrow-(1 a) \rightarrow & b_{2}=0 \\
\rightarrow & (10) \rightarrow \\
b_{2}=0, & b_{1} \rightarrow 1 \\
(2) \rightarrow & \text { ASEP } b_{1}, b_{2} \rightarrow 0 \\
& (\text { TASEP kak } \\
& \text { Частнийй сауний }
\end{aligned}
$$

bpeas $T \in \mathbb{Z} \geq 0$
(1a)


$$
\begin{gathered}
12345 \\
+1000000
\end{gathered}
$$

$\int_{0} 1-b_{1}$
upuraet reanyabo $c$ bep. 1-bl)
(18) u buex towner.

$$
\begin{aligned}
& b_{1}-1, \\
& T=\left\lfloor\tau / 1-b_{1}\right\rfloor
\end{aligned}
$$

$$
\tau \in \mathbb{R}_{\geqslant 0}
$$

© urtenc.1,
a tonraet bcex
(2)

2.3. Ctaluoraphblu pexum

Th. Crox $6 b$ coxparser $\operatorname{Ber}(\rho)$ ra $\mathbb{Z}$


$$
p \in[0,1]
$$

Pf. (1)


Ecm

$$
\begin{aligned}
& \left(1-b_{2}\right) \rho\left(1-\rho^{\prime}\right)= \\
& =\left(1-b_{1}\right) \rho^{\prime}(1-\rho)
\end{aligned}
$$

$\Downarrow$
pacup coxp.


| $\vdots$ |
| :---: |
| $\vdots$ |
| $\vdots$ |



个.. atta norut ko.
(2) Coreacobannocob

repyy rea myrax we $\mathbb{Z}^{2}$
गT и ест. cray. $6 b$ wgers e $\operatorname{Ber}(\rho)$

Pugroguramika/ upegenstar gropua toxe $\exists$.
2.4. Pyrkyus bblcotbl

$h(x, y)=$ \#nyter cupaba or kretkn $(x, y)$


UHTER рируемостb: hocipoum этот $\uparrow$ mapkobcunio war tha grymux ugerx, qge vor wotem lece nocrutatb.
Ho tyT mbl тoxe sygen nomozobinaces blque. uggenlun, JTo ygostee bcezo.
3. Bepuiиmная mogenv Xorra - Mutтıbga
("gegopmipobaritibix Sozorob")
3.1 Beca bepuum

$$
0 \leqslant t<1
$$


pukc.
(nogens bilcorozo cunka)

$$
\mathbb{C}\left[u_{1}, \ldots u_{N}\right]^{S(N)}
$$

$\lambda_{i} \in \mathbb{Z}$
$d=\left(\lambda_{1} \geqslant \lambda_{2} \geqslant \cdots \geqslant \lambda_{N} \geqslant 0\right) \longleftrightarrow$ Korqu2. cTpenok
Def (4epez nquepep) $\quad \lambda=(4,4,3,1,1,1,0,0)$


$$
\begin{aligned}
& \begin{array}{l}
m_{i}=m_{i}(\lambda) \\
(i \geq 0)
\end{array} \quad-\text { wратrocto } i \quad b \quad \lambda \\
& \quad m_{1}(\lambda)=3 \quad m_{2}(\lambda)=0
\end{aligned}
$$

Ruak. $\lambda$ - pazinerus

- guarp torza $C \leq N$
crpok
- curratypul (coapmue beca

$$
\left.u_{p-i}^{e} G L_{N}\right)
$$

$$
0 \leq t<1
$$


3.2. Mhoro4nerib) Xoma-Nuttabyga

Def.
ige $(t, t)_{k}:=(1-t)\left(1-t^{2}\right)-\left(1-t^{k}\right)$.

$$
\begin{gathered}
J=\left(\begin{array}{lll}
5 & 22 & 0
\end{array}\right) \\
N=4
\end{gathered}
$$



Def. $\quad F_{\lambda}\left(u_{1}, \ldots u_{N}\right)=P_{\lambda}\left(u_{1},-, u_{N}\right) \cdot \prod_{i \geqslant 0}\left(t_{j} t\right)_{m_{i}(\lambda)}$
Prop. (1) $P_{\lambda}$-ograp. nompram or $u_{i}$, ctenemen $|\lambda|=$
(2) $P_{\lambda}\left(u_{1}, \ldots, u_{N}\right)=\underbrace{u_{1}^{\lambda_{1}} u_{2}^{\lambda_{2}} \ldots u_{N}^{\lambda_{N}}+\ldots}$ $=\lambda_{1}+\cdots+\lambda_{N}$
nexcukorpap. Cragдий'́ чnek
(3) $\begin{aligned} P_{\left(\lambda_{1}, \ldots, \lambda\right)}\left(u_{1} \ldots u_{N-1}, 0\right)\end{aligned}=\left\{\begin{array}{c}P_{\lambda_{1} \ldots \lambda_{N-1}}\left(u_{1} \ldots u_{N-1}\right), \\ \lambda_{N}=0 \\ 0, \text { uraye }\end{array}\right.$ $(y \cap P$.

rekc. havc. yrex

[Makgorarog; "Cumus Qp. u urororrertal Xonka] $\mid$ Замечание. Многочлены $R_{\lambda}\left(x_{1}, \ldots, x_{n} ; t\right)$, определенные в § 1 , не обладают этим свойством стабильности, поэтому они представляют меньший интерес.

$$
R_{\lambda}\left(x_{1}, \ldots, x_{n} ; t\right)=\sum_{w \in s_{n}} w\left(x_{1}^{\lambda_{1}} \ldots x_{n}^{\lambda_{n}} \prod_{i<l} \frac{x_{i}-t x_{j}}{x_{i}-x_{j}}\right)
$$

(govaklem qapruyy $\Sigma$ nozgree)

Ex. $\quad t=0, \quad P_{\lambda}=S_{\lambda}, \quad \phi$. Whypar;

$$
s_{\lambda}\left(u_{1} \ldots u_{N}\right)=\operatorname{det}\left[u_{i}^{\lambda_{j}}+N-j^{0}\right]_{1}^{N}
$$

3.3. Cumиетрия $P_{\lambda}$ 4ерез ype 9-6.


Thus.
(yp), Arera-
Бakctepa)


$i_{123}, j+23$ pluke.

HL weights (RED)


```
m(|)= w[u_][i1_, j1_, i2_, j2_] := If[i1 + j1 == i2 + j2 && i1 \geq 0&& i2 \geq 0&& 1 >= j1 \geq 0 && 1 >= j2 \geq 0,
    If[i1 == i2 && j1 == j2 == 0, 1, 0] + If[i1 == i2 && j1 == j2 == 1,u, 0] + If[i1 +1 == i2 && j1 == 1&& j2 == 0, 1 - t^ i2, 0] +
    If [ i1-1 == i2 && j1 == 0&& j2 == 1,u, 0], 0]
```


## Cross weights

```
m(f)|=R[z_][i1_,j1_, i2_, j2_] := If [i1 + j1 == i2 + j2&& 1 >= i1 \geq0&& 1 >= i2 \geq 0&& 1 >= j1 \geq 0&& 1 >= j2 \geq 0,
    X[z][i1, j1, i2, j2], 0]
m(m)= }X[\mp@subsup{z}{_}{\prime}][0,0,0,0]:= 1
    X[z_][1, 1, 1, 1] := 1;
    X[z_][1, 0, 0, 1] := z (1-t)/(1-zt);
    X[z_][1, 0, 1, 0] := (1-z) / (1-z t);
    X[z_][0, 1, 0, 1] := t (1-z)/(1-zt);
    X[z_][0, 1, 1, 0] := (1-t) / (1-zt);
```

YBE
$m(s)=m:=3$
$m_{m p]}=\operatorname{Table}[\operatorname{Sum}[w[v][i 3, k 1, k 3, j 1] \times w[u][k 3, k 2, j 3, j 2] \times R[u / v][i 1, i 2, k 1, k 2],\{k 1,0,1\},\{k 2,0,1\},\{k 3,0, m+1\}]-$ Sum [w[v][k3, i1, j3, k1] $\times w[u][i 3, i 2, k 3, k 2] \times R[u / v][k 1, k 2, j 1, j 2],\{k 1,0,1\},\{k 2,0,1\},\{k 3,0, m+1\}]$, $\{i 1,0,1\},\{i 2,0,1\},\{i 3,0, m\},\{j 1,0,1\},\{j 2,0,1\},\{j 3,0, m\} / / S i m p l i f y$
ourtil $\{\{\{\{\{\{0,0,0,0\},\{0,0,0,0\}\},\{\{0,0,0,0\},\{0,0,0,0\}\}\}$,
$\{\{\{0,0,0,0\},\{0,0,0,0\}\},\{\{0,0,0,0\},\{0,0,0,0\}\}\},\{\{\{0,0,0,0\},\{0,0,0,0\}\}$,
$\{\{0,0,0,0\},\{0,0,0,0\}\}\},\{\{\{0,0,0,0\},\{0,0,0,0\}\},\{\{0,0,0,0\},\{0,0,0,0\}\}\}\}$, $\{\{\{\{0,0,0,0\},\{0,0,0,0\}\},\{\{0,0,0,0\},\{0,0,0,0\}\}\},\{\{\{0,0,0,0\},\{0,0,0,0\}\}$, $\{\{0,0,0,0\},\{0,0,0,0\}\}\},\{\{\{0,0,0,0\},\{0,0,0,0\}\},\{\{0,0,0,0\},\{0,0,0,0\}\}\}$, $\{\{\{0,0,0,0\},\{0,0,0,0\}\},\{\{0,0,0,0\},\{0,0,0,0\}\}\}\}\}$, $\{\{\{\{\{0,0,0,0\},\{0,0,0,0\}\},\{\{0,0,0,0\},\{0,0,0,0\}\}\}$, $\{\{\{0,0,0,0\},\{0,0,0,0\}\},\{\{0,0,0,0\},\{0,0,0,0\}\}\},\{\{\{0,0,0,0\},\{0,0,0,0\}\}$, $\{\{0,0,0,0\},\{0,0,0,0\}\}\},\{\{\{0,0,0,0\},\{0,0,0,0\}\},\{\{0,0,0,0\},\{0,0,0,0\}\}\}\}$, $\{\{\{\{0,0,0,0\},\{0,0,0,0\}\},\{\{0,0,0,0\},\{0,0,0,0\}\}\},\{\{\{0,0,0,0\},\{0,0,0,0\}\}$, $\{\{0,0,0,0\},\{0,0,0,0\}\}\},\{\{\{0,0,0,0\},\{0,0,0,0\}\},\{\{0,0,0,0\},\{0,0,0,0\}\}\}$, $\{\{\{0,0,0,0\},\{0,0,0,0\}\},\{\{0,0,0,0\},\{0,0,0,0\}\}\}\}\}$


4. Dboñcbenmble mк. $X \wedge$, тожgeetbo Koun $4.1 Q_{\lambda}$



Ex. $\quad P_{\lambda}=b_{\lambda} \quad Q_{\lambda}$ $b_{\lambda}=$ ? tee zale. or $v_{i}$

$$
\text { Ex. } \quad Q_{\lambda}\left(v_{1}-v_{N}\right)^{v_{i}}
$$ curnes. no $V_{i}$

Заmeyarue.

$$
Q_{\lambda}\left(v_{1} \ldots v_{M}\right)=\frac{\pi_{i \geqslant 0}(t, t)_{M_{i}(\lambda)}}{(t, t)_{N}} Q_{\lambda}^{*}\left(v_{1} \ldots v_{M}\right)
$$

4.2. Красно-синее


HL weights (BLUE)

```
mmpl) wstar[v_][i1_, j1_, i2_, j2_] := If [i1 + j2 == i2 + j1 && i1 \geq0&& i2 \geq0&& 1 >= j1 \geq 0&& 1 >= j2 \geq0,
    If[i1 == i2 && j1 == j2 == 0, 1, 0] + If[i1 == i2 && j1 == j2 == 1, v, 0] + If[i1 - 1 == i2 && j1 == 1&& j2 == 0, 1, 0] +
        If[i1 +1 == i2&& j1 == 0&& j2 == 1,v(1-t^ i2), 0], 0]
```

Weights R-tilde


```
            Xt[z][i1, j1, i2, j2], 0]
m|m= Xt[z_][0, 0, 0, 0] := 1;
Xt[z_][1, 1, 1, 1] := t;
Xt[z_][0, 0, 1, 1] := (1-t) z/(1-z);
Xt[z_][1, 0, 1, 0] := (1-tz)/(1-z);
Xt[z_][0, 1, 0, 1] := (1-tz)/(1-z);
Xt[z_][1, 1, 0, 0] := (1-t)/(1-z);
```

YBE 2
mpob Table[Sum[wstar [v][i3, k1, k3, j1] $\times \mathbf{w}[u][k 3, k 2, j 3, j 2] \times$ Rtilde[uv][i1, i2, k1, k2], \{k1, 0, 1\}, $\{k 2,0,1\},\{k 3,0, m+1\}]-\operatorname{Sum}[w s t a r[v][k 3, i 1, j 3, k 1] \times w[u][i 3, i 2, k 3, k 2] \times R t i l d e[u v][k 1, k 2, j 1, j 2]$, $\{k 1,0,1\},\{k 2,0,1\},\{k 3,0, m+1\}],\{i 1,0,1\},\{i 2,0,1\},\{i 3,0, m\},\{j 1,0,1\},\{j 2,0,1\},\{j 3,0, m\} / /$ Simplify
ай刀二= $\{\{\{\{\{\{0,0,0,0\},\{0,0,0,0\}\},\{\{0,0,0,0\},\{0,0,0,0\}\}\}$,
$\{\{\{0,0,0,0\},\{0,0,0,0\}\},\{\{0,0,0,0\},\{0,0,0,0\}\}\},\{\{\{0,0,0,0\},\{0,0,0,0\}\}$, $\{\{0,0,0,0\},\{0,0,0,0\}\}\},\{\{\{0,0,0,0\},\{0,0,0,0\}\},\{\{0,0,0,0\},\{0,0,0,0\}\}\}\}$, $\{\{\{\{0,0,0,0\},\{0,0,0,0\}\},\{\{0,0,0,0\},\{0,0,0,0\}\}\},\{\{\{0,0,0,0\},\{0,0,0,0\}\}$, $\{\{0,0,0,0\},\{0,0,0,0\}\}\},\{\{\{0,0,0,0\},\{0,0,0,0\}\},\{\{0,0,0,0\},\{0,0,0,0\}\}\}$, $\{\{\{0,0,0,0\},\{0,0,0,0\}\},\{\{0,0,0,0\},\{0,0,0,0\}\}\}\}\}$,
$\{\{\{\{\{0,0,0,0\},\{0,0,0,0\}\},\{\{0,0,0,0\},\{0,0,0,0\}\}\}$,
$\{\{\{0,0,0,0\},\{0,0,0,0\}\},\{\{0,0,0,0\},\{0,0,0,0\}\}\},\{\{\{0,0,0,0\},\{0,0,0,0\}\}$, $\{\{0,0,0,0\},\{0,0,0,0\}\}\},\{\{\{0,0,0,0\},\{0,0,0,0\}\},\{\{0,0,0,0\},\{0,0,0,0\}\}\}\}$, $\{\{\{\{0,0,0,0\},\{0,0,0,0\}\},\{\{0,0,0,0\},\{0,0,0,0\}\}\},\{\{\{0,0,0,0\},\{0,0,0,0\}\}$, $\{\{0,0,0,0\},\{0,0,0,0\}\}\},\{\{\{0,0,0,0\},\{0,0,0,0\}\},\{\{0,0,0,0\},\{0,0,0,0\}\}\}$, $\{\{\{0,0,0,0\},\{0,0,0,0\}\},\{\{0,0,0,0\},\{0,0,0,0\}\}\}\}\}\}$
4.3. Toxgectbo Koun

Th. $\sum_{\lambda} P_{\lambda}\left(u_{1, \ldots}, u_{N}\right) Q_{\lambda}\left(v_{1} \ldots v_{m}\right)=\prod_{i_{1} j} \frac{1-t u_{i} v_{j}}{1-u_{i} v_{j}}$
ecour $\left|u_{i} v_{j}\right|<1 \quad \forall i_{j}$
Pf. Dokaxem

$$
\sum_{\lambda} F_{\lambda}\left(u_{1}, \ldots, u_{N}\right) Q_{\lambda}^{*}\left(v_{1} \ldots v_{m}\right)=(t, t)_{N} \cdot \prod_{i, j} \frac{1-t u_{i} v_{j}}{1-u_{i} v_{j}}
$$

(2To arlabanenono $P_{\lambda} Q_{\lambda}$ tosgectley)
$n_{p}, 4$
$\lambda$

4.4. Mepbs Xома - Nurtnbyga

$$
\begin{gathered}
\left|u_{i} v_{j}\right|<1 \quad \forall i, j, \quad u_{i}, v_{j} \geqslant 0 \\
P_{r o b}(\lambda)=\frac{1}{z} P_{\lambda}\left(u_{1} \ldots u_{N}\right) Q_{\lambda}\left(v_{1} \ldots v_{m}\right) \\
Z=\prod_{i, j} \frac{1-t u_{i} v_{j}}{1-u_{i} v_{j}}
\end{gathered}
$$

$t=0$ : Mepal L Lypa (Okyrukob 1999)

Th. (govaxem b cilg.pay)

Cnocos gok-ba: bepestrocirtuis (Mapkobckud) anzopute, pazbliphblbarouguer mepy $X_{\lambda}\left(u\right.$ bce $\lambda^{(x, y)} \forall x, y$ ogrob panertro)

Tak, Moo $m_{0}\left(\lambda^{(x, y)}\right)$ meruetrs majkobckur ospagou u cobnagaet e $q$. lebicotbi b GB.



Лекция 5
ҰAпpeng
Hectubep山йhas MOAEлb
И mhoro4nemb Xonna - へuttnbyдa

1. Hanomuharus
1.1

$b_{1}, b_{2} \in[0,1]$

Ctoxaciuyeckas 66 mogenv


$$
\begin{aligned}
& \text { 1.2. } \\
& \text { MK-H } X-\Lambda \text {. }
\end{aligned}
$$

$$
\begin{aligned}
& m_{i}(\lambda) \text { - kpartsorts } \\
& i b \lambda \\
& (t, t)_{k}=(1-t)\left(1-t^{2}\right)-\left(1-t^{k}\right) .
\end{aligned}
$$


1.3. Toxgectbo Komn

$$
\begin{aligned}
& \sum_{\lambda} P_{\lambda}\left(x_{1} \ldots x_{N}\right) Q_{\lambda}\left(y_{1} \ldots y_{M}\right)=\prod_{i j} \frac{1-t x_{i} y_{j}}{1-x_{i} y_{j}} \\
& \left|x_{i} y_{j}\right|<1 \quad \forall i, j
\end{aligned}
$$

II

$$
\sum_{\lambda} F_{\lambda}\left(x_{1} \ldots x_{N}\right) Q_{\lambda}^{*}\left(y_{1} \ldots y_{M}\right)=(t, t)_{N} \prod_{i, j} \frac{1-t x_{i} y_{j}}{1-x_{i} y_{j}}
$$



$$
\left[\prod_{i, j} \frac{1-t x_{i} y_{i} j}{1-x_{i} y_{j}}\right]
$$


1.4. Pytekyus bbleotbl

$\uparrow\left(a_{1}\right)\left(u_{2}\right)$ ( $\pi_{3}$.

$$
x=0
$$

Heogropogras mogeno

$$
\underbrace{b_{1}\left(u_{x}, v_{y}\right)=\frac{t-u_{x} v_{y}}{1-t u_{x} v_{y}}}_{x-1} \longrightarrow \vdots \vdots
$$


(anpaguts no gubzoreande)

Teopena. $\forall x, y$-quxc.

$$
\begin{aligned}
& h(x, y) \stackrel{d}{=} m_{0}\left(\lambda^{(x, y)}\right)=y-l\left(\lambda^{(x, y)}\right), \\
& \lambda^{(x, y)} \sim \text { repe } \quad x-1 \\
& \operatorname{Rrob}(\lambda)=\frac{1}{z} P_{\lambda}\left(v_{1, \ldots,}, v_{y}\right) Q_{\lambda}\left(u_{1, \ldots,} u_{x}\right)
\end{aligned}
$$

(takke еств $\frac{\text { кevoropore cobu-pacup-s) }}{\text { bgore }}$

2. Dokazarenbctbo cooberctbus $6 b-X 1$
2.). Kocse toxg. Komu

$\Pi_{\text {negs }}$


$$
\left|x_{y}\right|<1
$$

$$
\begin{aligned}
& \underbrace{\sum_{\nu} f_{v / \lambda}(x) Q_{v / \mu}^{*}(y)=\frac{1-t x y}{1-x y}}_{\text {sece}} \cdot \\
& * \underbrace{\sum_{x e} F_{\mu / x}(x) Q_{\lambda / x e}^{*}(y)}_{\text {leore }}
\end{aligned}
$$

D-lo. Ha4unas c $æ$


2.2. Cnyयайнbй was

$$
\begin{aligned}
& \begin{array}{l|l}
h(x-1, y) & \frac{4(x, y)}{h(x-1, y-1)} \\
h(x, y-1) & \frac{4 \text { ceyzees }}{h \vdots}: \frac{h+2}{h+1} h
\end{array} \\
& \left.\begin{array}{c}
h+1 \\
h+1
\end{array}\right] h \\
& \text { Xotum: } \\
& h(x, y) \stackrel{d}{=} \operatorname{mo}\left(\lambda^{(x, y)}\right) \\
& \xrightarrow[h]{h}
\end{aligned}
$$

Cgerame:

$v$| $\mu$ | $v$ |
| :--- | :--- |
| $x$ | $\lambda$ |

(u)

Tak, 4 To
$n_{0}(v)$ zabucut OT $m_{0}(\mathscr{P}), m_{o}(\mu), m_{0}(\lambda)$ Tak *e kakb $6 b$.


$$
\begin{gathered}
\frac{1-\operatorname{tuv}}{1-u v} \sum_{x} P_{\mu \not x}(v) Q_{\lambda / x}(u) U(v \mid \mu, x, \lambda) \\
\frac{y_{n p \cdot p}}{}=P_{v)_{\lambda}}(v) Q_{v / \mu}(u) .
\end{gathered}
$$

$\left[\forall U(v \mid \lambda, x, y)\right.$ gaëт none $\lambda^{(x, y)}$ corn. e mepanu $\left.X n\right]$


(1)
(non)
$\operatorname{Prob}(\mu, x, \lambda)$

$$
=\frac{1}{z} Q_{\mu}\left(u_{1} \ldots u_{x-1}\right) P_{\mu / x}(v) Q_{\lambda / x}(u) P_{\lambda}\left(v_{1}, \ldots, v_{y-1}\right)
$$

(1) $\sum_{\mu_{1} \lambda}$ gaëт $\frac{1}{z} P_{x} Q_{x}$
(2) $\sum_{\mu, x}, \sum_{\lambda, x}$ gаёт мариин. paен.
(3) $U, \sum_{x}$ geièt cobm. pacup. $v, \lambda, \mu$.
(4) $U, \sum_{\lambda, x, \mu}$ gaèt $X_{\lambda}$ vepy no 0.

2,3. Элемектартвіи́ слуraírloús шаг
(bijectivisation, probabikstic bijection,
 Kanumiz)


$$
\begin{gather*}
+ \\
1-1-q
\end{gather*}
$$

Oolyúu upureyun. 95:

$$
\begin{aligned}
& \omega(A)+\omega(B)=\tilde{\omega}(C)+\tilde{\omega}(D) \quad \text { (uve ayè } \\
& \text { upaye) } \\
& p \omega(A)+q \omega(B)=\hat{\omega}(C) \\
& u+g \text {. }
\end{aligned}
$$

| 1 | 4 |  |  |
| :---: | :---: | :---: | :---: |
| hpawe bl | 2 | $1 / 5$ | $4 / 5$ |
|  |  | $1 / 5$ | $4 / 5$ |


|  | 1 | 4 |
| :--- | :---: | :---: |
| 2 | 0 | 1 |
| 3 | $1 / 3$ | $2 / 3$ |

Hpuwep un mo $\Rightarrow$ rexgecobo runa


0


$$
\omega(A)=\vec{\omega}(C)+\tilde{\omega}(D)
$$

egurcob. penerne

$$
\begin{aligned}
& p(A \rightarrow C)=\frac{\tilde{\omega}(C)}{\omega(A)} \\
& p(A \rightarrow D)=\frac{\tilde{\omega}(D)}{\omega(A)}
\end{aligned}
$$

Teoperar (1) Bepratrocon $U(\nu \mid \lambda, x, \mu)$ e ryxtrubies cb-lean cors- tu cyajectloynor.

$$
\begin{gathered}
\frac{1-\operatorname{tuv}}{1-u v} \sum_{x} P_{\mu / x}(u) Q_{\lambda / x}(v) \underset{\sim v}{u}(\nu, x, \lambda) \\
=P_{v)_{\lambda}}(u) Q_{v / \mu}(v)
\end{gathered}
$$

(2) Ha O-n ctorsye orm onp ograzpartho


prob $=1$

prob $=1$

$$
\text { prob }=b_{1}
$$


$p r o b=1-b_{1}$



$$
\text { prob }=b_{2}
$$

$$
p r o b=1-b_{2}
$$

$$
\begin{aligned}
& =\frac{1-t}{\frac{1-t u v}{1-u v}-\left(1-t^{g+1}\right)}=\frac{1-u v}{1-t u v}
\end{aligned}
$$

$$
\begin{aligned}
& =\frac{t(1-u v)}{1-t u v} \\
& =t b_{1}(u, v) \text {. }
\end{aligned}
$$

Bbilaog, $6 b-X_{\lambda}$

$$
\lambda^{(x, y)} \sim \frac{1}{z} P_{\lambda}\left(v_{1}, v_{y}\right) Q_{\lambda}\left(u_{1}, u_{x}\right)
$$

(no mogyлto yир-i с сенме. р-уиями)
3. Popryra g^s nomtromob Xへ

$$
\begin{aligned}
& \text { 3.1 Teopera }
\end{aligned}
$$

$$
\begin{aligned}
& \text { (2) } \prod_{i \geq 0}(t, t)_{m_{i}(\lambda)} P_{\lambda}\left(x_{1} \ldots x_{N}\right) \\
& \text { (3) } \frac{\left(t, t_{m_{0}} \prod_{i z 0}(t, t)_{m_{i}}(\lambda)\right.}{(t, t)_{N}} \cdot Q_{\lambda}^{*}\left(x_{1} \ldots x_{N}\right) \\
& \text { (4) }=Q_{\lambda}\left(x_{1}, \ldots x_{N}\right) \cdot(t ; t)_{\operatorname{mo}}(\lambda)
\end{aligned}
$$

(2),(4) - w оир-й $Q_{\lambda}$ через $Q_{\lambda}^{*}$

$$
P_{\lambda} \text { uepl3 } F_{\lambda}
$$

(3) $Y_{n p}$.

$$
\begin{aligned}
& W\left(j_{1} j_{i_{1}}^{j_{2}}\right)=\frac{(t, t)_{i_{2}}}{(t, t)_{i_{1}}} \omega\left(j_{2} j_{i_{2}}^{i_{1}}\right)
\end{aligned}
$$

Dokazatecucciba $\quad F_{\lambda}=\sum_{\delta}$ [Makgoranog]
$\rightarrow$ (1) Anzesfa Xoma
[Borobinn ${ }^{014]}$ (2) hpebepka/urgykumes $6 / H \sim V$
(3) Nocipocrue $F_{\lambda}$ yepej

Bepur. onveparopua (an2 arizay bere)
3.2 Bepжинныle onepatopb)

$$
\begin{aligned}
& V=l^{2}=\operatorname{span}\left(e_{0}, e_{1}, e_{2}, e_{3}, \ldots\right) ; \quad e_{k} \leftrightarrow \pi_{i}^{k} \\
& A(u) \cdots(u) \cdots \quad D(u) \frac{i}{i} \\
& B e_{k}=\omega\left(-\|_{k}^{k+1}\right) \cdot e_{k+1}=\left(1-t^{k+1}\right) e_{k+1}, \quad \text { uT.g. }
\end{aligned}
$$


$D(u) B(v)$

$$
\begin{aligned}
& V \otimes V \\
& e_{k} \otimes e e
\end{aligned}
$$



$V \otimes V$

$$
\begin{aligned}
B(u)\left(v_{1} \otimes v_{2}\right) & =B(u) v_{1} \otimes A(u) v_{2} \\
+ & D(u) v_{1} \otimes B(u) v_{2}
\end{aligned}
$$

hpleo Qoxa

$$
V \otimes V \otimes V \otimes \ldots
$$

$$
\binom{\text { notta bie vounorestra }}{=e_{0}}
$$

$$
\begin{aligned}
& e_{\lambda}=e_{n_{0}(\lambda)} \otimes e_{m_{1}(\lambda)} \otimes \\
& e_{\phi}=e_{0} \otimes e_{0} \otimes
\end{aligned}
$$


nemen.
Vonnytupytot $\left(A_{\bar{b}}\right)$

$$
\begin{aligned}
F_{\lambda}\left(u_{1} \ldots u_{N}\right)= & \left\langle e_{\lambda}, B\left(u_{1}\right) \ldots B\left(u_{N}\right) e_{\phi}\right\rangle \\
& \left\langle e_{\lambda 1} e_{\mu}\right\rangle=\delta_{\lambda \mu}
\end{aligned}
$$

Nemug. (1) B-komayTupyiot
(2)

$$
B\left(u_{1}\right) D\left(u_{2}\right)=\frac{u_{1}-u_{2}}{t u_{1}-u_{2}} D\left(u_{2}\right) B\left(u_{1}\right)+\frac{(1-t) u_{2}}{u_{2}-t u_{1}} B\left(u_{2}\right) D\left(u_{1}\right)
$$

(3)

$$
B\left(u_{1}\right) A\left(u_{2}\right)=\frac{u_{1}-u_{2}}{u_{1}-t u_{2}} A\left(u_{2}\right) B\left(u_{1}\right)+\frac{(1-t) u_{2}}{u_{1}-t u_{2}} B\left(u_{2}\right) A\left(u_{1}\right)
$$

Haypriep, (2):

3.3. Donazateubcilos opopuymsl gros $f_{\lambda} .1$.

$$
B\left(u_{i}\right) \equiv \beta_{i}, \quad u T \cdot g .
$$

$B_{N}-B_{1}\left(e_{0} \otimes e_{0}\right)-$ ms. кomSuncaynas

$$
\begin{aligned}
& B_{k_{1}} \ldots B_{k_{N-s}} \underbrace{D_{l_{1}} \cdots D_{l_{s}} e_{0}}_{D_{i} e_{0}=u_{i} e_{0}} \otimes B_{i_{1}} \cdots B_{i_{s}} \underbrace{A_{j_{2}} \ldots A_{j-s} e_{0}}_{A_{i} e_{0}=e_{0}} \\
& J=\left\{k_{1}<\cdots<k_{N-s}\right\}, \text { uT.g. }
\end{aligned}
$$

^емма. $\operatorname{In} K=\phi,(\Rightarrow I=L, K=J)$ L $\cap J=\phi$

D-leo.

$$
B_{N} \ldots B_{2}\left[B_{1} \otimes A_{1}+D_{1} \otimes B_{1}\right]\left(e_{0} \otimes e_{0}\right)
$$

$1 \in L \cap J \Rightarrow D_{1}, A_{1}$ cmpala $b$ senx raetrx, Tak me Subaer
no curvetpun, $L \cap J=\phi$

$$
\Rightarrow I \cap R=\varnothing
$$

Torge

$$
\begin{aligned}
B_{N} \ldots B_{1}\left(e_{0} \otimes e_{0}\right) & =\sum_{j<\alpha<N^{2}} f_{j}\left(u_{1} \ldots w\right) \times \\
& \times\left(\prod_{i \in J} B_{i} \prod_{l \notin k} D_{l}\right) e_{0} \otimes\left(\prod_{i \notin k} B_{i} \prod_{l \in \pi} A_{e}\right) e_{0}
\end{aligned}
$$

$f_{K}\left(u_{1}, \ldots u_{N}\right)$ weнл. no $u_{i} \& J K$ T.k. Bi nomer.

Лениа

$$
f_{K}\left(u_{1}-u_{N}\right)=\prod_{\substack{\alpha \in K \\ \beta \notin J}} \frac{u_{\beta}-t u_{\alpha}}{u_{\beta}-u_{\alpha}}
$$

D $x=\{1, \ldots, r\}$
$\left.B_{N} \cdots B_{r+1} D_{1} \ldots D_{1} \otimes\right)$

$$
\prod_{i}\left(B_{i} \otimes A_{i}+D_{i} \otimes B_{i}\right) \longrightarrow \otimes A_{N} \cdots A_{r+1} B_{r} \cdots B_{1}
$$

\& voum. B1.-Br 4ege3 $A_{r+1} \ldots A_{N}$

$$
A(u) B(v)=\frac{v-f u}{v-u} B(v) A(u)+\underbrace{\frac{(t-1) u}{v-u} B(u) A(v)}_{\text {re oסpaur b we.. } D}
$$

$$
\begin{aligned}
& \Downarrow \\
& B_{N}-B_{1}\left(l_{0} \otimes e_{0}\right)=\sum_{J \ll l|\ldots N\rangle}^{v_{1} \in V^{\otimes N}} \prod_{\substack{\alpha \in J \\
\beta \notin J}} \frac{u_{\beta}-t u_{\alpha}}{u_{\beta}-u_{\alpha}}
\end{aligned}
$$

3.4. Donagatecrciloo mopayus) gus $F_{\lambda} .2$.

$$
\begin{aligned}
& B_{N-1} B_{1}\left(e_{0} \otimes e_{0} \otimes e_{0} \otimes \ldots\right), D \\
& =\sum_{K_{0} \cup K_{1} \cup-} \prod_{0 \leqslant l<j}\left(\prod_{\alpha \in K_{j}} u_{\alpha}\right) \prod_{\substack{\alpha \in K_{i} \\
\beta \in K_{j}}} \frac{u_{\beta}-t u_{\alpha}}{u_{\beta}-u_{\alpha}} \\
& =\{1-N\} \\
& \text { - } B\left(K_{0}\right) e_{0} \otimes B\left(K_{1}\right) e_{0} \otimes \\
& F_{\lambda} \Leftrightarrow\left|K_{i}\right|=m_{i}(\lambda), i \geqslant 0 \quad B\left(z_{j}\right)=\prod_{i \in \mathcal{K}_{j}} B\left(u_{i}\right) \\
& \text { hpuvep. } \\
& \begin{array}{llllll}
1 & 2 & 3 & 4 & 5 & N=5 \\
1 & 1 & 1 & 1 & 1 & \\
0 & 1 & 0 & \mid & \\
& 1 & & 3 & 3 & \lambda=33100
\end{array} \\
& \pi_{1} \\
& K_{2}=\emptyset \\
& \prod_{\substack{0 \leqslant i<j}}^{\left.\prod_{\alpha \in K_{j}} u_{\alpha}\right)=u_{5}^{3} u_{4}^{3} u_{2}, m_{i}\left(m_{i}\right)} \\
& \text { B oryem, } \\
& \Theta \prod_{i} u_{i}^{r_{i}}, \quad r_{i}=j \Leftrightarrow i \in \mathcal{K}_{j}
\end{aligned}
$$

Mecr. 山lar:
разбнение

$$
\{1,2, \ldots, N\}=K_{0} \omega K_{1} \cup K_{2} \omega
$$

\& koapg. $(t, t)_{m_{j}}$ gas $K_{j}$
 borytpu $\pi_{i}$.
hpuvep.
76


$$
\begin{aligned}
& d=33100 \\
& \sigma=\frac{12345}{54231} \\
& 45231 \\
& 54213 \\
& 45213
\end{aligned}
$$

$$
K_{2}=\emptyset
$$

$$
M=\left|K_{s}\right|, s=0,1,2, \ldots
$$

$$
\begin{aligned}
& \left.\frac{\text { nемма }}{(y n p i)} \sum_{b \in S_{M}} \vec{b} \prod_{1 \leqslant i<j \leq M} \frac{u_{i}-t u_{j}}{u_{i}-u_{j}}\right)=\frac{(t ; t)_{M}}{(1-t)^{M}} \\
& (M=2): \frac{x-t y}{x-y}+\frac{y-t x}{y-x}=\frac{x-y+t(x-y)}{x-y}=1+t=\frac{(1-t)\left(1-t^{2}\right)}{(1-t)^{2}}
\end{aligned}
$$

$$
F_{\lambda}\left(x_{1}, \cdots x_{N}\right)=(1-t)^{N} \sum_{\partial \in S_{N}} b(\underbrace{x_{1}^{\lambda_{1}} \ldots x_{N}^{\lambda_{N}}}_{1} \prod_{1 \leqslant i<j \leqslant N} \frac{x_{i}-t x_{j}}{x_{i}-x_{j}})
$$

$$
\begin{array}{r}
\text { Teopeua gokergatia }]^{\downarrow} \\
\hbar_{i=1}^{N}\left(\frac{x_{i}-s}{1-s x_{i}}\right)^{\lambda_{i}}
\end{array}
$$

(spin Hall-Littlewood)
4. Duaroranbrbiú onepatop no $x_{i}$

$$
\begin{aligned}
& D(q, t)=\sum_{i=1}^{N} \prod_{j \neq i} \frac{t_{x_{i}-x_{j}}}{x_{i}-x_{j}} T_{q, x_{i}} \begin{array}{l}
\begin{array}{l}
\text { (neplowit) } \\
\text { oneratop } \\
\text { makgonaroga }
\end{array} \\
T_{g, x} f(x)=f(q x)
\end{array} \\
&
\end{aligned}
$$

Teop. $D(0, t) P_{\lambda}^{k}\left(x_{1} \ldots x_{N} \mid t\right)=$

$$
=\frac{1-t^{m_{0}(\lambda)}}{1-t} P_{\lambda}\left(x_{1}, \ldots, x_{N}\right)
$$

D. (1) $D(0, t) P_{\phi}=\frac{1-t^{N}}{1-t} P_{\phi} \quad\left(P_{\phi}=1\right)$

$$
\sum_{i=1}^{N}\left(\prod_{j \neq i} \frac{t x_{i}-x_{j}}{x_{i}-x_{j}}\right)=\frac{1-t^{N}}{1-t} \quad(y u p .)
$$

ykazarue: $\oint \prod_{j=1}^{N} \frac{t z-x_{j}}{z-x_{j}} \cdot \frac{d z}{(t-1) z}$
(2) $l=N-m 0 \quad T_{0, x_{i}}$ ovrymset wrle re tpopaet

$$
\begin{aligned}
& F_{\lambda}=\sum_{\zeta \in S_{N}}\left[\begin{array}{ccc}
\lambda_{1} & \lambda_{l} & \pi_{b(l)} \\
\frac{x_{b(i)}-t x_{b 1 j}}{\left.x_{b(i)}-x_{b 1 j}\right)}
\end{array}\right. \\
& i \in b(\{l+1, ., N\}) \\
& \cdot \prod_{a}^{\prod_{j=1} \prod_{i=l+1}^{N} \frac{x_{b(j)}-t x_{b(i)}}{x_{b(j)}-x_{b(i)}}} \cdot \prod_{l+1 \leq i<j \leq N} \frac{x_{b(i)}-t x_{b(j)}}{x_{b(i)}-x_{b}(j)}
\end{aligned}
$$

$i \geqslant l+1 T_{0, x_{g}(i)}$ gaicbeyer $T_{k}:$

$$
-x_{\sigma(i)}=0
$$



T. $1 h(x, y) \stackrel{d}{=} y-l(\lambda)$,
$\lambda \sim \quad \frac{1}{z} P_{\lambda}\left(v_{1},-, v_{y}\right) Q_{\lambda}\left(u_{1}, \ldots, u_{x}\right)$

T.2. $\quad F_{\lambda}=P_{\lambda} \cdot \prod_{i \geqslant 0}(t, t)_{m_{i}(\lambda)}$

$$
0 \leq t<1, \quad\left|n_{i} v_{j}\right|<1
$$

$$
(t, t)_{n}=
$$

$$
\begin{aligned}
& t, \tau_{n}= \\
& =(1-t)\left(1-t^{2}\right) . .\left(1-t^{n}\right)
\end{aligned}
$$

$$
\begin{aligned}
& F_{\lambda}\left(u_{1} \ldots u_{N}\right)= \\
& =(1-t)^{N} \sum_{j \in S_{N}} \sigma\left(u_{1}^{\lambda_{1}}-u_{N}^{\lambda_{N}} \prod_{i<j} \frac{u_{i}-t u_{j}}{u_{i}-u_{j}}\right)
\end{aligned}
$$

T3.

$$
\begin{gathered}
D=\sum_{i=1}^{N} \prod_{j \neq i} \frac{t u_{i}-u_{j}}{u_{i}-u_{j}} T_{0, u_{i}} \\
T_{0, x} f(x)=f(0) \\
D F_{\lambda}^{\left(u^{u} u^{\mu n}\right.}=\frac{1-t_{0}(\lambda)}{1-t} F_{\lambda}^{\left(u u_{i}-u^{\mu}\right)}
\end{gathered}
$$ upureerye

coso
.

$$
Z=\sum_{\lambda} P_{\lambda}(\vec{k}) Q_{\lambda}(\vec{v})=\prod_{i, j} \frac{1-t_{i} v_{j}}{1-u_{i} v_{j}} \quad \text { (koulu) }
$$

Oterynnerue :

$$
\begin{aligned}
& D^{q, t}=\sum_{i=1}^{N} \prod_{j \neq i} \frac{t_{u_{i}-u_{j}}}{u_{i}-u_{j}} T_{q_{1} u_{i}} \\
& \text { brepocop } \\
& \text { margorange } \\
& D^{q, t} P_{\lambda}\left(u_{2}-u_{N} \mid g, t\right)= \\
& =\left(\sum_{i=1}^{N} q^{\lambda_{i}} t^{N-i}\right) P_{\lambda}^{1}\left(u_{1}-u_{N} \mid g, t\right) \\
& \sum_{\lambda} P_{\lambda}(\vec{u}) Q_{\lambda}(\vec{v})=\prod_{i_{1} j} \frac{\left(t u_{i} v_{j} ; q\right)_{\infty}}{\left(u_{i} v_{j} ; q\right)_{\infty}} .
\end{aligned}
$$

2. Onepatop D u тожgectlo Koun

$$
\begin{array}{r}
\tilde{D}=t^{-N}[1+(t-1) D] \quad(q=0, x-1) \\
{\left[\tilde{D} P_{\lambda}=t^{-l(\lambda)} P_{\lambda}\right.} \\
\left(t^{-N}\left(1+(t-1) \frac{1-t^{N-l}}{1-t}\right)=t^{-l}\right) \\
\mathbb{E}_{H L} f(\lambda)=\frac{1}{z} \sum_{\lambda} f(\lambda) P_{\lambda}\left(u_{1} \ldots u_{N}\right) Q_{\lambda}\left(s_{1} \ldots v_{M}\right)
\end{array}
$$

$$
\mathbb{E}_{H L} t^{-k l(\lambda)}=\frac{1}{z} \sum_{\lambda} f(\lambda) P_{\lambda}\left(u_{1} \ldots u_{N}\right) Q_{\lambda}\left(v_{1} \ldots v_{M}\right)
$$

$k \geq 0$ qukc

$$
\begin{aligned}
& =\frac{1}{z} \sum_{\lambda} t^{-k l l(\lambda)} P_{\lambda}\left(u_{1} \ldots u_{N}\right) Q_{\lambda}\left(v_{1} \ldots v_{M}\right) \\
& =\frac{1}{z} \sum_{\lambda}\left[\tilde{D}^{k} P_{\lambda}\left(u_{1} \ldots u_{N}\right)\right] Q_{\lambda}\left(v_{1}, v_{N}\right) \\
& =\frac{\tilde{D}_{(\vec{u}}^{k}, Z(\vec{l}, \vec{v})}{Z(\vec{u}, \vec{v})}
\end{aligned}
$$

T.4. $E_{s 6 r} t^{k h(x, y)}=t^{k y} \frac{\tilde{D}_{\left(v_{1} . . v_{y}\right)}^{k} Z\left(\overrightarrow{u_{e}}, \vec{v}\right)}{Z(\vec{u}, \vec{v})}$

$$
v_{v_{y}}{\underset{q}{h}}^{v_{1}(x, y)} u_{1}-u_{x} l^{2}(\vec{u}, \vec{v})=\prod_{i=1}^{x} \prod_{j=1}^{y} \frac{1-t u_{i} v_{j}}{1-u_{i} v_{j}}
$$

$$
f(z)=\prod_{i} \frac{1-t u_{i} z}{1-u_{i} z}
$$

3. Onepatop $\widetilde{D}$ u bbiuetiol

$$
\begin{aligned}
& D=\sum_{i=1}^{N} \prod_{j \neq i} \frac{t v_{i}-v_{j}}{v_{i}-v_{j}} T_{0, v_{i}} \\
& \tilde{D}=t^{-N}[1+(t-1) D]
\end{aligned}
$$

T. 5.

$$
D\left[f\left(v_{1}\right) \ldots f\left(v_{N}\right)\right]=f\left(v_{1}\right) \ldots f\left(v_{N}\right)
$$

$$
\begin{aligned}
& f(0)=1, \\
& \text { 20ciomopgraa } \\
& \text { ovoro } \\
& 0, v_{1}, \ldots, s_{N}
\end{aligned} \left\lvert\, \frac{1}{2 \pi i} \oint_{j=1} \prod_{j}^{N} \frac{t z-v_{j}}{z-v_{j}} \frac{d z}{(t-1) z} \frac{1}{f(z)}\right.
$$



$$
\begin{aligned}
\text { D. } & \frac{1}{2 \pi i} \oint_{v_{1} \cdot v_{N}} \prod_{j=1}^{N} \frac{t z-v_{j}}{z-v_{j}} \frac{d z}{(t-1) z} \frac{1}{f(z)}= \\
= & \sum_{i=1}^{N} \prod_{j_{j} \neq i} \frac{t v_{i}-v_{j}}{v_{i}-v_{j}} \cdot \frac{t v_{i}-v_{i}}{(t-1) v_{i}} \cdot \frac{1}{f\left(v_{i}\right)} \\
& T_{0, v_{i}}\left(f\left(v_{1}\right)-f\left(v_{N}\right)\right)=\frac{f\left(v_{1}\right) \ldots f\left(v_{N}\right)}{f\left(v_{i}\right)}
\end{aligned}
$$

$$
\begin{aligned}
& \text { Cn. } \tilde{D}=t^{-N}[1+(t-1) D] \\
& \tilde{D}\left[f\left(v_{1}\right) \ldots f\left(v_{N}\right)\right]=f\left(v_{1}\right) \ldots f\left(v_{N}\right) \\
& \cdot \frac{1}{2 \pi i} \oint_{0, v_{1}, v_{N}} \prod_{j=1}^{N} \frac{z-v_{j} / t}{z-v_{j}} \frac{d z}{z} \frac{1}{f(z)}
\end{aligned}
$$


4. Onepatop $\widetilde{D}^{k} u \oint_{k} . . \oint^{-}$

Ha4rëm c $k=2$

$$
\begin{aligned}
& \tilde{D}\left[f\left(v_{1}\right) \ldots f\left(v_{N}\right)\right]=f\left(v_{1}\right) \ldots f\left(v_{N}\right) . \\
& \cdot \frac{1}{2 \pi i} \oint_{0, v_{1} \cdot v_{N}} \prod_{j=1}^{N} \frac{z-v_{j} / t}{z-v_{j}} \frac{d z}{z} \frac{1}{f(z)} \\
& f(v) \frac{z_{1}-v / t}{z_{1}-v}=f_{1}(v) \quad \text { (3abucut or } z \text { ) } \\
& \tilde{D}\left[f\left(v_{1}\right) \ldots f\left(v_{N}\right)\right]= \\
& \cdot \frac{1}{2 \pi i} \oint_{0, v_{1} \cdot v_{N}} \prod_{j=1}^{N} f_{1}\left(v_{j}\right) \frac{d z}{z} \frac{1}{f(z)}
\end{aligned}
$$

hpumerver $\tilde{D}$ owsto $k \quad f_{1}\left(v_{1}\right) \ldots f_{1}\left(v_{N}\right)$

$$
\begin{aligned}
& \tilde{D}^{2}\left[f\left(v_{1}\right) \ldots f\left(v_{N}\right)\right]= \\
& -\frac{1}{2 \pi i} \oint_{0, v_{1} \cdot v_{N}}\left\{\begin{array}{l}
\prod_{j=1}^{N} f_{1}\left(v_{j}\right) \frac{d z_{1}}{z_{1}} \frac{1}{f\left(z_{1}\right)}
\end{array}\right. \\
& \left.\begin{array}{l}
0 \sum_{2 \pi_{i}} \oint_{0, v_{1} \ldots v_{N}} \prod_{j=1}^{N} \frac{z_{2}-v_{j} / t}{z_{2}-v_{j}} \frac{d z_{2}}{z_{2}}\left(\frac{1}{f_{1}\left(z_{2}\right)}\right)
\end{array}\right\} \\
& \text { baver } b \quad z_{2}=z_{1} t \\
& k a_{k} \text { ? } \\
& z_{2}=t z_{1}, \quad z_{1}=t^{-1} z_{2}
\end{aligned}
$$



Vtb. $\quad k=2$.

$$
\begin{gathered}
\frac{\tilde{D}^{2}\left(f\left(v_{1}\right) \sim f\left(v_{N}\right)\right)}{f\left(v_{1}\right) \ldots f\left(v_{N}\right)}=\frac{1}{(2 \pi i i)^{2}} \oint_{j 1} \frac{d z_{1}}{z_{1}} \oint_{\gamma_{2}} \frac{d z_{2}}{z_{2}} \frac{z_{1}-z_{2}}{z_{1}-z_{2} / t} \\
\prod_{j=1}^{N} \frac{z_{1}-v_{j} / t}{z_{1}-v_{j}} \frac{z_{2}-v_{j} / t}{z_{2}-v_{j}} \frac{1}{f\left(z_{1}\right) f\left(z_{2}\right)}
\end{gathered}
$$



Tib. (Dogué coyrars)

$$
\begin{aligned}
& \frac{\tilde{D}^{K}\left(f\left(v_{1}\right)-f\left(v_{N}\right)\right)}{f\left(v_{1}\right) \ldots f\left(v_{N}\right)}=\frac{t^{\frac{K(k-1)}{2}}}{(2 \pi i)^{k}} \oint_{j} \frac{d z_{1}}{z_{1}} \cdots \oint_{\gamma_{k}} \frac{d z_{k}}{z_{k}} . \\
& \left.\cdot \prod_{1 \leqslant A \angle B \leqslant K} \frac{z_{A}-z_{B}}{z_{A}-t z_{B}} \prod_{j=1}^{K} \oint \frac{1}{f\left(z_{j}\right)} \prod_{i=1}^{N} \frac{z_{j}-v_{i} / t}{z_{j}^{\prime}-v_{i}}\right]
\end{aligned}
$$



$$
\begin{align*}
& C_{1-e} \\
& t_{S 6 V} t^{k \cdot h(x, y)}=t^{k y} \frac{\tilde{D}^{K}\left(f\left(v_{1}\right) \ldots f\left(v_{N}\right)\right)}{f\left(v_{1}\right) \ldots f\left(v_{N}\right)} \\
& {\left[f(z)=\prod_{i=1}^{x} \frac{1-t u_{i} z}{1-u_{i} z}\right]} \\
& \theta \frac{t^{\frac{k(k-1)}{2}}}{(2 \pi i)^{k}} \oint_{\gamma_{1}} \frac{d z_{1}}{z_{1}} \cdots \oint_{\gamma_{k}} \frac{d z_{k}}{z_{k}} \cdot \prod_{1 \leqslant A \angle B \leqslant K} \frac{z_{A}-z_{B}}{z_{A}-t z_{B}} \\
& \text { - } \prod_{j=1}^{k}\left\{\prod_{i=1}^{x} \frac{1-z_{j} u_{i}}{1-t z_{j} u_{i}} \prod_{i=1}^{y} \frac{t z_{j}-v_{i}}{z_{j}^{j}-v_{i}}\right]
\end{align*}
$$

$\left(T e * e\right.$ voragp|1) + ycuabur ta $\left.v_{i} / t^{-1} u_{i}^{-1}\right)$

$$
\text { max } v_{i}<t^{-1} \min _{i}^{-1}
$$

5. Иposrema moneктоb
$V$-blp.repa ra $[a, b]$

$$
\alpha_{k}=\int_{a}^{b} x^{k} \nu(d x) \quad, \quad \alpha_{0}=1
$$

Torge $\left\{\alpha_{k}\right\}_{k \geqslant 1}$ ogrogrartio oupegens vor $\nu$
hovery? Beíepmipack

$$
f \in C[a, b]
$$

I $\int_{a}^{b} f d \nu \quad$ oup-repre $\left\{\alpha_{k} k\right.$
npmoxtrue $k \quad 6 b$ rogeren $(x, y$ qukc.)

$$
\begin{aligned}
& \mathbb{E} t^{k \cdot h(x, y)}=\oint_{L_{k}} \ldots \oint \\
& E^{h(x, y)} \in(0,1] \\
& \text { Komn. Hocutru } \\
& \Downarrow \\
& \left\{\mathbb{E} \mathbb{E}^{k h(x, y)}\right\}_{k \geqslant 1}-\underset{\substack{\text { ogrozroantio oup } \\
\text { pacup-e }}}{ } h
\end{aligned}
$$

Ho: kak ncuouzobaro dro gus асеयntorukn $h(x, y)$

$$
x, y \rightarrow \infty \quad ?
$$


6. q- - иноииаnbias теорена (q 3 avernuen

$$
(a ; t)_{n}=(1-a)(1-a t) \ldots\left(1-a t^{n-1}\right)
$$

$$
n=\infty \text { toxe or }
$$

$$
\frac{1-t^{n}}{1-t} \rightarrow n, t \rightarrow 1
$$

T.7. $|y|<1,|t|<1$

$$
\sum_{n=0}^{\infty} \frac{s^{n}}{(t ; t)_{n}}=\frac{1}{(\tau ; t)_{\infty}}
$$

$$
\sum_{n=0}^{\infty} \frac{3^{n}}{n!}=e^{5}
$$

D-60. Hycos $B A=t A B+(1-t) B^{2}$

$$
\begin{aligned}
& (p A+(1-p) B)^{n}= \\
& =\sum_{k=0}^{n} C_{n k} A^{k} B^{n-k} \\
& C_{n k}=?
\end{aligned}
$$

$$
\begin{gathered}
\left(\sum_{k=0}^{n-1} C_{n-1, k} A^{k} B^{n-1-k}\right) \cdot(p A+(1-p) B) \\
=\sum_{k=0}^{n} C_{n k} A^{k} B^{n-k} \\
B A=t A B+(1-t) B^{2} \\
C_{n, k}=C_{n-1, k} \cdot(1-p)+C_{n-1, k-1} p t^{n-k} \\
\left.A^{k} B^{n-k-1}(1-p) B\right) \quad\left(A^{k-1} B^{n-k} p A\right) \\
+C_{n-1, k}^{n-k} p \cdot\left(1-t^{n-k-1}\right) \\
B^{l} A=t^{l} A B^{l}+(1-t)\left[1+t+\ldots B^{n-k-1} p A\right) B^{l+1}
\end{gathered}
$$

$t=1$-no4in Sunam. korpp. To4kel - Surnam.p-e.

$$
p=\frac{1}{2} \quad C_{n, k}=\frac{1}{2^{n}} C_{n}^{k}=\frac{\binom{n}{k}}{2^{n}}
$$

OTbT (upobepeerse no ungyskyme)

$$
\begin{array}{r}
C_{n k}=p^{k}(p ; t)_{n-k} \frac{(t ; t)_{n}}{(t ; t)_{k}(t ; t)_{n-k}} \\
t=1 ; p^{k}(1-p)^{n-k} C_{n}^{k} \\
\bigoplus_{\binom{n}{k}}
\end{array}
$$

Tak*e no ung. onebugtro, noo

$$
\begin{gathered}
\sum_{k=0}^{u} c_{n k}=1 \\
c_{\infty, k}=p^{k} \frac{(p ; t)_{\infty}}{(t ; t)_{k}}, \sum_{k=0}^{\infty} c_{n, k}=1
\end{gathered}
$$

$$
\frac{(1-t) p^{k}}{(t, t)_{k}} \rightarrow p^{k} / k!, t \rightarrow \infty
$$

7. $q$ - §иномиапиtas $^{\text {teopema }}$ u acumпtotuka

$$
\begin{aligned}
& j \in \mathbb{C}, \\
& |y|<1
\end{aligned}
$$

$$
\begin{aligned}
& \sum^{\infty} \\
& \begin{array}{l}
\sum_{k=0}^{\infty} s^{k} \mathbb{E} t^{k h(x, y)} /(t ; t)_{k}=\text { Cake }_{\text {no }} \\
=\mathbb{E}\left[\sum_{k=0}^{\infty}\left(\xi t^{h(x, y))^{k}} /(t ; t)_{k}\right]\right.
\end{array} \\
& =\mathbb{E}\left[\frac{1}{\left(3 t^{h(x, y)} ; t\right)_{\infty}}\right]
\end{aligned}
$$


pabromeptio ra

$$
x \in(-\infty, 0)
$$

$$
\begin{aligned}
& \mathbb{E}\left[\frac{1}{\left(\zeta t^{h(x, y)} ; t\right)_{\infty}}\right] \xrightarrow{?} \quad x, y \rightarrow \infty \\
& \left.\left.\begin{array}{c}
\text { Punoreza/ } \\
\text { bequer }
\end{array}: h(L L x\rfloor, L L y\right\rfloor\right) \simeq \\
& L^{\alpha} \begin{array}{l}
x, y \in \mathbb{R}_{\geq 0} \\
d=\frac{1}{2} \\
u_{h} \cap T
\end{array} \\
& 354 \\
& \begin{array}{cc}
354 & \simeq L \\
\frac{L(L x, L y)}{L}+F H(x, y) & (\alpha>0)
\end{array} \\
& \zeta=t^{-L \nVdash(x, y)+s L^{\alpha}} \\
& \simeq L \mathcal{H}(x, y)-L_{\eta}^{\alpha} \eta \\
& s \in \mathbb{R} \\
& \frac{1}{(0, t) \infty}=1 \\
& \mathbb{E}[\underbrace{\frac{1}{\left(t^{L^{\alpha}(s-\eta)} ; t\right)}}_{c \wedge \cdot b \cdot G}]
\end{aligned}
$$

$$
\begin{aligned}
& \eta<s ; \quad t^{L^{\alpha}(s-\eta)} \rightarrow 0 ; \quad G=1 \\
& \eta>s ; \quad t^{L^{\alpha}(s-\eta)} \rightarrow+\infty \quad j G=0 \\
& t \in(0,1) \\
& \Rightarrow G \underset{L \rightarrow \infty}{\longrightarrow} \text { unguikropy }(\eta<s) \\
& \mathbb{E}\left[\frac{1}{\left(\zeta t^{h(x, y)} ; t\right)_{\infty}}\right] \xrightarrow{L \rightarrow \infty)} \mathrm{P}(\eta<s)
\end{aligned}
$$

中. parmpors
prykogayuer $\eta$.
8. - - суниироbание моиентов. 3 aga4a


$$
=\sum_{k=0}^{\infty} \frac{3^{k}}{(t ; t)_{k}}
$$

$\left(\begin{array}{cc}\text { paztule } & \text { korrypbl, } \\ \text { a } & \text { eenu } \\ \text { sun slannu } \\ & \text { ogurakobu.-. }\end{array}\right)$
$\cdot \frac{t^{\frac{k(k-1)}{2}}}{(2 \pi i)^{k}} \oint_{\gamma_{1}} \frac{d z_{1}}{z_{1}} \cdots \oint_{\gamma_{k}} \frac{d z_{k}}{z_{k}} \cdot \prod_{1 \leqslant A \angle B \leq K} \frac{z_{A}-z_{B}}{z_{A}-t z_{B}} \cdot f\left(z_{1}\right)-f\left(z_{k}\right)$

$$
f(z)=\prod_{i=1}^{x} \frac{1-z u_{i}}{1-t z u_{i}} \prod_{i=1}^{y} \frac{t z-v_{i}}{z-v_{i}}
$$

9. Dypegenuteme Pregrourua

$$
\begin{aligned}
& \operatorname{det}(A)=\cdots \quad A=\left(a_{i j}\right) \quad n \times n \\
& \operatorname{det}(1+z A)=1+z \cdot \sum_{i} a_{i i}+\quad \begin{array}{l}
1=I d \\
\text { Toxgect } b, \\
\text { mejpurg }
\end{array} \\
& +z^{2} \sum_{i<j} \operatorname{det}\left[\begin{array}{ll}
a_{i i} & a_{i j} \\
a_{j i} & a_{j j}
\end{array}\right]_{2 \times 2} \\
& +z^{3} \sum_{i<k<j} \operatorname{det}_{3 \times 3}+\cdots \\
& =1+\sum_{k=1}^{n} z^{k} / k!\sum_{i_{j, i}=1}^{n} \operatorname{det}_{k \times, k}\left[a_{i_{\alpha} i_{\beta}}\right]
\end{aligned}
$$

$$
\begin{aligned}
& \text { hycto } K: L^{2}(\Omega) \rightarrow L^{2}(\Omega) \\
& (k f)(x)=\int_{\Omega} K(x, y) f(y) \overparen{d y} \\
& {[\Omega=\{1, \ldots, n\} \text { : iplegagyagaies coyral }]}
\end{aligned}
$$

$K$-trace class / egeprowí oneparop
/ nuclear operator
$\operatorname{ecm} \quad \int_{\Omega}|k(x, x)| d x<\infty$

Onp.

$$
\begin{aligned}
& \operatorname{det}(1+z k) \\
& =1+\sum_{m=1}^{\infty} \frac{z^{m}}{m!} \int_{\Omega^{m}} \int_{m \times m} \operatorname{det}_{m \times m}\left[k\left(x_{i}, x_{j}\right)\right] d x_{1} \cdot d x_{m}
\end{aligned}
$$

$$
\sum_{k=0}^{\infty} \frac{3^{k}}{(t ; t)_{k}}
$$

to mor Sol nomyructous onp. Ppegroabua!

$$
\begin{aligned}
& \prod_{1 \leqslant A \angle B \in K} \frac{z_{A}-z_{B}}{z_{A}-z_{B}}=\operatorname{det} \text { ? } \\
& \begin{array}{r}
\operatorname{det}\left[\frac{1}{x_{i}-y_{j}}\right]_{i, j=1}^{N}=V(x) v(y) \\
= \pm \frac{\prod_{i<j}\left(\left(x_{i}-x_{j}\right)\left(y_{i}-y_{j}\right)\right)}{\prod_{i j}\left(x_{i}-y_{j}\right)}
\end{array} \\
& \operatorname{dt}\left[\frac{1}{z_{i}-t z_{j}}\right]= \pm \frac{V(z) V(t z)}{\prod_{i j}\left(z_{i}-t z_{j}\right)} \\
& \sum_{z} z\left(\prod_{1 \leqslant A \angle B \in K} \frac{z_{A}-z_{B}}{z_{A}-t_{B}}\right) \\
& \ \prod_{A \neq B} \frac{z_{A}-z_{B}}{z_{A}-t z_{B}} \cdot \sum_{b} b\left(\prod_{B>A} \frac{z_{A}-t z_{B}}{z_{A}-z_{B}}\right)
\end{aligned}
$$

$$
\begin{aligned}
& \text { 10. Cтarubaние Kortypob }\left\{\begin{array}{r}
\text { Heckman-Opdak } \\
\begin{array}{r}
1997 \\
\text { Borodin-Corwin } \\
2011
\end{array}
\end{array}\right. \\
& f(z)=\prod_{i=1}^{x} \frac{1-z u_{i}}{1-t z u_{i}} \prod_{i=1}^{y} \frac{t z-v_{i}}{z-v_{i}}=g(z) / g(t z) \\
& g(z)=\prod_{i=1}^{x}\left(1-z u_{i}\right) \prod_{i=1}^{y} \frac{1}{z-v_{i}} \\
& \frac{t^{\frac{k(k-1)}{2}}}{(2 \pi i)^{k}} \oint_{\gamma 1} \frac{d z_{1}}{z_{1}} \cdots \oint_{\gamma_{k}} \frac{d z_{k}}{z_{k}} \cdot \prod_{1 \leqslant A \angle B \leqslant K} \frac{z_{A}-z_{B}}{z_{A}-t z_{B}} \cdot f\left(z_{1}\right)-f\left(z_{k}\right) \\
& \text { ( }
\end{aligned}
$$

3afara: cteryor be $z_{i} k$
b nopregue $z_{k}, z_{k-1,} \ldots, z_{1}$

Fto gaèt gon. $\Theta l a b l y e r$

$$
b \quad z_{A}=t z_{B} \quad A<B
$$

$$
k=2
$$



$$
\begin{aligned}
& \frac{t}{\left(2 \pi x_{i}\right)^{2}} \oint_{\gamma_{1}} d z_{1} \oint_{\gamma_{2}} d z_{2} \quad \frac{z_{1}-z_{2}}{z_{1}-t z_{2}} \frac{f\left(z_{1}\right) f\left(z_{2}\right)}{z_{1} z_{2}} \\
& \frac{t}{(2 \pi i)^{2}} \oiint_{\gamma}+\text { curu. } \\
& \frac{t}{2 \pi i} \oint_{\gamma} d z_{2} \begin{array}{r}
-\operatorname{Res} \\
z_{1}=t z_{2}
\end{array} \\
& \left(\frac{z_{1}-z_{2}}{z_{1}-t z_{2}}+\frac{z_{2}-z_{1}}{z_{2}-t z_{1}}\right) \frac{1}{z_{1} z_{2}} \cdot \frac{1}{2} \\
& \frac{1}{2} \cdot \operatorname{det}\left[\frac{1}{z_{i}-t z_{j}}\right]_{2 \times 2} \cdot\left(t_{j}, t_{2} \cdot t^{-1}\right. \\
& \frac{1}{2 \lambda_{i}^{i}} \oint_{\gamma} d z_{2} \frac{1-t}{z_{2}} f\left(z_{2}\right) f\left(t z_{2}\right) \\
& 2 \times 2 \text { det } \operatorname{nog} \oint \oint! \\
& (t ; t)_{2} \cdot \operatorname{det}\left[\frac{1}{z_{1}-t^{2} z_{1}}\right]_{1 \times 1} f\left(z_{2}\right) f\left(t z_{2}\right) \\
& \text { (TOT *e det, no aymerby) }
\end{aligned}
$$

B obyen eytal nogciatuban:

$$
\begin{aligned}
& f\left(w_{1}\right) f(w, t) \ldots f\left(w_{1} t^{\lambda_{1}-1}\right) \\
& |\lambda|=k \\
& f\left(\omega_{z}\right) f\left(\omega_{2} t\right) \ldots f\left(\omega_{2} t^{\lambda_{2}-1}\right) \\
& f(\omega)=\frac{g(\omega)}{g(t \omega)} \quad \operatorname{def} \\
& f\left(\omega_{e}\right)^{i} f\left(v_{e} t\right)-f\left(\omega_{e} t^{\lambda \ell-1}\right) \\
& f(\omega \circ \lambda)=\frac{g\left(\omega_{1}\right) \cdots g(\omega e)}{g\left(t^{\lambda_{1}} \omega_{1}\right)--g\left(t^{\lambda \ell} \omega e\right)}
\end{aligned}
$$

$\frac{(t, t)_{2}}{(2 \pi i)^{2}} \oint \oint \frac{1}{2} \operatorname{det}\left[\frac{1}{w_{i}-t^{\lambda j \omega_{j}}}\right]_{2 \times 2} f(w \Delta \lambda)$
kpatroctu

$$
\begin{array}{r}
+\frac{(t, t)_{2}}{2 \pi i} \oint \operatorname{det}\left[\frac{1}{\omega_{1}^{0}-t^{\lambda j} \omega_{j}}\right]_{1 \times 1} f(\omega 0 \lambda) \\
\lambda=(2)
\end{array}
$$

T. 8 (Borodin-Corwin) $\leftarrow$ osugus wyras rez gak.

$$
\begin{aligned}
& \mathbb{E} t^{k h(x, y)}=(t, t)_{k} \sum_{\substack{\lambda: \\
|\lambda|=k}} \prod_{i \geqslant 1} \frac{1}{m_{i}(\lambda)!} \\
& \left.\cdot \frac{1}{(2 \pi i}\right) l(\lambda) \oint_{\gamma} \oint_{\gamma} \operatorname{det}\left[\frac{g\left(\omega_{i}\right) / g\left(t^{\lambda_{i}^{i}} \omega_{i}\right)}{\omega_{i}-t^{\lambda_{j}} \omega_{j}}\right]_{l(\lambda) \times l(\lambda)} d \vec{\omega} .
\end{aligned}
$$

Мекуия 7 (b zamen)

1. Peшаен соохастческуг 68 ноgень
T. $1 h(x, y) \stackrel{d}{=} y-l(\lambda)$,
$\lambda \sim \quad \frac{1}{z} P_{\lambda}\left(v_{1},-, v_{y}\right) Q_{\lambda}\left(u_{1}, \ldots, u_{x}\right)$
2. Uro mbl y*e govazam

$$
\begin{aligned}
& \mathbb{E} t^{k h(x, y)}=(t, t)_{k} \sum_{\substack{\lambda: \\
|\lambda|=k}} \prod_{i \geqslant 1} \frac{1}{m_{i}(\lambda)!} . \\
& \left.\cdot \frac{1}{(2 \pi i}\right) l(\lambda) \oint_{\gamma} \oint_{\gamma} \operatorname{det}\left[\frac{g\left(\omega_{i}\right) / g\left(t^{\lambda i} \omega_{i}\right)}{\omega_{i}-t^{\lambda_{j}} \omega_{j}}\right]_{l(\lambda) \times l(\lambda)} d \vec{\omega} .
\end{aligned}
$$

(Teop. o ciembarum kortignob)

$$
\begin{aligned}
& g(z)=\prod_{i=1}^{x}\left(1-z k_{i}\right) \prod_{g=1}^{y}\left(z-v_{j}\right) \\
& f(z)=g(z) / g(t z)
\end{aligned}
$$



Pyrarbobano bo bcex ypeoprapobarues vorotypro
3. Qpegrama 1
$K\left(\omega, \omega^{\prime}\right) \quad \omega, \omega^{\prime} \in \gamma$
Onp.

$$
\begin{aligned}
& \operatorname{det}(1+z k) \\
& =1+\sum_{m=1}^{\infty} \frac{z^{m}}{m!} \int_{\gamma \gamma} \int_{\gamma \gamma} \operatorname{det}_{m \times m}\left[k\left(x_{i}, x_{j}\right)\right] \frac{d x_{1} \cdot d x_{m}}{\left(2 x_{i}\right)^{m}} \\
& \pi \\
& \mathbb{E} \frac{1}{\left(\zeta t^{h(x, y)} ; t\right)_{\infty}}=\operatorname{det}\left(1+k_{\zeta}^{1}\right) \\
& |s|<\varepsilon-c x-c s \\
& K_{3}^{1}\left(n, \omega j n^{\prime}, w^{\prime}\right)=\frac{3^{n}}{w^{\prime}-t^{n} \omega} g(\omega) / g\left(t^{n} \omega\right) \\
& l_{\text {rgpe }} b L^{2}\left(Z_{21} \times \gamma\right) \quad \int_{\Omega} \neq \sum_{n} \oint_{\gamma}
\end{aligned}
$$

$$
\begin{aligned}
& \text { D. } \mathbb{E} \frac{1}{\left(3 t^{h}, t\right)_{\infty}}=\sum_{k=0}^{\infty} \frac{3^{k}}{(t, t)_{k}} \mathbb{E} t^{k \cdot h} \\
& \text { (q-रuram. } \left.T \cdot+\operatorname{orp} t^{k h}\right) \\
& =\sum_{k=0}^{\infty} \frac{3^{k}}{(t, b)_{k}}(t, t)_{k} \sum_{\lambda:} \prod_{i \geqslant 1} \frac{1}{m_{i}(\lambda)!} . \\
& s^{5}=\pi 5 \\
& |\lambda|=k \\
& \cdot \frac{1}{(2 \pi i)} l(\lambda) \oint_{\gamma} . \oint_{\gamma} \operatorname{det}\left[\frac{g\left(\omega_{i}\right) / g\left(t^{\lambda_{i}^{i}} \omega_{i}\right)}{\omega_{i}-t^{\lambda_{j}} \omega_{j}}\right]_{l(\lambda) \times e(\lambda)} d \vec{\omega} . \\
& \lambda_{1 \geqslant, \ldots 1 l}^{\lambda_{1}} \sum_{\substack{\operatorname{lom} \\
\lambda}} \prod_{i \geqslant 1} \frac{1}{m_{i}(\lambda)!} \cdot \frac{1}{(2 \pi i)} l(\lambda) \\
& \oint_{\gamma} \oint_{\gamma} \operatorname{det}\left[\frac{j^{\lambda i} g\left(\omega_{i}\right) / g\left(t^{\lambda_{i}^{i}} \omega_{i}\right)}{\omega_{i}-t^{\lambda_{j}} \omega_{j}}\right]_{l(\lambda) \times e(\lambda)} d \vec{\omega} . \\
& =\sum_{l \geqslant 0} \sum_{n_{1}, \ldots, n_{l} \geqslant 1} \frac{1}{l!} \frac{1}{(2 n i)^{l} l} \oint_{\gamma} \oint_{\gamma} \operatorname{det}_{e x l} K_{\zeta}^{1}\left[n_{i}, w_{i j} n_{j,}, w_{j}\right]
\end{aligned}
$$

\# repecr. =

$$
L^{2}\left(\mathbb{z}_{21} \times \gamma\right)
$$

Ny м Thran. $\operatorname{koz\varphi \varphi }$

$$
K_{3}^{1}\left(n, w j n^{\prime} w^{\prime}\right)=\frac{3^{n}}{w^{\prime}-t^{n} w} g(w) / g\left(t^{n} w\right)
$$

4. Dpegroroн 2. / $\sum$ Менинна- Бaprеса

Деша

$$
\sum_{n=1}^{\infty} F(n) \zeta^{n}=\frac{1}{2 \pi i} \int_{C} \Gamma(-s) \Gamma(1+s)(-3)^{s} F(s) d s
$$

que xarounix $F$

$|\zeta|<1, \zeta \notin \mathbb{R}_{\geqslant 0}$

$$
(-3)^{s}=e^{s \log (-3)}
$$

pasper $\frac{\log }{-3 \leq 0}$
wan $-3 \leqslant 0$
Pf, $\left(\begin{array}{l}\text { Res } \Gamma(s)=(-1)^{k} / k! \\ s=-k\end{array}<\operatorname{ymp}\right.$.

$$
k=0,1, \ldots
$$

$$
\Longrightarrow \begin{aligned}
& - \text { Res } M(s+1) r(-s)=(-1)^{n} \\
& s=n \\
& n=1,2, \ldots
\end{aligned}
$$

$$
K_{3}^{1}\left(n, w j n^{\prime}, w^{\prime}\right)=\frac{3^{n}}{w^{\prime}-t^{n} w} g(w) / g\left(t^{n} w\right)
$$

T. 2

$$
\operatorname{det}\left(1+K_{3}^{1}\right)^{\operatorname{Tar}^{2}\left(Z_{1} \times \gamma\right)}=\operatorname{det}\left(1+K_{3}^{2}\right)^{L^{2}(\gamma)}
$$

$K_{5}^{2}$ - egporo na $\gamma$

$$
K_{s}^{2}\left(w, w^{\prime}\right)=\frac{1}{2 x^{i}} \int_{C} \Gamma(-s) \Gamma(1+s)(-\zeta)^{s} \frac{g(w)}{g\left(t^{s} w\right)} \frac{d s}{w^{\prime}-t^{s} w}
$$

D. $\quad \operatorname{det}\left(1+k_{5}^{1}\right)=$

$$
=\sum_{l \geqslant 0} \sum_{n_{1}, \ldots, n_{l} \geqslant 1} \frac{1}{e!} \frac{1}{(2 a i)} l \oint_{\gamma} \oint_{\gamma} \operatorname{det}_{l \times l} k_{\zeta}^{1}\left[n_{i}, w_{i} ; n_{j}, w_{j}\right]
$$

$$
\begin{aligned}
& \sum_{n \geqslant 0} k_{3}^{1}\left(n, w, n^{\prime}, w^{\prime}\right)= \\
& =\sum_{n \geqslant 0} \frac{3^{n}}{w^{\prime}-t^{n} w} g(w) / g\left(t^{n} w\right) \\
& =\frac{1}{2 \pi^{i}} \int_{C} \Gamma(-s) \Gamma(1+s)(-\zeta)^{s} \frac{g(w)}{g\left(t^{s} w\right)} \frac{d s}{w^{\prime}-t^{s} w}
\end{aligned}
$$

5. Асинитотическии́ A Агянз.

$$
\begin{aligned}
\frac{1}{\left(z t^{h}, t\right)_{\infty}} \rightarrow \text { ungukarogy } \\
\frac{\frac{h-L J t}{L^{\alpha}}}{<-r}
\end{aligned}
$$

(\& pabronegros no

$$
\left.3 t^{h} \leq 0\right)
$$

$$
\begin{aligned}
& \mathbb{E} \frac{1}{\left(\zeta t^{h(L x, L y)} ; t\right)_{\infty}} \\
& J=-t^{-L H+L^{\alpha} \rho} \\
& r \in \mathbb{R} \\
& \text { H(x,y) } \\
& \text { - upegerbras } \\
& \text { popue } \\
& L \rightarrow \infty \\
& -t^{-L J+r L^{\alpha}+h} \underbrace{0, \quad \frac{h-L J L}{L^{\alpha}}>-r}_{-\infty} \begin{array}{l}
\frac{h-L \not C L}{L^{\alpha}}<-r
\end{array}
\end{aligned}
$$

$$
\begin{aligned}
& \Rightarrow \\
& \mathbb{E} \frac{1}{\left(\zeta t^{h(L x, y)} ; t\right)_{\infty}} \underset{L_{\rightarrow \infty}}{\simeq} \mathbb{P}\left(\frac{h(L x, L y)-L \mathcal{H}(x, y)}{L^{\alpha}}>-r\right)
\end{aligned}
$$

6. Aramiz ma ypobre Apegronbma

$$
u_{i} \equiv 1, \quad v_{i} \equiv v \in(0,1), \quad t \in(0,1)
$$

(2 napa a merifa)

$$
g(z)=(1-z)^{L x}(z-v)^{-l y}
$$

$$
\left[\frac{1}{h((L x, L y), 1)}=\quad \zeta=-t^{-L H+L^{\alpha \rho}}\right.
$$

$$
=\sum_{l \geqslant 0} \frac{1}{l!} \frac{1}{(2 \pi i)^{2}} \underbrace{\oint}_{c} \underbrace{\oint}_{\gamma} \underbrace{\oint \oint}_{\Gamma 1} \prod_{i=1}^{\ell} r\left(-s_{i}\right) r\left(s_{i}+1\right)(-3)^{s_{i}} \frac{g\left(w_{i}\right)}{g\left(t^{s_{i}} w_{i}\right)}
$$

$$
\operatorname{det}\left[\frac{1}{w_{j}-t^{s_{i}} w_{i}}\right]_{i, j=1}^{l} d \vec{\omega} d \vec{s}
$$



$$
\begin{aligned}
& S_{i}=\frac{1}{\log t} \log \left(z_{i} / \omega_{i}\right) \\
& =\sum_{l \geqslant 0} \frac{1}{l!} \frac{1}{(2 n i)} \underbrace{2} \underbrace{\oint}_{\gamma} \underbrace{\oint}_{\gamma} \underbrace{\oint} \oint \pi_{i=1}^{l} \Gamma\left(-s_{i}\right) \Gamma\left(s_{i}+1\right)(-3)^{s_{i}} \frac{g\left(w_{i}\right)}{g\left(z_{i}\right)} \\
& \operatorname{det}\left[\frac{1}{\omega_{j}-z_{i}^{c}}\right]_{i . j=1}^{l} \frac{d \vec{\omega} d \vec{z}}{{\underset{i}{i}}_{\left.\eta_{i} \log t\right)}^{l}}
\end{aligned}
$$

$z=t^{s}(w, w$, no oxpy*troin

$$
\begin{array}{r}
|w|=a, \quad v<a<1 \\
|z|=\left|t^{s}\right| \cdot|w|=\left|t^{\operatorname{Res}}\right| \cdot a=t^{\delta} \cdot a
\end{array}
$$

To*l wo ongy xHocru (re umbro jay!)

vouszozrentray log $b$

$$
S_{i}=\log \left(z_{i} / w_{j}\right)
$$

$$
\begin{aligned}
& e^{L\left[\delta\left(\omega_{i}\right)-S\left(z_{i}\right)\right]+r L^{\alpha}\left[\log z_{i}-\log \omega_{i}\right]} \\
& S(\omega) \stackrel{\operatorname{def}}{=}-y \log (\omega-v)+x \log (1-\omega) \\
& +\underbrace{孔(x, y) \log \omega} \\
& w_{z}(-\zeta)^{s}=e^{(\log z-\log \omega)\left(-L J L+M L^{\alpha}\right)} \\
& \zeta=-t^{-L J+r L^{\alpha}}, t^{s}=z / \omega
\end{aligned}
$$

․ $\oint e^{L S(z)} d z, L \rightarrow \infty$ netog merebana /cray. pa3b1 / steepest descut

$$
\begin{aligned}
& \left|e^{L S(t)}\right|=\left|e^{L \cdot \operatorname{Re} S(z)}\right| \cdot\left|e^{i L \operatorname{In} S(t)}\right| \\
& \Rightarrow e^{L S(z)} \rightarrow \begin{cases}0 & \operatorname{Re} S<0 \\
\infty & \operatorname{Re} S>0 \\
\text { (oyynmepyet } T, & \operatorname{Re} S=0\end{cases}
\end{aligned}
$$

Ecmi $S^{\prime}\left(z_{0}\right)=0$, T0, ocyunaeymu re $l_{\text {axicun! }}$

$$
\begin{aligned}
& \delta(z)=S\left(z_{0}\right)+S^{\prime \prime\left(z_{0}\right) \cdot \frac{1}{2}\left(z-z_{0}\right)^{2}+\ldots} \text { Koltyp } \\
& \quad \operatorname{Re}\left(z-z_{0}\right)^{2}:-\theta
\end{aligned}
$$

$$
\begin{aligned}
& z-z_{0}=\frac{i}{\sqrt{L}} \tilde{z}, \tilde{z} \in \mathbb{R} \text { b oup -u } z_{0} . \\
& L S(z)=L S\left(z_{0}\right)-\frac{1}{2} \tilde{z}^{2} \cdot S^{\prime \prime}\left(z_{0}\right)+O\left(L^{-1 / 2}\right) . \\
& \Rightarrow \oint e^{L S(z)} d z \sim e^{L S\left(z_{0}\right)} \cdot \frac{C}{\sqrt{L}}
\end{aligned}
$$

Mopars: bus acurintotura oupegengetes no kput. Toukam $\delta(z)$

Y HaC:

$$
e^{L\left[S\left(\omega_{i}\right)-S\left(z_{i}\right)\right]}
$$

$$
\Rightarrow \quad \begin{gathered}
z_{i}, w_{i}-b \text { oxp-iu } \\
\text { k/uт. toyek. }
\end{gathered}
$$

крит. точек.
8. Kakne kpri-Toyke y rac?

$$
\begin{align*}
& S(w)=孔 \log w+x \log (1-w)-y \log (w-v) \\
& S^{\prime}(w)=0 \quad \text { clagp } \cdot y p-e \tag{1}
\end{align*}
$$

zabuc. or H:


2 II kopme, pagrule
$2 \mathbb{C}$ conguet kopiths
groíréá koperis
(1): $\iint e^{L S(\omega)-L S(z)}$

$$
\begin{equation*}
\sim e^{\operatorname{Re}\left(L\left(S\left(z_{1}\right)-S\left(z_{2}\right)\right)\right)} \tag{3}
\end{equation*}
$$

име $\infty$
(max0)
ra camoen gere O, y rac te beportmorat (7T0 wogckaxys kortypbl)

$$
\text { (2) } \iint e^{L S(\omega)-L S(z)}=\frac{1}{(\sqrt{L})^{2}} e^{L \operatorname{Re}\left[S\left(z_{1}\right)-S\left(\overline{z_{1}}\right)\right]}
$$

Ho torga $z$,w b pajtila $\left.\operatorname{det}\left[\frac{1}{\omega_{i}-z_{j}}\right]_{1}^{\text {On pectrocill }}\right]_{i}^{l} \frac{1}{z_{i} \log t}, d \vec{z} d \vec{\omega}$
hoteruguaer no 300 urite pecro, ho mova गг octabue, FTo craxnel rem glo. kpuri: $T$.
(3) hycso gloôthers крит-Touka, $S^{\prime}=S^{\prime \prime}=0$ - 2 ypabrerves tha $w, f(x, y)$

$$
\Rightarrow \text { Serér Torbke } u=0
$$

$$
\begin{aligned}
& L S(z)=L S\left(z_{0}\right)+0+0+L \frac{S^{\prime \prime \prime}\left(z_{0}\right)}{6}\left(z-z_{0}\right)^{3}+\ldots \\
& z-z_{0}=\tilde{z} / L^{1 / 3} \quad\left(z_{0} \neq 0\right. \\
& \begin{array}{l}
\text { syfé lagno) parsegiencs } \\
\text { nowozke }
\end{array} \\
& d z=d \tilde{z} / L^{1 / 3}
\end{aligned}
$$

$$
\begin{aligned}
& \underbrace{\prod_{i}\left[-\frac{\pi}{\sin \left[\frac{\pi \log z_{i}-\pi \log \omega_{i}}{\log t}\right]}\right]}_{n \in \mathbb{Z}} . \\
& \arg z \in(0,2 \pi)+2 \pi n \\
& \arg \omega \in(0,2 \pi) \text {, } \\
& h \neq 0 \\
& \sim L^{l \cdot 1 / 3} \text { ecm } \\
& n=0 \text {. }
\end{aligned}
$$

Daree,

$$
\begin{gathered}
L\left[S\left(\omega_{i}\right)-S\left(z_{i}\right)\right]+r^{\alpha}\left[\log z_{i}-\log \omega_{i}\right] \\
\sim\left(\widetilde{w}_{i}^{3}-\widetilde{z}_{i}^{3}\right)+r L^{\alpha} \cdot L^{-1 / 3}(\tilde{z}-\tilde{\omega}) \cdot C^{\prime} \\
+o(L)
\end{gathered}
$$

$\Rightarrow \quad \alpha=1 / 3 \quad$ - upaburbiers witcare prykryayur
9. Mpegenbreas gopua

$$
\begin{aligned}
& S(w)=H \lg w-y \log (w-v)+x \log (1-w) \\
& S^{\prime}(w)=\frac{H}{w}-\frac{x}{1-w}-\frac{y}{w-v} \\
& S^{\prime \prime}(w)=-\frac{H}{w^{2}}-\frac{x}{(1-w)^{2}}+\frac{y}{(v-v)^{2}}
\end{aligned}
$$

$$
\begin{aligned}
S^{\prime}(w)=S^{\prime \prime}(w) & =0 \quad \Rightarrow \\
w S^{\prime \prime}+S^{\prime} & =0, \quad \frac{v y}{(v-w)^{2}}=\frac{x}{(1-w)^{2}}
\end{aligned}
$$

ge.

$$
W_{c}=\text { kpmir-Torka }
$$

$$
\left\{\begin{array}{l}
w_{c}=\frac{v(x-y) \pm(1-v) \sqrt{v x y}}{x-v y} \\
\nVdash\left(w_{c}\right) \\
\not(x, y)=\frac{w_{c}\left(y-v x+\omega_{c}(x-y)\right)}{\left(1-w_{c}\right)\left(v-w_{c}\right)}
\end{array}\right.
$$



kaptukna vee zabneni ot (t)
$S(z)$ тoke the zabucles or $t$.
10. Эbpucruka c korycom

$\left(1-b_{1}\right) b_{1}^{a}$, \& apgrean gura
 upposera rayalo

$$
=\frac{1}{1-b_{1}}
$$

lo up. Guerer undera blopt $=\frac{1}{1-t b_{1}}$


$$
\begin{aligned}
& \text { raceron }=\frac{1-b_{1}}{1-t b_{1}} \\
& =\frac{1-t v-1+v}{1-t b-t+t b}=v
\end{aligned}
$$

11. A cumntotuka

$$
\begin{aligned}
& S(w)=H \log w-y \log (w-v)+x \operatorname{leg}(1-w) \\
& H(x, y)=\frac{(\sqrt{y}-\sqrt{\sqrt{x}})^{2}}{1-v} \\
& v<\frac{y}{x}<\frac{v}{v}-\infty<\omega_{c}(x, y)<0 \\
& e^{i\left(S\left(v_{i}\right)-s\left(z_{i}\right)\right)}
\end{aligned}
$$

$\operatorname{Re} S(z)>\operatorname{Re} S\left(\omega_{c}\right)$


$$
\begin{gathered}
(v=1 / 3, \quad x=1, \quad y=3 / 2) \\
w_{c}=-1.27
\end{gathered}
$$



$$
e^{L\left(S\left(w_{i}\right)-S\left(z_{i}\right)\right)}
$$

T nepegleanem woroypol coga, Terge lunag bre oxppou $\omega_{c}$

$$
\sim e^{-\beta L} \xrightarrow{\beta>0} 0
$$

B grysos zore

$$
\begin{aligned}
& \frac{y}{x}<v \\
& u m \\
& \frac{y}{x}>\frac{1}{v}
\end{aligned}
$$



Paynoxime:

$$
\begin{aligned}
& z=\omega_{c}+L^{-1 / 3} \tilde{z} / \sigma \\
& \omega=\omega c+L^{-1 / 3} \tilde{\omega} / \sigma \\
& \hat{\sigma}=-\sqrt[3]{\frac{S^{\prime \prime \prime}\left(w_{c}\right)}{2}}>0 \quad S^{\prime \prime \prime}\left(\omega_{c}\right)<0 \\
& \sum_{l \geqslant 0} \frac{1}{l!} \frac{1}{(2 \pi i)^{2 l}} \oint_{\gamma^{\prime}}-\oint_{\gamma} d \vec{z} \oint_{\gamma}-\oint_{k} d \vec{o} d e t\left[\frac{1}{w_{i}-z_{j}}\right]_{1}^{l} \\
& \times \prod_{i=1}^{l}\left[-\frac{\pi}{\sin \left[\frac{n \log z_{2}-n \log \omega_{i}}{\log t}\right]}\right] \cdot \frac{e^{L\left(S\left(\omega_{i}\right)-s\left(z_{i}\right)\right)+L^{1 / 3}\left(\log _{i}-\log _{i} \omega_{i}\right)}}{z_{i} \log t}
\end{aligned}
$$

$$
\begin{aligned}
& \frac{d w_{i} d z_{j}}{w_{i}-z_{j}}=\frac{d \tilde{w}_{i} \tilde{z}_{j}}{\tilde{w}_{i}-\tilde{z}_{j}} \cdot L^{-1 / 3} \\
& p L^{1 / 3}\left[\rho_{g}\left(z_{i}\right)-\log \left(w_{i}\right)\right]=\frac{\Gamma}{3 \omega_{c}}(\tilde{z}-\tilde{\omega})+O\left(L^{-1 / 3}\right) \\
& L\left(S\left(w_{i}\right)-S\left(z_{i}\right)\right)= \\
& =\frac{\tilde{w}^{3}-\tilde{z}^{3}}{6 z^{3}} \cdot S^{\prime \prime \prime}+O\left(L^{-1 / 3}\right) \\
& =\frac{\tilde{z}^{3}-\tilde{w}^{3}}{3}+O\left(L^{-1 / 3}\right) \\
& B 1 / z_{i} \\
& z_{i}=
\end{aligned}
$$

$$
\begin{aligned}
\frac{-\pi}{\sin \left[\frac{\pi \log z-\pi \log \omega}{\log t}\right]} & \simeq \frac{-\pi}{\sin \left[\frac{\pi}{\log t}\left(L^{-1 / 3} / \sigma \omega_{c}(\tilde{z}-\tilde{\omega})\right)\right]} \\
& \simeq \frac{-1 \cdot \log t \cdot}{(\tilde{z}-\hat{\omega})} \cdot 6 \omega c L^{+1 / 3}
\end{aligned}
$$

Pegyutar $\operatorname{det}\left(1+k_{s}^{2}\right)=\mathbb{E} \frac{1}{\left(3 t^{h}, t\right)_{\infty}}$
b renver pexwe
$x l, y l$

$$
\begin{aligned}
& x L, y L \\
& F\left((x, y)=\frac{(\sqrt{y}-\sqrt{v x})^{2}}{1-v}\right. \\
& \zeta=-t^{-L H+L^{1 / 3} r}
\end{aligned}
$$

cxoguecs $k$ bot tarkay pregy:
no4rentio

$$
\begin{aligned}
& \sum_{l \geq 0} \frac{1}{l!(2 \pi i)^{2 l}} \int-\iint_{-\int} d \overrightarrow{\tilde{z}} d \vec{w} \\
& \tilde{\omega}) k \tilde{w_{i}} \\
& \cdot \operatorname{det}\left[\frac{1}{\tilde{w}_{i}-\tilde{z}_{j}}\right]_{1}^{l} \cdot\left(w_{c}<0\right) \\
& , \prod_{i=1}^{l} \exp \left[\frac{\tilde{z}_{i}^{3}}{3}-\frac{\vec{w}_{i}^{3}}{3}+\frac{p}{\left(-2 \omega_{c}\right)}\left(\tilde{w_{i}}-\tilde{z}_{i}\right)\right] \\
& \cdot \frac{1}{\tilde{w}_{i}-\tilde{z}_{i}}
\end{aligned}
$$

12. Orber

$$
\begin{aligned}
& \operatorname{det}\left(1+K_{s}^{2}\right)=\mathbb{E} \frac{1}{\left(s t^{h}, t\right)_{\infty}} \\
& \downarrow L \rightarrow \infty \\
& * \operatorname{det}\left[1+\widetilde{K}_{r / 6 w_{c}}\right] \\
& \widetilde{K}_{-r}\left(u, u^{\prime}\right)=-\frac{1}{2 \pi i} \int e^{\omega^{3} / 3-u^{3} / 3+r(u-\omega)} \\
& u, u^{\prime} \in \dot{\gamma} \cdot 0 \quad:\left\langle\omega \quad \cdot \frac{d w}{(\omega-u)\left(\omega-u^{\prime}\right)}\right. \\
& \text { * }=\lim _{h \rightarrow \infty} \mathbb{P}\left[\frac{h(L x, L y)-L \mathcal{H}(x, y)}{L^{1 / 3}}>-r\right]
\end{aligned}
$$

$$
\text { legine gus } v<\frac{y}{x}<\frac{1}{v}
$$

hekyms 8

$$
28 \text { anpers }
$$

1. Hawomntaruce: cipyktypa

$$
\mathbb{E} \frac{1}{\underbrace{\left(3 t^{h(x, y)}\right.}_{\text {r }} ; t)_{\infty}}=\underbrace{\operatorname{det}\left(1+K_{s}^{2}\right)}_{\text {oup. 中pegromome }}
$$

$h(x, y)=p \cdot$ bancoin $\cos x \cdot 6 b$ mogem

$$
\frac{y-\begin{array}{l}
u_{i} \equiv 1, \quad v_{j} \equiv v \\
\rightarrow l_{1}=\frac{1-v}{1-v t}
\end{array}}{x}
$$

$$
t \in(0,1))
$$

$$
v \in(0,1)
$$

Ppegroubue na kotūype $\gamma$

$$
\operatorname{det}\left(1+k_{5}^{2}\right)=\sum_{l \geqslant 0} \frac{1}{l!} \underbrace{\oint}_{l} \oint_{l} \operatorname{det}\left[k_{5}^{2}\left(x_{i}, x_{j}\right)\right]_{1}^{l} \cdot d x_{1} \ldots d x_{l} .
$$



$$
\int e^{L f(z)} d z
$$

$$
\begin{gathered}
K_{s}^{2}\left(w, w^{\prime}\right)=\frac{1}{2 x^{2}} \int_{C}^{C} \Gamma(-s) \Gamma(1+s)(-\zeta)^{s} \frac{g(w)}{g\left(t^{s} w\right)} \frac{d s}{w^{\prime}-t^{s} w} \\
w, w^{\prime} \in \gamma \underset{0}{c} \quad g(w)=(1-z)^{x} /(z-s)^{y}
\end{gathered}
$$

Pezyrutat (hupomras rekyus)

(72)

Фaykyayin reetpub. gues $v<\frac{y}{x}<\frac{1}{v}$ \& was noyzalem

$$
\mathbb{E} \frac{1}{\left(\zeta t^{h(L x, L y)}, t\right)_{\infty}} \sim \mathbb{P}\left(\frac{h(L x, L y)-L J t(x, y)}{L^{1 / 3}}>-r\right)
$$

$$
\begin{aligned}
& \zeta=-t^{-L \mathcal{F}(x, y)+\Gamma L^{1 / 3}}, \quad r \in \mathbb{R} \\
& L \rightarrow \infty \\
& >\text { cxognics } k \quad \operatorname{det}\left[1+\tilde{K}_{p / \text { wwc }}\right]_{y>} .0
\end{aligned}
$$

Bwc<0, revorotas elracs $p$-yus or $x, y$
mpegeusiol agro b ong. Ppegravoma:

$$
\begin{aligned}
& \widetilde{K}_{-r}\left(u, u^{\prime}\right)=-\frac{1}{2 \pi i} \int e^{\omega^{3} / 3-u^{3} / 3+r(u-\omega)} \frac{d \omega}{(\omega-n)\left(\omega-u^{\prime}\right)} \\
& u, u^{\prime} \in \forall_{0} \quad 0 \& \frac{\psi^{\omega}}{u^{\prime} \mid k_{w}^{\omega / 3}}
\end{aligned}
$$

$$
\operatorname{Re}\left(z^{3}\right)=0
$$


2. Npegeruras 甲opua TASEP/ASEP




$$
H(x, y)=\frac{(\sqrt{y}-\sqrt{v x})^{2}}{1-v}
$$

$x, y$ wikampyglu ororo grazorank

$$
\begin{aligned}
& x=\frac{\tau}{1-v}+\alpha, \quad y=\frac{\tau}{1-v}+\beta, \quad, \quad \tau \in \mathbb{R} \geqslant 0 \\
& H \rightarrow \frac{(\tau+\beta-\alpha)^{2}}{4 \tau} \quad \begin{array}{l}
\text { TasEP } \\
h(\tau, x) \\
h(\tau, x)=\frac{x^{2}+\tau^{2}}{2 \tau}
\end{array}
\end{aligned}
$$

3. Ngemтиquкаушs upegera

$$
\begin{aligned}
& \widetilde{K}_{-r}\left(u, u^{\prime}\right)=-\frac{1}{2 \pi i} \int e^{\omega^{3 / 3}-u^{3} / 3+r(u-w)} \frac{d \omega}{(w-u)\left(w-u^{\prime}\right)} \\
& u, u^{\prime} \in \forall_{0} \quad 0_{0} \quad 0 \forall
\end{aligned}
$$

$$
\begin{aligned}
P\left(\frac{h(L x, L y)-L J(x, g)}{\left(-b w_{c}\right) L^{1 / 3}}>-P\right) \rightarrow & \operatorname{det}\left[1+\widetilde{K_{-r}}\right] \\
& F_{2}(r), r \in \mathbb{R}
\end{aligned}
$$

$F_{2}(r)$ - pipacupegeserses cu, bemurura $\xi$

$$
h(L x, L y) \sim L J C(x, y)+r\left(-6 w_{c}\right) L^{1 / 3} \cdot \xi
$$

$$
\left.\begin{array}{rll}
F_{2}(r) & \rightarrow 1, & r \rightarrow+\infty \\
>0, & r \rightarrow-\infty
\end{array}\right\} \text { ylougune }
$$

Hagblbarices pacupegerevuler
Tpaícu-Bugona (1993, er. matp)


Ylengun engé 2 mpegcrabrerves gus $F_{2}$ (ofro sez gou-ba)
(1) Agpo Júpu

$$
A(i, s)=\frac{1}{(2 \pi)^{2}} \iint_{\operatorname{sem}} \frac{e^{u^{3 / 3}-v^{3 / 3}-r u+s v}}{u-v} d u d v
$$

$A_{i}(x)-p$. Эúpu (Airry) $\quad A_{i}^{\prime \prime}=x \cdot A_{i}$


$$
\begin{aligned}
\operatorname{Ai}(x) & =\frac{1}{\pi} \int_{-\infty}^{+\infty} \cos \left(t^{3} / 3+t x\right) d t \underbrace{t}_{0} \underbrace{t}_{2} \\
& =\frac{1}{2 \pi} \int_{-\infty}^{+\infty} e^{i t^{3} / 3+i t x} d t
\end{aligned}
$$

nemua. (gok).

$$
e^{-r u+s v}=\frac{1}{s-r}\left(\frac{\partial}{\partial u}+\frac{\partial}{\partial v}\right) e^{-r u+s v}
$$

$$
\frac{e^{u^{3} / 3-v^{3} / 3}\left(\frac{\partial}{\partial u}+\frac{\partial}{\partial v}\right) e^{-r u+s v}}{(u-v)(s-r)}
$$

\& no yactiln, moryralm $(u+v)$
Ynp. - Зavon4nio gon-lo

- Unobeguris guge. ype Эípu
Sinpu


D-80. $\operatorname{det}(1+x y)=\operatorname{det}(1+y x)$
(osa $X Y, Y X$ - ageprule onep)

$$
\begin{aligned}
& \widetilde{K}_{-r}\left(u, u^{\prime}\right)=-\frac{1}{2 \pi i} \int e^{\omega^{3} / 3-u^{3} / 3+r(u-w) d w} \frac{(\omega-u)\left(w-u^{1}\right)}{\left(u^{1}\right)} \\
& u, u^{\prime} \ngtr 0 \\
& \theta-x y
\end{aligned}
$$

$$
\frac{1}{w-u}=\int_{0}^{\infty} d \lambda \cdot e^{-\lambda(w-u)} \quad \operatorname{Re}(w-u)>0
$$

(6)

$$
\begin{align*}
& X: L^{2}\binom{\eta}{K} \rightarrow L^{2}(0, \infty) \\
& Y: L^{2}(0, \infty) \rightarrow L^{2}\binom{X}{K} \\
& X(u, \lambda)=e^{-u^{3} / 3+u(\lambda+r)} \\
& Y\left(\lambda, u^{\prime}\right)=\frac{1}{2 \pi i} \int \frac{d z}{z-u^{\prime}} e^{z^{3} / 3-z(\lambda+r)} \\
& \langle z
\end{align*}
$$

$\Downarrow$

$$
\begin{aligned}
& Y X: L^{2}(0, \infty) \circlearrowleft \\
& Y X\left(\lambda, \lambda^{\prime}\right)=A\left(\lambda+r, \lambda^{\prime}+r\right)
\end{aligned}
$$

$$
\begin{aligned}
& \Rightarrow F_{2}(r)= \operatorname{det}(1-A)(r,+\infty) \\
& r \rightarrow+\infty \\
& r \rightarrow-\infty 1
\end{aligned}
$$

$$
\log \operatorname{det}(1-A)=\operatorname{tr} \log (1-t) \simeq \operatorname{tr}(-A+\cdots)
$$

(3)


Painleve II

$$
(I, I I, \ldots \bar{I})
$$

Bameranue: Hopu. p-e kak Ppegrovom (поменнии степень)

$$
G_{\Gamma}\left(\omega, \omega^{\prime}\right)=-\frac{1}{2 \pi_{i}} \int e^{\frac{\text { (поменепи степень) }}{-\omega^{2} / 2+z^{2} / 2}+\Gamma(z-\omega)} d z
$$

$\operatorname{det}\left(1+G_{r}\right)=\phi(r), \phi \cdot$ pacup. vayccobctor $c, b$.
Tpuisu- $B y^{\circ} \mathrm{o}^{n} \mathrm{f} f(x) \rightarrow$ rayec


$$
\int_{1} e^{-\omega^{3} / 3+z^{3} / 3+\cdots} \cdot\left(\frac{z}{\omega}\right) \ldots
$$

BBP phase transition (2005)
4. Bonbure ykrorethus $b 6 b$ uggen 4.1 Knaccu4ecras teog. bep.

$$
\begin{aligned}
& S_{n}=x_{1}+\ldots+x_{n} \text {, wes, } x_{i}=0,1 \text { bep } \frac{1}{2} \\
& \frac{S_{n}}{n} \rightarrow \frac{1}{2} \\
& \mathbb{P}\left(\frac{S_{n}-n / 2}{\frac{1}{2} \sqrt{n}}<x\right) \rightarrow \Phi(x)=\int_{-\infty}^{x} \frac{1}{\sqrt{27}} e^{-z^{2} / 2} d z \\
& \mathbb{P}\left(S_{n}-n / 2>\alpha_{n}\right) \asymp e^{-n I_{+}(\alpha)} \\
& \mathbb{P}\left(S_{n}-n / 2<-\alpha n\right) \asymp e^{-n I_{-}(\alpha)} \\
& \alpha>0 \\
& \lim _{n} \frac{1}{n} \log \mathbb{P}=-I_{ \pm}(\alpha)
\end{aligned}
$$

Lim sup, liminf -urriga gocarolto.

$$
\begin{aligned}
& I_{ \pm}(\alpha) \text { - leozpactanot } \\
& \mathbb{P}\left(S_{n}>\frac{n}{2}+\alpha n\right) \asymp \mathbb{P}\left(S_{n}=n\left(\alpha+\frac{1}{2}\right)\right) \\
& =\frac{1}{2^{n}}\binom{n}{n \alpha+n / 2} \quad\left(\Rightarrow I_{+}=I_{-}\right) \\
& =\left(G_{\text {Tupmitz }}\right) \quad N!\sim e^{N \log N-N} \\
& =e^{-n\left(\left(\alpha+\frac{1}{2}\right) \log \left(\alpha+\frac{1}{2}\right)+\left(\alpha-\frac{1}{2}\right) \log \left(\alpha-\frac{1}{2}\right)\right)}
\end{aligned}
$$

4.2 TASEP


$$
\frac{1}{t} X_{c t}(t) \rightarrow 0
$$

$$
\begin{aligned}
& \frac{1}{t} X_{t / 4}(t) \rightarrow 0, \quad \frac{1}{t^{1 / 3}} X_{t / 4}(t)-\text { coyrastoo } \\
& \alpha>0 \\
& \text { (Tracy-Widom) } \\
& \text { cmurnom } \\
& \text { wegrenaro: } \\
& \mathbb{P}\left(X_{t / 4}(t)<-\alpha t\right) \\
& e^{-t \cdot I_{-}(\alpha)} \\
& \mathbb{P}\left(X_{t / 4}(t)>\alpha t\right) \asymp e^{-t^{2} \cdot I_{+}^{(\alpha)}} \\
& \text { (Johnson 2000) }
\end{aligned}
$$

Theorem 1.1. For each $q \in(0,1)$ and $\gamma \geq 1$,

$$
\begin{equation*}
\lim _{N \rightarrow \infty} \frac{1}{N} \mathbb{E}[G([\gamma N], N)]=\frac{(1+\sqrt{q \gamma})^{2}}{1-q}-1 \doteq \omega(\gamma, q) \tag{1.4}
\end{equation*}
$$

Also, $G([\gamma N], N)$ has the following large deviation properties. There are functions $i(\epsilon)$ and $\ell(\epsilon$ ) (which depend on $q$ and $\gamma$ ), so that, for any $\epsilon>0$,

$$
\begin{equation*}
\lim _{N \rightarrow \infty} \frac{1}{N^{2}} \log \mathbb{P}[G([\gamma N], N) \leq N(\omega(\gamma, q)-\epsilon)]=-\ell(\epsilon) \tag{1.5}
\end{equation*}
$$

and

$$
\begin{equation*}
\lim _{N \rightarrow \infty} \frac{1}{N} \log \mathbb{P}[G([\gamma N], N) \geq N(\omega(\gamma, q)+\epsilon)]=-i(\epsilon) \tag{1.6}
\end{equation*}
$$

The functions $\ell(x)$ and $i(x)$ are $>0$ if $x>0$.
Note that the existence of the limit (1.4) follows by a subadditivity argument, so it is the explicit form of the constant that is interesting. The large deviation result (1.6) has been obtained in [Se2]. The theorem will be proved in section 2.
4.3. Ctox. 6B mogenb

$$
\begin{aligned}
& \alpha>0 \\
& \mathbb{P}(h(L x, L y)>L H(x, y)-\alpha L) \approx 1-e^{-L \cdot I+(\alpha)} \\
& +\alpha L \longrightarrow 0 \\
& \uparrow \operatorname{acnm} n \text {. эквиe. } \\
& \mathbb{E} \frac{1}{\left(-t^{h(L x, L y)-L \notin \oplus) \alpha L} ; t\right)_{\infty}} \\
& =\operatorname{det}\left(1+K_{\zeta}^{2}\right) \quad \zeta=-t^{-L J L \pm \alpha L} \\
& \rightarrow 1:(1)+\int K_{3}^{2}(x, x) d x+\frac{1}{2} \int \infty \\
& \rightarrow 0: 1+\int K_{3}^{2}(x, x) d x+\frac{1}{2} \iint-
\end{aligned}
$$

oylret 10

$$
\iint e^{L(S(\omega)-S(z))} \cdot d z d \omega \circ \text { (koreypwe) }
$$

$$
S(w)=(f l-\alpha) \lg w-y \log (w-v)+x \lg (1-w)
$$

$$
\alpha=0
$$

$$
s^{\prime}=s^{\prime \prime}=0
$$

mexgur 7,

$$
\begin{aligned}
& w=w_{c} \\
& S^{\prime}=0-k b \cdot y p+w
\end{aligned}
$$


hyers, $x=y=1, v=1 / 4$

$$
\begin{aligned}
& f(1,1)=\frac{1-\sqrt{v}}{1+\sqrt{v}}=\frac{1}{3} \\
& W_{c}(\alpha)=\frac{4+15 \alpha \pm 3 \sqrt{3} \sqrt{\alpha(8+3 \alpha)}}{24 \alpha-8}
\end{aligned}
$$

$\alpha>0,2$ pa.jholx bay. kopthes $\alpha<0,2$ kounn. coup. koptus

5. Hветнble mogeme (robar тема)

$\longrightarrow$ Mberhoun TASEP,
zagana o yactuye II Kracca
 uporixk

$$
I, \quad \frac{U(t)}{t} \rightarrow \underset{\text { ver }}{T}[-1,1]
$$

Nekymes 9
(1) R matpuybl us kbarctobux yrywn ("burconars Hayka" ga bepm. uggensum)

Anreiger Xoupa - koanzorps + gou. cb-ba

- $A \otimes A \rightarrow A$
$\left[\begin{array}{l}\Delta: A \rightarrow A \otimes A \\ \text { corberoblyer gecurbum }\end{array}\right.$

| $A$ | $B$ | $C$ | $D$ | $b$ |
| :---: | :---: | :---: | :---: | :---: |
| $\rightarrow$ | $\triangle$ | $\cdots$ | $\cdots$ |  |

$$
\begin{aligned}
& B(v \otimes W)= A v \otimes B \omega+B v \otimes D w \\
& \uparrow \frac{1}{1} \uparrow \uparrow+ \\
& \Delta(B)=A \otimes B+B \otimes D
\end{aligned}
$$

(t packpoitue (kovora)
koyunco xeruce
oneratopob heck. crousyax
"Klarrobai yyynna" - korkparkas anzesja Xoupa, "noxaxans" ma $q$ - geopopuay to kaccuy. sen (nayprocioú/apputros./..)

1) a1z. Mu

$$
S L(2, \mathbb{C})=\left\{\left(\begin{array}{ll}
a & b \\
c & d
\end{array}\right): \begin{array}{c}
a, b, c, d \in \mathbb{C} \\
a d-b c=1
\end{array}\right\}
$$

$$
\downarrow \text { an2. Au } \operatorname{det}(1+x)=1 \Rightarrow \operatorname{tr} X=0
$$

$$
s l_{2}=\left\{\left(\begin{array}{cc}
a & b \\
c & -a
\end{array}\right): \quad a, b, c \in \mathbb{C}\right\}
$$

Sagac $e=\left(\begin{array}{ll}0 & 1 \\ 0 & 0\end{array}\right) \quad f=\left(\begin{array}{ll}0 & 0 \\ 1 & 0\end{array}\right) \quad h=\left(\begin{array}{cc}1 & 0 \\ 0 & -1\end{array}\right)$
onepasus $[A, B]=A B-B A$
coothon. $\quad[e, f]=h, \quad[h, f]=-2 f$

$$
[h, e]=2 e
$$

2) Appuntras

$$
\begin{aligned}
& e_{i}, f_{i}, h_{i} \\
& i=0,1
\end{aligned}
$$

$\left(\right.$ Kac-Moody $\left.\leftrightarrow A=\left(\begin{array}{cc}2 & -2 \\ -2 & 2\end{array}\right)\right)$
cootrom.

$$
\begin{aligned}
& {\left[h_{i}, f_{j}\right]=-a_{i j} f_{j}} \\
& {\left[h_{i}, e_{j}\right]=a_{i j} e_{j}} \\
& {\left[e_{i}, f_{j}\right]=\delta_{i j} h_{j}}
\end{aligned}
$$

Cootrour. Ceppa

$$
\begin{aligned}
& i \neq j \quad a d e_{i}^{1-a_{i} j} e_{j}=0, \quad a d f_{j}^{1-a_{i j}} f_{j}=0 \\
& \hat{s l_{2}}=\left(s l_{2} \otimes \mathbb{C}\left[t, t^{-1}\right]\right) \oplus \underbrace{\mathbb{C} c}_{\text {yertp. }}
\end{aligned}
$$

"Oomurule" anzespol gesobyer b terezofinums

$$
\begin{aligned}
& \text { uppouge } \quad n p-i \pi: A \rightarrow \operatorname{End}(V) \\
& 0: A \rightarrow E u d(w)
\end{aligned}
$$

$$
3: A \rightarrow \operatorname{End}(w)
$$

$$
\pi \otimes \rho: A \rightarrow \operatorname{End}(V \otimes W)
$$

$$
(\pi \otimes \rho)(a)=\pi a \otimes 1+1 \otimes \rho a
$$ $a \in A$ (upaburs reúśrulya gus gup9. guar. gex́erbus upyunbl $/ 1 u$

$$
(\pi \otimes \rho)(g)=g \otimes g)
$$

$\longleftrightarrow$ koy uroxerulo

$$
\Delta(a)=a \otimes 1+1 \otimes a
$$

\& иринествеш $\pi \otimes \rho$
3) kb. app. $u_{q}^{q}\left(\hat{s l}_{2}\right)$.

- bie" geqpoprupzerna, biknoras koymmokerne.
ravep. $\underbrace{e_{0}, e_{1},}_{e_{0}^{+}, e_{1}^{+}} \underbrace{f_{0}, f_{1}}_{e_{0}^{-}, e_{1}^{-}}, k_{0}, k_{1}$

$$
\left[k_{0}, k_{1}\right]=0
$$

$$
\begin{gathered}
K_{i} K_{i}^{-1}=K_{i}^{-1} K_{i}=1, \quad(\text { Chari-Pressley } 1991) \\
K_{0} K_{1}=K_{1} K_{0}, \\
K_{i} e_{i}^{ \pm} K_{i}^{-1}=q^{ \pm 2} e_{i}^{ \pm}, \\
K_{i} e_{j}^{ \pm} K_{i}^{-1}=q^{\mp 2} e_{j}^{ \pm}, \quad i \neq j, \quad q^{h_{i}}-q^{-h i} / q-q^{-1} \longrightarrow h_{i}, q \rightarrow 1 \\
{\left[e_{i}^{ \pm}, e_{i}^{-}\right]=\frac{K_{i}-K_{i}^{-1}}{q-q^{-1}},} \\
{\left[e_{0}^{ \pm}, e_{1}^{\mp}\right]=0,}
\end{gathered}
$$

Koymokerue

$$
\begin{aligned}
& \begin{array}{l}
\Delta\left(e_{i}^{+}\right)=e_{i}^{+} \otimes K_{i}+1 \otimes e_{i}^{+}, \\
\Delta\left(e_{i}^{-}\right)=e_{i}^{-} \otimes 1+K_{i}^{-1} \otimes e_{i}^{-},
\end{array} \quad \text { "geqодри. иравияя } \\
& \Delta\left(K_{i}\right)=K_{i} \otimes K_{i}, \\
& \text { "pynnober zakon" }
\end{aligned}
$$

4so takce $A$ matpua?
Upegcorbietues $\pi, \rho, \quad A \rightarrow$ End $(V), E n d(\omega)$
$A \xrightarrow{\Delta} A \otimes A \xrightarrow{n \otimes \rho}$ End $(V \otimes W)$


Haygumes,

$$
\begin{aligned}
& A \xrightarrow{\Delta} A \otimes A \xrightarrow{I \otimes \rho} \operatorname{End}(V \otimes W) \\
& \left\langle\underset{\text { veperarub- } \downarrow \text { ka }}{ } \downarrow \widetilde{R}_{\pi, \rho}\right. \\
& \Delta \Delta \searrow A \otimes A \xrightarrow{\pi \otimes \rho} \operatorname{End}(V \otimes W) \\
& \widetilde{R}_{\pi, \rho}: V \otimes W \rightarrow V \otimes W \\
& R_{\pi, \rho}=\widetilde{R}_{\pi, \rho} \circ b: \quad V \otimes W \rightarrow W \otimes V
\end{aligned}
$$

Ecoo youbepcacovas $R$ verpnga

$$
\begin{aligned}
R & \in A \otimes A, \text { T.4r0. } \\
(\pi \otimes \rho) R & =R_{\pi, \rho}
\end{aligned}
$$

Ypabrerme lmz-bakcrepa gus $R$ uponexegur uy nfumeruous cuncranovgux oneparopob gus 3 up-6


$$
(\mid \otimes R)(R \otimes \mid)(\mid \otimes R)=(R \otimes \mid)(1 \otimes R)(R \otimes \mid)
$$

$k b \cdot r p \cdot u_{q}\left(\hat{s}_{n+1}\right)$

youb. $R$ veijuya

$$
\forall 2 \text { upp--ex ग. } \rho
$$ aguè oveparop hea $V \otimes W$

(2) 450 gäer joa "bucokars reayka" gua etox. bepmurmuix usgeces?
Gox. beca bapmens gs $6 b$ rogeun - 200 (C TOUROCToto go couprexervens) эreverore $R_{V, V}$, givs $U_{q}\left(\widehat{s Q_{2}}\right)$ $V=\mathbb{C}^{2}=\operatorname{sjan}\left(v_{0}, v_{1}\right)$ "cramgantrol" "p-e q-sez)

Slews q, crans $t$

Beca bepumin $\longleftrightarrow U_{t}\left(\hat{s_{t}}\right)$


1

$$
\frac{1}{b_{1}=\frac{1-u}{1-t u}}
$$


$1-b_{1}$
$t b_{1}$ $1-t b_{1}$
$t$ spure.
u cuekjo rapar.


Onepatop rea $V \otimes V$

$$
V=\mathbb{C}^{2}=\operatorname{span}\left\{v_{0}, v_{1}\right\}
$$

9-sl2 bV,V

$$
R_{W}^{\text {stoch }}\left[\begin{array}{cccc}
00 & 01 & 10 & 11 \\
1 & 0 & 0 & 0 \\
0 & b_{1} & 1-b_{1} & 0 \\
0 & 1-t b_{1} & t b_{1} & 0 \\
0 & 0 & 0 & 1
\end{array}\right]_{10}^{\infty} \quad b_{1}=\frac{1-u}{1-u t}
$$

$$
R_{u}^{\text {stoch }}\left(i_{1}, j_{2} ; i_{2}, j^{2}\right)
$$



Ypabr. $8-6 . \quad$ (yup.)


$$
\text { (bce } \left.3 \text { beca }-R_{\ldots}^{\text {stoch }}\right)
$$

Uges: bnokrw forr nodere 3 pastux spegcrabrekus $q$-sl !
$(2 d, 2 d) \rightarrow$ coox. $6 b$ uggent 6. Ecro eye" reckousto bapuamob
(3) $q$-sl2, 3 амена upegcal rermé. (a wTou - zavens $q-s l_{2}$ Ha $q-s l_{n}$ ).

$\rightarrow \quad \cos x .6 b$


$$
\mathbb{C}^{\infty}=\operatorname{span}\left(v_{0}, v_{1}, v_{2}, \ldots\right)
$$

"uoggub Bepua"

$$
s l_{2} i \quad h v_{i}=(\lambda+2 i) v_{i}
$$

$$
f v_{i}=i(1-i-\lambda) v_{i-1}
$$

$$
e v_{i}=v_{i+1}
$$

$$
i=0,1,2 \ldots
$$

$$
\begin{aligned}
& \lambda \in \mathbb{C} \quad-\infty \text { dim } \\
& \lambda=-2 n<0 \\
& \text { koneynomep } \\
& \text { kempub. } \\
& \text { koumorehia }
\end{aligned} \quad \begin{aligned}
& {[e, f]=h, \quad[h, f]=-2 f } \\
& {[h, e]=2 e }
\end{aligned}
$$

Bepumirna

crox beea
(higher spin stoch. Gv

$$
s u=-\tilde{u}
$$

$s \rightarrow 0$
X^ beypu. mogend

$$
\frac{1+\hat{u} q^{g}}{1+\tilde{u}} \quad \frac{\tilde{u}(1-q}{} \frac{g}{1+\tilde{u}^{2}}
$$

$$
\frac{1}{1+n}
$$

$$
\frac{\tilde{u}}{1+\tilde{u}^{2}}
$$ (coxacrufecras)

Pant: $s^{2}=1 / q$, Bepua $\leadsto \mathbb{C}^{2}$
(1)
\& nongrabm crox. $6 b$ regers
(yups.)

$$
s^{2}=1 / q I, \frac{I=2,3, \ldots \Rightarrow}{\substack{\text { unvicobans } \\ \text { nosect }}}
$$

(2) $6 b \rightarrow$ laucorkú cnur : Potokte (fusion)

$$
\begin{gathered}
\text { (Kymur-Pemetuxum- } \text { Ckreitun } 1983 \\
\text { Corwin-P. 2015, coxacruk9) }
\end{gathered}
$$

$$
t^{I-1} \quad t_{4}^{2} \text { tu } u
$$

 vepexefor b t-nepeciarcoboltode.

$$
t \text {-nepe } \sigma_{1, p-e ~}^{\text {p- }} \quad \mathbb{P}(\ldots 10 \ldots)=t \cdot \not P(\ldots 01 \ldots)
$$

$(\leftrightarrow$ Mallows measure)


1


$$
\frac{1-u}{1-t u} \frac{u(1-t)}{1-t u} \frac{t(1-u)}{1-t u} \quad \frac{1-t}{1-t u}
$$

Deciatoutco gonagart goes $I=2$ (garome valgykubus)
(a)

$\frac{1-t u}{1-t^{2} u} \cdot \frac{u(1-t)}{1-t \pi}$

$\frac{t u(1-t)}{1-t^{2} u}$


$$
\begin{gathered}
t \text {-nepect. } \\
10 \\
11=t \cdot 11
\end{gathered}
$$




Ha $t$-nepecrarobo4ribix pacup-ax gocraroutlo 3 rett olyep renco g crpesok

Yוb. (yus.) leeca srokob ma $t$-regrecs. pacup. genot


$$
\frac{1-\operatorname{su} t^{g}}{1-s u}
$$


$\frac{-\operatorname{su}\left(1-t^{8}\right)}{1-s u} \frac{1-s^{2} t^{8}}{1-s u}$


$$
\frac{-s h+s^{2} t^{g}}{1-s h}
$$

$\pi \frac{1-z y_{i}}{1-z t y_{i}} \leadsto$
$S^{2}=t^{-I} \quad$ ( u moxer gurt
$\leadsto \pi \frac{1-z y_{i}}{1-z y_{i} t^{I}}$
reago zamerulb $u$, hlo 7 to naxnco gearr)

Euei us $u_{p}\left(\hat{s t}_{2}\right)$


верие $\mathbb{C}^{\infty}$

$$
\begin{aligned}
& i_{2}, j_{1}, i_{2}, j_{2} \in \mathbb{Z}_{\geqslant 0} \\
& \mathrm{~L}_{u, \mathrm{~s}}^{(J)}\left(i_{1}, j_{1} ; i_{2}, j_{2}\right)= \mathbf{1}_{i_{1}+j_{1}=i_{2}+j_{2}} \frac{(-1)^{i_{1}} q^{\frac{1}{2} i_{1}\left(i_{1}+2 j_{1}-1\right)} u^{i_{1}} \mathbf{s}^{j_{1}+j_{2}-i_{2}}\left(u \mathrm{~s}^{-1} ; q\right)_{j_{2}-i_{1}}}{(q ; q)_{i_{2}}(\mathrm{~s} u ; q)_{i_{2}+j_{2}}\left(q^{J+1-j_{1}} ; q\right)_{j_{1}-j_{2}}} \\
& \times \bar{\phi}_{3}\left(\left.\begin{array}{c}
q^{-i_{2}} ; q^{-i_{1},}, \mathrm{Suq}^{J}, q \mathbf{q} / u \\
\mathrm{~s}^{2}, q^{1+j_{2}-i_{1}}, q^{J+1-i_{2}-j_{2}}
\end{array} \right\rvert\, q, q\right) . \\
&
\end{aligned}
$$

-I q-mnepreme $p$-yus
$s^{2}=q$ - sake. bepr cums
$q^{J}$ - sogke. sopry. cank
(4) $q$-si $l_{n+1}$ (tho upocive upegerabr.)
(Borodin-wheeler 2018)

The $U_{q}\left(\widehat{\mathfrak{s k}_{n+1}}\right) R$-matrix acts in a tensor product $W_{a} \otimes W_{b}$ of two $(n+1)$-dimensional vector spaces, and takes the form

$$
\begin{align*}
& R_{a b}(z)=\sum_{i=0}^{n}\left(R_{z}(i, i ; i, i) E_{a}^{(i i)} \otimes E_{b}^{(i i)}\right)  \tag{2.1.1}\\
& +\sum_{0 \leqslant i<j \leqslant n}\left(R_{z}(j, i ; j, i) E_{a}^{(i i)} \otimes E_{b}^{(j j)}+R_{z}(i, j ; i, j) E_{a}^{(j j)} \otimes E_{b}^{(i i)}\right) \\
& +\sum_{0 \leqslant i<j \leqslant n}\left(R_{z}(j, i ; i, j) E_{a}^{(i j)} \otimes E_{b}^{(j i)}+R_{z}(i, j ; j, i) E_{a}^{(j i)} \otimes E_{b}^{(i j)}\right) \\
& R_{z}(i, i ; i, i)=1, \quad i \in\{0,1, \ldots, n\}, \\
& \left.\begin{array}{ll}
R_{z}(j, i ; j, i)=\frac{q(1-z)}{1-q z}, & R_{z}(i, j ; i, j)=\frac{1-z}{1-q z} \\
R_{z}(j, i ; i, j)=\frac{1-q}{1-q z}, & R_{z}(i, j ; j, i)=\frac{(1-q) z}{1-q z}
\end{array}\right\} \quad i, j \in\{0,1, \ldots, n\}, \quad i<j . \\
& \operatorname{c}^{u+1} \\
& \text { inti } \\
& \operatorname{span}\left(v_{0}, v_{1}, \ldots, V_{n}\right) \\
& V_{0}=m y \operatorname{cota} \\
& v_{i}=\text { user } i
\end{align*}
$$

Def.


$$
b_{1}=\frac{1-u}{1-t u}
$$

$$
1-b_{1}=\frac{(1-t) u}{1-t u}
$$

CToxacrujeckas
Hleeinas $6 b$ mogerb
(Tyr soxe bozroxer poroxm...)


(5) Oszegunerue yberob

$$
\underbrace{0,1,2, \ldots, k}_{\text {ylet } 0}, \underbrace{k+1, \cdots n}_{\text {yber } 1} \text { (cpenku) }
$$

$\Rightarrow$ reyberurs $6 b$ vogeit

$$
\begin{aligned}
& \text { c pare } y \text { cn. } \quad \underbrace{(0, \ldots, 0}_{k}, 1,1, \ldots 1) \\
& \text { whero }
\end{aligned}
$$

HeT cukero
(6) Bыprozgerus b yelrian ASEP /TASEP

$$
(t \neq 0) \quad(t=0)
$$



$$
b_{1}, t b_{1} \rightarrow 0
$$

$$
b_{1}=\varepsilon \rightarrow 0
$$

yburturs ASEP:
y kaxgon 4act.
her yber (yber $0=$ mycoota)


$$
j^{0}>i^{\sigma}
$$


have. yca.

$$
-2-10
$$

$\cdots$ (6)(4)(3)(2) $1 \cdots \cdots$,

TASEPb c KAПпинZOu

(4) (1)
(1) $\downarrow$ un urenc-1
(7) 4actuya II masce
$\rightarrow$ Mbertion TASEP
zagana O yaciuye II Kracca (ramancrasme)


$$
I=\frac{U(t)}{t} \rightarrow \underset{\text { vea }}{T}[-1,1]
$$

(8) Cuмметpus sbera-nozmun b ybernoir 6b

| Angel-A - Valko <br> (TASEP) | 2008 |
| :--- | :--- |
| Berodin-Wheeler | 2018 |
| Borodin-Bufetor | 2019 |
| Bufetor | 2020 |
| Galashin | 2020 |

Oup.

$$
P_{J}^{c d}=P_{10 b}
$$



$$
\begin{aligned}
& J=\left\{J_{1}<\ldots<J_{e}\right\} \subseteq \\
& \leq\{1 \ldots N\}
\end{aligned}
$$

$t, u$
$b_{1}, t b_{1}$

eco ctperku b nozaymex J, ...Je cleprxy

Teopene. (BW 2018) $\quad P_{J}^{c o t}=P_{J}^{\text {incol }}$
(govatim 12 mas repe3 ar2. Tekke)
undelpita "ta buerbocto"

$$
\begin{aligned}
& P^{c o d}(\underbrace{h_{1}(M, N)}_{\substack{\text { uncwo bax } \\
\text { cipecok } \\
\text { ruske }(M, N)}} \geqslant l)=\sum_{\substack{J \\
|J|=l}} P_{J}^{c o l}= \\
&=\sum_{\substack{J,|J|=l}} P_{J}^{\text {unct }}=P^{u \phi c o l}(h(M, N) \geqslant l)
\end{aligned}
$$

Cregeibue (reifubuarbroc)


$$
=P^{\operatorname{racot}}\left(\begin{array}{l}
\overrightarrow{3} \\
\underset{\rightarrow}{\overrightarrow{3}}
\end{array} \underset{\substack{\text { ecro } \\
\text { Tyr } \\
\text { cperka }}}{\substack{3 \\
\hline}}\right)
$$

(9) TASEP $\forall x \in \mathbb{D}, \forall t \in \mathbb{R}_{\geqslant 0}$ yacruya nockegreezo Cregcob $\mathbb{P}_{(t)}^{\text {col }}$ yber 1 cuba or $\left.x\right)$ $=\|_{(t)}^{\text {uncol }}$ (ent raciuya $b \quad x+1$ )

G ybeirnom TASEP
$\ldots$ (4)(3)(2)(1) $\ldots .$.

yber 2222221

$$
\mathbb{P}\left(\frac{u(t)}{t} \geqslant \alpha\right) \simeq \mathbb{P}_{(t)}^{\text {uncol }}\left(\begin{array}{ll}
\text { racs. } b & d t
\end{array}\right)
$$

Acumentoiuka:

- erporas repeg uproyector Nuypa
- "reectpozal" tepez zugpogurevuky
$t \rightarrow \infty$,

$$
\begin{aligned}
& \mathbb{P}_{(t)}^{\text {incol }}(\text { racs } b d t) \simeq \text { niorrocos } \\
& \rho_{1} \text { b rayke } \alpha \\
& \nearrow \frac{\partial}{\partial t} \rho+\frac{\partial}{\partial x}(\rho(1-\rho))=0
\end{aligned}
$$

$\rho_{\tau}(\alpha)$ - undrito eto yavous $b$ Torke $\alpha L \quad l$ wovent $t L$,



$$
\begin{gathered}
\rho_{2}(\alpha)=1 / 2-\alpha / 2=\lim _{t \rightarrow \infty} \mathbb{P}\left(\frac{u(t)}{t} \geqslant \alpha\right) \\
\Downarrow \\
\frac{u(t)}{t}
\end{gathered}>\text { Unif }[-1,1] . \quad .
$$

12 mas 2021 Oyerky: igabaйтe заgaңи go 30 mas
nekyus 10 , $b$ koropoun anzespa leake nomoraer pozorparas e ugerama ogrum maxom.
(1) Cтохастинескаs yleiress mogerb




$$
b_{1}=\frac{1-u}{1-t u}
$$

$$
1-b_{1}=\frac{(1-t) u}{1-t u}
$$

(2) Cumnerprus yber-n zulyus

$$
\begin{aligned}
& \text { Oup: } \\
& P_{J}^{c d}=P_{\text {lob }}
\end{aligned}
$$



$$
\begin{aligned}
& J=\left\{J_{1}<\ldots<J_{2}\right\} \leq \\
& \leq\{1 \ldots N\}
\end{aligned}
$$

$J_{1}, \ldots, J_{e}$
$\forall$ ropragne

evo opperkn b nozuymex J, ...Je cleproy

Tegpune.(Bw 2018) $\quad P_{J}^{c o t}=P_{J}^{\text {ancol }}$


Yup: Her sorerymu no rgurypa
yui
(3) Anzespa Tekke, (hogrotobka / $S_{n}$ )

Hammien c $S_{n}$.

$$
s_{i}=(i, i+1) \quad i=1 \ldots n-1
$$

71. Taracmozииии

Yus. (1) $S_{n}$ nproxgeres $\left\{S_{i}, i=1 \ldots-n-1\right\}$
(2) $S_{n}=\left\langle S_{i}: i=1 \ldots n / \begin{array}{l}\text { coornomerns }\end{array}\right.$
$\left\{\begin{array}{l}S_{i} S_{j}=S_{j} S_{i} \quad|i-j| \geqslant 2 \quad \text { cooinom. } \\ S_{i} S_{i+1} S_{i}=S_{i+1} S_{i} S_{i+1} \\ S_{i}^{2}=e\end{array}\right.$
( $S_{n}$ kan yyunna komcipa)

$=e \quad b S_{n}$ ro te $b$ synne koc
reduced word 15 (KPaT yas neee eno)
(3) $\forall \sigma \in S_{n}, \exists G=S_{i_{1} \ldots}-S_{i(6)}$,
ze $l(b)=\#$ urbepcur

$$
=\nRightarrow\left\{i<j: \quad \delta_{i}>\sigma_{j}\right\}
$$

$\forall 2$ кратиarumx cuoba gus ó nepologetas gruy b groge c nom. LOATHM, KOC
$\left(\begin{array}{ll}\operatorname{sez} & S_{i}^{2}=e\end{array}\right)$
hдимер. (bepostrourinas uoge-s crezka b (toprory)

$$
b=(n, n-1, \ldots, 2,1) .
$$

Bozbmën bce uparly. woba, ux

$$
\frac{\left(C_{n}^{2}\right)!}{1^{n-1} 3^{n-2} 5^{n-3}-(2 n-5)^{2}(2 n-3)^{1}}
$$

Boznuèn ofro endo eyrasro!


## random sorting networks

https://gilkalai.wordpress.com/ 2018/05/20/dubcan-dauvergne-and-balint-virag-settled-the-random-sorting-networks-conjectures/


Morulaupolora

$$
S_{i} S_{i+1} S_{i}=S_{i+1} S_{i} S_{i+1} \longleftrightarrow y_{p}-e 9 b
$$



$$
\begin{aligned}
& S_{i} S_{j}=S_{j} S_{i} \quad|i-j| \geqslant 2 \\
& S_{i} S_{i+1} S_{i}=S_{i+1} S_{i} S_{i+1} \\
& S_{i}^{2}=e
\end{aligned}
$$

4, ug arozo rego "ueneptuto" (geqpopinpobar), roobu cbezarr $C$ $R$ notpryaien?

Orber: kocw syreme re троzarb!

$$
S_{i}^{2}=e \quad S_{i}^{2}=\text { numeirture Kows. }
$$

Si u $e$
anzespa buecro rpyanbl
(име: geopopm. yignnoborангегрия $\left.\mathbb{C}\left[S_{n}\right]\right)$
(4) Anzespa Pekke (q)
(Iwahori-) Hecke algebra

- avzeopr $H=H\left(S_{n}\right)$
mopokgèrras $\quad T_{i}, \quad i=1-h-1$ u cootromerns un

$$
\begin{aligned}
& \left\{\begin{array}{l}
T_{i} T_{j}=T_{j} T_{i} \quad|i-j| \geqslant 2 \\
T_{i} T_{i+1} T_{i}=T_{i+1} T_{i} T_{i+1} \\
T_{i}^{2}=(1-q) T_{i}+q \cdot T_{e}
\end{array}\right\} \text { ko(b) } \quad\left(T_{e}=1\right) \\
& \sim_{\text {ovalino } q-1, ~ \text { ro tro ok bub. }} \quad \mathbb{C}_{1}\left[S_{n}\right] .
\end{aligned}
$$

Yn. (1) dim $H_{n}=n$ !
(2) sagnc $T_{2}$, ye $T_{2}=T_{i_{1}} \ldots T_{i_{e(6)}}$ gas (mosozo) kpatuavinezo eraba

$$
b=S_{i_{1}}-S_{i_{e(6)}}
$$

(3) Ha sagnel $T_{6}$ uneem:

$$
\left\{\begin{array}{l}
T_{i} T_{\omega}=T_{s_{i} w} \quad \ell\left(s_{i} w\right)=\ell(\omega)+1 \\
T_{i} T_{\omega}=(1-q) T_{\omega}+q T_{s_{i} w} \\
\ell\left(s_{i} \omega\right)=\ell(\omega)-1 ;
\end{array}\right.
$$

$w \in S_{h}$
grus biopozo cyyzase, $w=s_{i} w^{\prime}$

$$
\begin{aligned}
& \ell\left(w^{\prime}\right)=\ell(w)-1 \\
& T_{i} T_{s_{i} w^{\prime}}=T_{i}^{2} T_{w^{\prime}}=(1-q) T_{i} T_{w^{\prime}}+q T_{\omega^{\prime}} \\
& \omega^{\prime}=s_{i} w
\end{aligned}
$$

(4) $T_{i}^{-1}=$ ?
(5) Unborto yия b Hn

Oup

$$
\begin{array}{ll}
I=H_{n} \rightarrow H_{n} \\
I\left(T_{\sigma}\right)=T_{\sigma^{-1}} \\
\quad(\text { Ha } \text { squce })
\end{array} \quad \forall b \in S_{n}
$$

Banera: $\quad T_{b^{-1}} \neq\left(T_{b}\right)^{-1}$, cur $T_{i}^{-1} \neq T_{i}$
Teopema. I - ureboenotubruis aron- 20mouply.

$$
\text { T.e. } I^{2}=1
$$

$$
\begin{aligned}
& I\left(T^{1} T^{2} \ldots T^{(k)}\right)= \\
& \quad=I\left(T^{(k)}\right) I\left(T^{(k-1)}\right) \ldots I\left(T^{(1)}\right) .
\end{aligned}
$$

Yup. Dokagaro no ungyegum no $l(\omega)$.
(6) Ot aurespal 「ekke $k$ bepoltitocu

$$
h=\sum_{b} c_{b} T_{b}<H_{\text {lep. uspa ra } S_{n}}^{\left(c_{3} \geq 0, \quad \sum c_{z}=1\right)}
$$

$g, h \in H_{p r o b}$

$$
h \longmapsto g h=\sum_{b} c_{b}^{\prime} T_{b}
$$

mepa $\left\{c_{b}\right\} \longmapsto$ mepa $\left\{c_{b}^{\prime}\right\}$

Mapkobckar уеиb $\Leftrightarrow$ ctox. матрияa ( $P_{b \tau}$ )

$$
P_{2 \tau} \geqslant 0, \quad \sum_{\tau} P_{z \tau}=1
$$

$\Leftrightarrow$ meth. (crox.) onepcoop rea mopax
Y Hal $\quad h \mapsto g h$, mapuobckas yenb

Zgede g noxtr shoro quxcupobaris, un businparius cuyravito uz kavo20-to reasopa Irevercteb.

Nrunep. $J_{3}$

$$
\delta_{(213)} \quad \delta_{(312)}
$$

$$
h=e \xrightarrow{T_{1}} T_{1} \xrightarrow{T_{2}} T_{2} T_{1}
$$

$$
\begin{aligned}
& L T_{2} \\
& (1-q) T_{2} T_{1}+q T_{1} \\
& (1-q) \delta_{(3 \mid 2)}+q \delta_{(2,3)}
\end{aligned}
$$

hpunep. ASEP C $n$ yedraven

$$
e=(1) \text { (2) (3) } \cdots n \text { (n) } \quad i=1 \ldots-1
$$

Zbonoyms: Sepein $Y_{i, x}=x T_{i}+(1-x)$
coyreasto $(i=1, \ldots, n-1)$
wh kaxgom waze

NB. 2oo ASEP © guckp bpevenev, a gas кешр. врешеки Sepien $x=d t \rightarrow 0$ ( $n$ upegen $k$ treup-bp.)

$$
\begin{aligned}
& l(s i \omega)=l(\omega)+1 \\
& y_{i, x} T_{w} x T_{s i \omega}+(1-x) T_{w} \\
& f l(s ; \omega)=l(\omega)-1 \\
& x\left[(1-q) T_{\omega}+q T_{s_{s}, \omega}\right]+(1-x) T_{\omega} \\
& =\left((1-q x) T_{\omega}+q x T_{s ; \omega}\right.
\end{aligned}
$$

$\Rightarrow$ Bonnyus

(a)
(b)
(b) (a)
yyber a joubue yem y b"


NB! $x \sim x_{i}$ (Ha katgou peSple $i-i+1$ cbos "cropoctr"; u zgene beasue met popreye
(7) CTox. 6 b rogenb и a a . Tekke (ogta bepurerta)

$a<b$ kav oneveroton ( $\frac{j}{\text { ubler a somome }}$

$$
j>i
$$



$$
\left(x=\frac{1-u}{1-q u}\right)
$$

Dercilontemeho, $(n=3$ ylara)
$R: V \otimes V \rightarrow V \otimes V$ varpuca $9 \times 9$

$$
\begin{aligned}
& \left.Q=11 \begin{array}{ccccccccc}
00 & 01 & 02 & 10 & 11 & 12 & 20 & 21 & 22 \\
00 \\
01 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
02 & 0 & \frac{(-1+q) u}{-1+q u} & 0 & \frac{-1+u}{-1+q u} & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{(-1+q) u}{-1+q u} & 0 & 0 & 0 & \frac{-1+u}{-1+q u} & 0 & 0 \\
0 & \frac{q(-1+u)}{-1+q u} & 0 & \frac{-1+q}{-1+q u} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{(-1+q) u}{-1+q u} & 0 & \frac{-1+u}{-1+q u} & 0 \\
20 & 0 & 0 & \frac{q(-1+u)}{-1+q u} & 0 & 0 & 0 & \frac{-1+q}{-1+q u} & 0 \\
21 & 0 \\
22 & 0 & 0 & 0 & 0 & \frac{q(-1+u)}{-1+q u} & 0 & \frac{-1+q}{-1+q u} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right) \\
& x=\frac{1-u}{1-q u} \\
& \frac{1}{x}(R-(1-x)-I d)=T \quad \text { yoobr. } \quad T^{2}=(1-q) T+q \cdot I_{d} \\
& \Rightarrow \quad R=Y_{i, x}=x T_{i}+(1-x) \text {, }
\end{aligned}
$$

vge $x=\frac{1-u}{1-q u}$

Yup. Yi,x take ygobl. kbagp. coortom. \& $R^{2}, R, I d$ Toxe numeinto $3^{\text {aboucumen }}$ (woxtwo Howith cinykgym Tekke "b sos")
$\underline{\underline{u n}}$

$$
T=\frac{1}{x}(R(x)-(1-x)-I d)
$$

colnagale $\left.C \quad R\right|_{x=1}$.
(8) CTox. $6 b$ nogeur $u$ a^respa Sekke (решїтtкa)

Oup. $W_{b, a}=Y_{(b-1, b), x} Y_{(b-2, b-1), x} \cdots Y_{(a, a+1), x}$ $(b<a)$
geácrloger $c$ zitor csoprorla


$$
\begin{array}{r}
\Leftrightarrow(k+1) \times(a-b+2) \quad \text { crox. ybertans } \\
6 b \text { rogenb }
\end{array}
$$



$$
T_{e}=1
$$

$$
\Leftrightarrow
$$

6 (5) (6) (7) (8)
5 (1) (2) (3) (4)
4321

NB!


Bgece ver $\phi$ ybeta,
Ho mokro osregurusto
Ubera 1-14 u orsbeno $n_{x} \phi$


1
( 300 yactionte ceypars)
(9) Cимметрия mozичии - ybета

Teopewar. $\forall$ qukc. $b \in S_{n}$,


$$
\begin{aligned}
= & \mathbb{P}\left[\begin{array}{cccc}
1 & \sigma_{1}^{-1} & \sigma_{2}^{-1} & \sigma_{3}^{-1} \\
2 & & & \sigma_{4}^{-1} \\
3 & & & \sigma_{5}^{-1} \\
4 & 5 & 6 & 7
\end{array}\right] \\
& {\left[\begin{array}{lll}
\text { Borodin - Bufetov } & & \\
\begin{array}{lll}
\text { Bufetov } & 2019 \\
\text { Galashin } 2020
\end{array}
\end{array}\right] }
\end{aligned}
$$

$D-60$.


$$
\begin{aligned}
& \text { Prob }=\text { koэ } 99 . \\
& \text { nym } T_{6} f \\
& \text { мpumereruen } \\
& \text { jioro k Te }
\end{aligned}
$$

Prob $=$ ko 999 .
uthloutasus

$$
=\quad b_{2}^{-1}
$$

1 sypu $T_{b}-1$ b мримеретеие


$$
\begin{equation*}
\text { jioro } k T_{e^{-1}} \tag{10}
\end{equation*}
$$



ロ

60

$$
J(Y)=Y
$$



$$
\begin{aligned}
& \left\langle T_{b}, Y_{2} Y_{3} Y_{1} Y_{2} T\right. \\
& \left\langle T_{b}{ }^{+1}, Y_{2} Y_{1} Y_{3} Y_{2} T_{e}\right\rangle
\end{aligned}
$$

(10) hpuso*eruss / g-bo теореub) $^{\text {- }}$

1) Hact 220 uracka

$$
=\mathbb{P}\left(\sigma_{5}^{-1} \in\{1,2,3\}\right)
$$

$$
=\mathbb{P}_{\text {uncol }}\binom{\text { mro louxoger }}{\text { zgevo }}
$$

2) 



D-bo Teoperum


$$
\begin{aligned}
& =\mathbb{P}\left(\left\{\sigma_{1, \ldots}, \partial_{N}\right\} \supset\left\{J_{1}+M, \ldots, J_{l}+M\right\}\right) \\
& \text { II } \\
& \mathbb{P}\left(\{1, \ldots N\} \supset\left\{\sigma^{-1} J_{1}+M, \cdots, \sigma^{-1} J_{e}+M\right\}\right) \\
& \text { II }
\end{aligned}
$$

(11) Cobu. p-e ybertuix p. baricotion


теореша $\Rightarrow$

$$
\begin{aligned}
& \left\{H_{\geqslant 1}(M, N), H_{\geqslant 2}(M, N), \ldots, H_{\geqslant N}(M, N)\right\} \\
& \| d \\
& \left\{H^{u n c d}(M, N), H^{u n c o l}(M, N-1), \ldots, H^{u n c o l}(M, 1)\right\}
\end{aligned}
$$

$$
\underline{d}
$$


(12) Xpryeccul Xosıa-Muttrbyga

$$
\begin{aligned}
h(M, N) & \stackrel{d}{=} N-l(\lambda) \\
\lambda & \sim \frac{1}{z} P_{\lambda}\left(u_{2}, \ldots u_{N}\right) Q_{\lambda}\left(v_{1} \ldots v_{M}\right)
\end{aligned}
$$

(Joano parcome)


Teopama.


$$
\left\{k-l\left(\lambda^{(k)}\right)\right\}_{k=1 \ldots N}
$$

$\lambda^{(j)} \sim$ nproyecs $\times \imath$

$$
\begin{aligned}
& \frac{1}{z} P_{\lambda^{(1)}}\left(u_{1}\right) P_{\lambda^{(2)} / \lambda^{(1)}}\left(u_{2}\right) \ldots P_{\lambda^{(v)} / \lambda^{(\omega-1)}}\left(u_{N}\right) \\
& z_{z=11-t u_{1} v_{j}} \quad Q_{\lambda^{(N)}}\left(v_{1} \ldots v_{M}\right)
\end{aligned}
$$

 ~ асииипотка
(13) $\oint$ gnv upoyeccob $X \wedge$

$$
\begin{aligned}
& D(t, 0)=\sum_{i=1}^{N} \prod_{j \neq i} \frac{t_{x_{i}-x_{j}}}{x_{i}-x_{j}} T_{0, x_{i}} \\
& D(t, 0) P_{\lambda}\left(x_{1} \ldots x_{N} \mid t\right)= \\
& =\frac{1-t^{N-l(\lambda)}}{1-t} P_{\lambda}\left(x_{1}, \ldots, x_{N}\right) \\
& \tilde{D}=t^{-N}[1+(t-1) D(t, 0)] \\
& \tilde{D} P_{\lambda}=t^{-l(\lambda)} P_{\lambda} \\
& \tilde{D} f\left(x_{1}\right) \cdots f\left(x_{N}\right) \\
& f\left(x_{1}\right) \ldots f\left(x_{N}\right) \\
& \quad f(0)=1
\end{aligned}
$$

neruag. $\mathbb{E} t^{-\sum_{i=1}^{K} l\left(\lambda^{\left(n_{i}\right)}\right)}$

$$
\frac{\tilde{D}_{n_{k}} \cdots \tilde{D}_{n_{2}} \tilde{D}_{n_{1}} Z}{Z}
$$

$$
z=\pi \frac{1-u_{i} v_{j}}{1-u_{i} v_{j}}
$$

D-6

$$
\begin{aligned}
& \quad\left(\text { npunep } \quad N=3, \quad n_{1}=3,\right. \\
& Z\left.n_{2}=2\right) \\
& \sum_{\lambda \mu \nu} P_{\lambda}^{(3)}\left(u_{1}\right) P_{\mu / \lambda}\left(u_{2}\right) \quad P_{\nu / \mu}\left(u_{3}\right) \cdot \begin{array}{l}
\lambda^{(2)}=\mu \\
\lambda^{(1)}=\lambda
\end{array} \\
&= Q_{\nu}(\vec{v}) \\
&= \sum_{\nu \nu \nu}\left(u_{1}, u_{2}\right) P_{v} / \mu\left(u_{3}\right) Q_{\nu}(\vec{v}) \\
& P_{v}\left(u_{1}, u_{2}, u_{3}\right) Q_{\nu}(\vec{v})
\end{aligned}
$$

Нриметим $\tilde{D}_{2} \tilde{D}_{3}:$

$$
\begin{aligned}
& \tilde{D}_{2} \tilde{D}_{3} z \\
= & \tilde{D}_{2} \sum_{v} \tilde{D}_{3} P_{v}\left(u_{1}, u_{2}, u_{3}\right) Q_{v}(\vec{v}) \\
= & \tilde{D}_{2} \sum_{v} t^{-l(\nu)} P_{v}\left(u_{1}, u_{2}, u_{3}\right) Q_{v}(\vec{v}) \\
= & \tilde{D}_{2} \sum_{v, \mu} t^{-l(\nu)} P_{\mu}\left(u_{1}, u_{2}\right) \quad P_{v / \mu}\left(u_{3}\right) Q_{v}(\vec{v}) \\
= & \sum_{v, \mu} t^{-l(\nu)} \tilde{D}_{2} P_{\mu}\left(u_{1}, u_{2}\right) P_{v / \mu}\left(u_{3}\right) Q_{v}(\vec{v}) \\
= & \sum_{v, \mu} \quad t^{-l(\nu)-l(\mu)} P_{\mu}\left(u_{1}, u_{2}\right) P_{v / \mu}\left(u_{3}\right) Q_{v}(\vec{v})
\end{aligned}
$$

Teop.

$$
n_{1} \geqslant \ldots \geqslant n_{k}
$$

$$
E t^{\sum_{i=1}^{k}\left(n_{i}-l\left(\lambda^{\left(n_{i}\right)}\right)\right)}=\frac{t^{k(k-1)}}{(2 \pi i)^{k}} \oint \ldots \oint \frac{d z_{1}-d z_{k}}{z_{1} \ldots z_{k}}
$$

$$
\prod_{1 \leq A \angle B \leq k}^{z_{A}-z_{B}} \prod_{i=1}^{k}\left[\prod_{j=1}^{n_{i}} \frac{t z_{i}-U_{j}}{z_{i}-u_{j}} \prod_{j=1}^{m} \frac{1-z_{i} V_{j}}{1-t z_{i} V_{j}}\right]
$$



ЭTO $\delta_{b \mid N}$ cobu. parus. $H_{\geqslant k}^{\operatorname{col}}(M, N)$ b ogroú to4ke.

Bufetov - Korotkikh 2020

- oryare poperyea gus yb. rogen
(14) Osyar go-1a (5ez g-ba)

$$
\begin{aligned}
& T_{i} f\left(w_{1}, \ldots, \omega_{N}\right)= \\
= & q f\left(w_{1} ;-\omega_{N}\right)+\frac{w_{i+1}-q w_{i}}{\omega_{i+1}-w_{i}}\left[S_{i}-I_{d}\right] f\left(\omega_{1}, \ldots \omega_{N}\right)
\end{aligned}
$$

$q=1$ : geiscrbue самл. ууynиы

Yup. $\left\{T_{i}\right\}$-an-rekke $\quad\left(T_{i}^{2}=(q-1) T_{i}+q\right)$



Figure 2. Left: an example of the data for a SC6V model, namely, a skew domain, rapidities of rows and columns and a monotone coloring. Here the coloring is (1, 2, 2, 3, 4, 5, 5, 5). Right: an example of a path configuration satisfying the boundary condition on the left picture, as well as the values of the height function $h_{\geq 4}^{(\alpha, \beta)}$ with respect to this configuration.

(Q):
yber

$$
\text { Touka }\left(\gamma(c)_{2} \delta(c)\right)
$$

go reï ybera $\leqslant c$, nocne ybera>c


$$
\begin{aligned}
& \alpha_{1} \leq \ldots \leq \alpha_{k} \\
& \beta_{1} \geqslant \cdots \beta_{k}
\end{aligned}
$$



Torku

Uldra:

$$
0 \leq c_{1} \leq \ldots \leq c_{k}
$$

hatoc, $\pi \in S_{K}\left(\right.$ mak $c_{i}$ cbez. $\left.c\left(\alpha_{i}, \beta_{i}\right)\right)$
hycot:

Torga

$$
\begin{aligned}
& \mathbb{E} q^{\eta}=q^{\frac{k(k-1)}{2}-l(\pi)} \oint_{\Gamma} d w_{1} \cdots \oint_{r_{1}} d w_{k} \prod_{a<b} \frac{w_{b}-w_{a}}{w_{b}-q w_{a}} \\
& \times\left[T_{\pi}\left(\prod_{a=1}^{k} \prod_{i=1}^{i<\delta\left(c_{a}\right)} \frac{1-x_{i} w_{a}}{1-q x_{i} w_{a}} \prod_{j>\gamma\left(c_{a}\right)}^{M} \frac{1-y_{j} w_{a}}{1-q y_{j} w_{a}}\right)\right] \prod_{a=1}^{k}\left(\prod_{i=1}^{i<\beta \beta_{1}} \frac{1-q x_{i} w_{a}}{1-x_{i} w_{a}} \prod_{j>a_{j}}^{M} \frac{1-q y_{j} w_{a}}{1-y_{j} w_{a}} \frac{\left.\frac{1}{2 \pi i w_{a}}\right)}{2}\right) \\
& \text { gns osyero } r \\
& \text { JTO cloxitoe loupathue }
\end{aligned}
$$

C1. (1) $C_{1}=\cdots=c_{k}=1, \quad \lambda=e$
(2) $\left(\alpha_{i}, \beta_{i}\right)$ cobwagaer, $\quad \tau=e$


Cnum. nognkus -ybera.
nekyus 11.
(unocreghss)
Oyerky: cgabaíte zagarn go 30 mas
「ауссовские иределы $b$ интегрируемых

1. TASEP /SGV $\rightarrow$ acumnt. Tuna KPZZ Kardar-Parrisi - Zhang


- анализ по существу ограничен точными формулами (за исключением случайных матриц и некоторых совсем новых результатов)
- в случае TASEP / детерминантных процессов есть многоточечная асимптотика (KPZ fixed point, Airy sheet, directed landscape) $\xi(x, y, t, s)$
- в случае стохастической шестивершинной модели и других недетерминантных ситуаций многоточечный анализ очень затруднен

2. Yprbrestue $k P Z_{x \in \mathbb{R}}$ u rayccobectr

$$
\frac{\partial}{\partial t} h(t, x)=-\frac{1}{2} \frac{\partial^{2}}{\partial x^{2}} h(t, x)+\frac{1}{2}\left(\frac{\partial}{\partial x} h(t, x)\right)^{2}+\eta(t, x)
$$

$\eta$-semens nyw $b \mathbb{R}^{2}$

$$
\left\{\begin{array}{l}
\mathbb{E} \eta=0 \\
\mathbb{E} \eta(t, x) \eta\left(t^{\prime}, x^{\prime}\right)=\delta\left(x-x^{\prime}\right) \delta\left(t-t^{\prime}\right) \\
\text { raycobocuín upoyec }
\end{array}\right.
$$

$h$ - upegen ASEP/SGV мй accumverpun $q \rightarrow 1$



$$
h(0, x)=h_{0}(x)
$$

$t \rightarrow \infty, \quad h(t, x) \sim \operatorname{TASEP} /$ SGV rea Somomom Opeveren

$$
\begin{aligned}
& \frac{h(t, x)+t / 24}{c t^{1 / 3}} \rightarrow F_{2} \\
& t \rightarrow 0, \quad h(t, x) \approx \text { ASEP/S6V, } \quad \text { rayccobek wi upryect }
\end{aligned}
$$

3. Гayecobexue upegeus - pacupegerenus
$(d=1) \quad f(x)=\frac{1}{\sqrt{2 \pi}} e^{-x^{2} / 2}$


$$
X=3 Z+\mu
$$

$$
x \sim N\left(\mu, \sigma^{2}\right)
$$

$$
\left\{\begin{array}{l}
E x=\mu \\
\operatorname{Var} x=3^{2}
\end{array}\right.
$$

np-e Pypbe $\mathbb{E} e^{i t X}=e^{i t \mu-t^{2} b^{2} / 2}$

0 On.
$\left(\mathbb{R}^{d}\right)\left(X_{1}, \ldots, X_{d}\right)$ - $\begin{aligned} & \text { aycc. be voop } \\ & \text { ectu }\end{aligned}$
$\underbrace{\sum_{i=1}^{d} d_{i}^{i} X_{i}}$ - rayec gus no5020 $\left(\alpha_{1} \ldots \alpha_{d}\right)$
zabneut of $E$, Var

$$
\operatorname{Var}\left(\sum \alpha_{i} X_{i}\right)=\sum \alpha_{i}^{2} \operatorname{Var} X_{i}+2 \sum_{i<j} \alpha_{i} \alpha_{j} \operatorname{Cov}\left(X_{i} X_{j}\right)
$$

$\Rightarrow$ colen p-e zabncut toubko or cpefrezo u vobap. uatpmyb)

$$
M=\left[\operatorname{Cov}\left(x_{i}, x_{j}\right)\right]_{i, \hat{j}=1}^{d}
$$

mp-e pypbe $\mathbb{E} e^{i \underbrace{\sum t_{k} x_{k}}_{\begin{array}{c}\text { oftra zayccoba } \\ c \cdot e .\end{array}}}$
4. Tayecobexue upegersl

UIRT $\xi_{i} \sim$ mez.og.pacup.

$$
E \xi=0, \quad \operatorname{Var} \xi=1
$$

yng.

$$
\delta_{n}=\frac{\delta_{1}+\cdots+\xi_{n}}{\sqrt{n}} \rightarrow \mathcal{V}(0,1) \quad n \rightarrow+\infty
$$

Epayroberve glux.

$$
t \in \mathbb{R} \geqslant 0
$$

$$
\frac{\xi_{1}+\cdots+\xi_{[n t]}}{\sqrt{n}}=\sqrt{t} \frac{\xi_{2}+\cdots+\xi_{[n t]}}{\sqrt{n t}} \rightarrow W_{t}
$$

$$
W_{t} \sim N(0, t)
$$

nareet rezzlanc. zagec. nplugaryerus \&



HIO cobvectride pacup-e me rakoe, $\operatorname{kak}\{\sqrt{t} N(0,1)\}$

5. TASEP $\rightarrow$ rayce. npegen

$k=1$

$$
\frac{X_{1}(L L)-\tau L}{\sqrt{L}} \rightarrow W_{\tau}
$$

$$
\mathbb{E} X_{1}(t)=t-1
$$

$$
\operatorname{Var} X_{1}(t)=t
$$

(nyacl.c.b.)
Sp. Slangerme


$$
\left\{\frac{x_{i}(\tau L)-\tau L}{\sqrt{L}}\right\}_{i=1}^{k} \longrightarrow\left\{w_{i}(\tau)\right\}_{i=1}^{k}
$$

отрахёнrыle op.gous.

(bce cobu pos b pagrule nom. לppea. ebrude, ro yate ke zayccobck.ue)

Pyltkyuoral oo rejalunc. ofr. gloux Crus?
(cblzb es cey4. यатйyaru)
GUE $=\frac{X+\bar{X}^{T}}{2}\left(\begin{array}{c}\text { Gaussian } \\ \text { Unitary } \\ \text { Ersembles. }\end{array}\right)$
$G U E(t)$ - ofo glout le vecto

$$
N(0,1)
$$



$$
\begin{aligned}
& w_{1}(t)=\lambda_{\min , 1}(t) \\
& W_{2}(t)=\lambda_{\min , 2}(t) \\
& w_{3}(t)=\lambda_{\min _{3} 3}(t)
\end{aligned}
$$

$\forall$ qukc.t
6. Pazobole mepexogbl tuna BBP
(Baik - Ben Arous - Peche)
a) TASEP a regretmón ractugér (curymasuor)
https://wt.iam.uni-bonn.de/ferrari/research/jsanimationtasep

$F_{2} \leftrightarrow$ egnno Airy

$$
\begin{aligned}
& \quad \mathrm{K}_{\mathrm{Ai}}\left(v, v^{\prime}\right)=\frac{1}{(2 \pi \mathbf{i})^{2}} \int_{e^{-2 \pi \mathrm{i} / 3} \infty}^{e^{2 \pi \mathrm{i} / 3} \infty} d w \int_{e^{-\pi \mathrm{i} / 3} \infty}^{e^{\pi \mathrm{i} / 3} \infty} d z \frac{1}{z-w} \exp \left\{\frac{z^{3}}{3}-\frac{w^{3}}{3}-z v+w v^{\prime}\right\} \\
& \tilde{F}_{2}: \operatorname{gos}{ }^{2} b_{k a} \quad(Z / w)
\end{aligned}
$$

8) (yyr. матриуb)
quyriyaym

$$
\begin{array}{rlrl}
\lambda_{\max }(M) & \sim F_{2} & |\alpha|<2 \\
& \sim F_{2} & & |\alpha|=2 \\
& \sim \text { Raycc., } & & |\alpha|>2
\end{array}
$$

n gryroe enegree
7. laycc. cbosogroe none
(GFF, Gaussion Free Field)


Lbognyngerve zakpluлéstros egrayroi)

$$
t \in \mathbb{R}^{2}
$$


$\operatorname{GFF}(x, y)$ - cmzairase rayccobcurs ooosyethnas pyorkus
$\operatorname{GFF}(x, y)$ re weeer cuacia,
tho $\operatorname{Cov}\left(\operatorname{GFF}(x, y), \operatorname{GFF}\left(x^{\prime}, y^{\prime}\right)\right)=\ldots$
$u \forall \operatorname{gos}{ }^{\imath}, \quad \phi_{j}, \quad \phi_{j} / \partial \Omega=0, \quad \phi_{j} \in C^{\infty}$

$$
\begin{array}{r}
\left\{\iint \phi_{j}(x, y) \operatorname{Gff}(x, y) d x d y\right\}_{j=1}^{k} \\
- \text { zayec. bekTop } \quad, \quad E=0
\end{array}
$$



Th.(kempon, P.2012,...)

$$
\begin{aligned}
& h(L x, L y)-\mathbb{E} h(L x, L y) \rightarrow \phi(x, y) \\
& \phi\left(w^{-1}(z)\right)=\operatorname{GFF}(z), \quad z \in \mathbb{C}_{+}\binom{\operatorname{bepxrest}}{\operatorname{cosyn} .} \\
& E(\operatorname{GFF}(z) \operatorname{GFF}(w))=-\frac{1}{2 \pi} \log \left|\frac{z-w}{z-\bar{w}}\right| \\
& z=w \text { gaët to } \\
& \text { b } \oint \oint \\
& z \quad \mathbb{C}_{+} \\
& {[0,1]^{2}} \\
& \text { (korgopuras uobap -T, GFF) }
\end{aligned}
$$

8. Heurozo o cucriuax c $q=1$

$$
\text { ASEP } \rightarrow \text { SSEP }
$$

$$
h\left(t_{1}\right) \sim t \cdot H_{0}+\cdots
$$

$$
h(t, 0) \sim \sqrt{t} H_{0}+t^{1 / \varphi} \cdot(\text { 2aycc. })
$$

Te te u gus 56 V

$$
\xrightarrow{b_{1}}=b_{2} \uparrow
$$

Haver spopryes:

$$
\mathbb{E} q^{k h(x, y)}=\oint \ldots \oint
$$

$q=1$ He nueer cmacha

Daume: noemo tpiner ru korikpetruns Mpumep
$c$ $6 b$ rogentro u zayссоbским mpegeron.
9. $t \rightarrow 1$, njeger $\mathbb{E} t^{\cdots}=\oint$

$$
t \rightarrow 1, \quad b_{1}, b_{2} \rightarrow 1
$$

$$
\begin{aligned}
& b_{1}=\frac{1-v}{1-t v} \\
& b_{2}=t \frac{1-v}{1-t v}
\end{aligned}
$$

$$
\begin{aligned}
b_{1} & =e^{-\beta_{1} / L}, \quad b_{2}=t b_{1}=e^{-\beta_{2} / L}, & \beta_{1}, \beta_{2}>0 \\
t & =\tau^{1 / L}, & \beta_{1} \neq \beta_{2}
\end{aligned}
$$

$$
\log \tau=\beta_{1}-\beta_{2}
$$

quкс. $v=\beta_{1} / \beta_{2}=\frac{1-b_{1}}{1-t b_{1}}$
( cucters co 1asos acu nuerpues $\approx k P z$, mave bpeus)

Cheyber ${ }^{\text {ras }}$

$$
S 6 V
$$

wogerb)
$\Downarrow$

$$
\begin{gathered}
\text { (354) } \frac{1}{L} h(L x, L y) \rightarrow J(x, y), L \rightarrow \infty \\
t^{h}=\tau^{h / L} \rightarrow \tau^{J L}
\end{gathered}
$$

(unT)

$$
\text { 7) } \left.\begin{array}{rl}
\left\{\frac{h(L x, L y)-\mathbb{E}}{}\left(L_{x}, L_{y}\right)\right. \\
\sqrt{L}
\end{array}\right\} \rightarrow \begin{gathered}
\begin{array}{c}
\text { yerotwpobanke } \\
\text { 2aycc, wore } \\
b \mathbb{R}_{2}^{2} \geqslant 0
\end{array} \\
L \rightarrow \infty \\
\phi(x, y)
\end{gathered}
$$

$$
\forall\left(x_{i}, y_{i}\right), \quad\left\{\phi\left(x_{i}, y_{i}\right)\right\}_{i=1}^{k}
$$

- raycc.lekTop, $\mathbb{E} \phi=0$
(T-e. JTo ke wak GFF, tro obayraie eyre. becururior)
cregyer us paccuorperny

$$
L \operatorname{Cov}\left(t^{h^{\prime}}, t^{h^{\prime}}\right) \rightarrow \ldots
$$

(n bue Cragume vamernour
rago buipayuos repez
E, Cov Ti.k. rayecolocir)
10. § qориуга

$$
\begin{aligned}
\sum h\left(M, n_{i}\right) & t^{\frac{k(k-1)}{2}} \\
\sqrt{2 \pi i})^{k} & \oint . \oint \frac{n_{1} \geqslant \ldots \geqslant n_{k} \geqslant 1}{z_{1} \ldots z_{k}}
\end{aligned}
$$

$$
\prod_{\substack{1 \leq A<B \leq k}}^{z_{A}-z_{B}} \prod_{i=1}^{k}\left[\prod_{j=1}^{z_{i}} \frac{t z_{B}}{z_{i}-u_{j}} \prod_{i} \frac{1-w_{j}}{\prod_{j=1}} \frac{1-z_{i} v_{j}}{1-t z_{i} v_{j}}\right]
$$


(govagam b upromabies pay $c$ nonougbro upoys. X1)
11. Tyegenstars gopue

$$
\begin{aligned}
& u_{j} \equiv 1, v_{j} \equiv v \\
& \mathbb{E} t^{h(L x, L y)}=\frac{1}{2 \pi i} \oint \frac{d z}{z}\left(\frac{t z-1}{z-1}\right)^{L y}\left(\frac{1-z v}{1-t z v}\right)^{L x} \\
& \left.\begin{array}{ll}
0 & 1 \\
0 & 1 \\
0 & 1
\end{array}\right) \\
& =1+\frac{1}{2 \pi i} \oint \frac{d z}{z}\left(\frac{t z-1}{z-1}\right)^{L y}\left(\frac{1-z v}{1-t z v}\right)^{L x} \\
& \rightarrow 1+\frac{1}{2 \pi i} \oint \frac{d z}{z} \tau^{\frac{y z}{z-1}+\frac{x v z}{1-v z}} d z \\
& \odot_{1} \\
& \geqslant \nexists(x, y)
\end{aligned}
$$

upes. qopue

Yup. $f(x, y)=\tau^{\text {It }(x, y)}$
ygeberb. "ypitoreruno теме мрора"
(Klein-Gordon)

$$
\partial_{x y} f+\alpha \partial_{y} f+\beta \partial_{x} f=0
$$

(gues nofxefalyux $\alpha, \beta$ Jabuces ugux or $\tau, v$ )

Ovagubaers, qryktyaym yfoler. crox. ypabrecmo tese zpapa
12. Koryc b 6 b nogem $t$-quac. $\sim$ moxake ra

$$
\tau=t^{L} \rightarrow 0
$$

$$
\begin{aligned}
& \tau=e^{\beta}, \beta \rightarrow-\infty \\
& e^{\beta み}=\oint e^{\beta j^{\beta(z)}} \frac{d z}{z} \text { T.e. } \oint \simeq e^{\beta \cdot \text { apegen } x} \\
& \delta(z)=y z / z-1+\frac{x v z}{1-v z} \\
& S^{\prime}(z)=0
\end{aligned}
$$

$$
\begin{aligned}
& z_{c}=\frac{\sqrt{x v / y}-1}{\sqrt{\frac{x v}{y}}-v} \\
& \oint e^{\beta s(z)} \sim e^{\beta s\left(z_{c}\right)} \\
& s\left(z_{c}\right)=\frac{(\sqrt{y}-\sqrt{x v})^{2}}{1-v} \\
& \text { upeger. p. lemcorm S6 V } \\
& \text { sypu quec. } t \text {., } \\
& \text { lemyopul koryca }
\end{aligned}
$$




$$
\pi \sum^{\sum t\left(M, n_{i}\right)}=\frac{t^{\frac{k(k-1)}{2}}}{(2 \pi i)^{k}} \oint \ldots \oint \frac{d z_{1}-d z_{k}}{z_{1}-z_{k}}
$$

$$
\underbrace{\prod_{1} \frac{z_{A}-z_{B}}{z_{A}-t z_{B}}}_{\substack{1 \leq A<B \leq k}} \prod_{\substack{i=1 \\ 1, m o \text { 2ro }}}^{k}\left(\frac{t z_{i}}{z_{i}}\right.
$$ rengalarno!

Uso geratb? Moguqnyupobaro pppuyny!

$$
\begin{aligned}
& \mathbb{E} \prod_{i=1}^{k}\left(t^{h\left(M, n_{i}\right)}-t^{i-1}\right) \\
& \quad=\frac{t^{k \frac{(k-1)}{2}}}{(2 \pi i)^{k}} \oint_{1} \cdots \oint_{1} \frac{d z_{1}-z_{k}}{z_{1}-z_{k}} \\
& \prod_{1 \leq A<B \leq k}^{\frac{z_{A}-z_{B}}{z_{A}-t z_{B}}} \prod_{i=1}^{k}[\underbrace{\prod_{j=1}^{n_{i}} \frac{t z_{i}-1}{z_{i}-1} \prod_{j=1}^{M} \frac{1-z_{i} v}{1-t z_{i} v}}]
\end{aligned}
$$

$\oint 0$ vorl 1 , sez yacres bonpyn 0

$$
\begin{aligned}
& f_{n_{i}}\left(z_{i}\right) \\
& f_{n}(0)=1
\end{aligned}
$$

D-lo.

$\sum$ no $I=\{1 \ldots k\}$
$z_{i}, i \notin I$ urverp learpys $O$

$$
\begin{aligned}
& I=\left\{i_{1}<\ldots<i_{l}\right\} \\
& I^{c}=\left\{p_{1}<\ldots<p_{k}-l\right\}
\end{aligned}
$$

$z_{p_{k-l}}, \cdots, z_{p_{2}}, z_{p_{1}}$ crezubalm $k 0$

$$
\Rightarrow \quad t^{-\Sigma\left(k-p_{j}\right)}
$$

$$
\prod_{\alpha<\beta} \frac{w_{\alpha}-w_{\beta}}{w_{\alpha}-t w_{\beta}} \prod_{\alpha=1}^{\ell} f_{n_{i}}\left(w_{\alpha}\right)
$$

Mycor $X_{j} \leftrightarrow t^{h\left(M, n_{j}\right)}$
Torge $\sum_{I}$ - cymere $\mathbb{E}$ nompromole or $x_{2}, \ldots, x_{k}$.

Bee torga cugyer uy remul?
^emes

$$
\begin{aligned}
& \sum_{l=0}^{k} \sum_{|I|=l}\left(t^{-1}\right)^{(k-l)(k-l+1) / 2} \\
& 0\left(x_{i_{1}}-\left(t^{-1}\right)^{i 1}\right)\left(x_{i_{2}}-\left(t^{-1}\right)^{i_{2}-1}\right) \ldots\left(x_{i_{l}}-\left(t^{-1}\right)^{i_{l}-l+1}\right) \\
& =X_{1} x_{2} \ldots x_{k} \\
& \left(y_{\text {np }} .\right)+y_{\text {kajarue }} .
\end{aligned}
$$

Nemes. Moxtro busferb kotoypur bor rak:

$$
\begin{aligned}
& \mathbb{E} \prod_{i=1}^{k}\left(t^{h\left(n, n_{i}\right)}-t^{i-1}\right) \\
& =\frac{t^{\frac{k(k-1)}{2}}}{(2 \pi i)^{k}} \oint \ldots \oint \frac{d z_{1}-d z_{k}}{z_{1} \ldots z_{k}} \\
& \prod_{\leqslant A \angle B \leq k} \frac{z_{A}-z_{B}}{z_{A}-t z_{B}} \prod_{i=1}^{k}\left(\frac{t z_{i}-1}{z_{i}-1}\right)^{n_{i}^{0}}\left(\frac{1-z_{i} v}{1-t z_{i} v}\right)^{M}
\end{aligned}
$$

D-loo (ugels)
(supu lebrecernu voroypplo, gowonk. buelletor ysoget:

$$
\begin{aligned}
z_{2}=t^{-1} z_{1}, & \text { Torga } \\
& \frac{t z_{1}-1}{z_{1}-1} \cdot \frac{t z_{2}-1}{z_{2}-1} \longrightarrow \frac{t z_{1}-1}{z_{1}-1} \cdot \frac{z_{1}+1}{t^{-1}-1}
\end{aligned}
$$

T.e-lамие $z_{1}=1$ yxegro, $\oint=0$.
14. Robapuayus
(gow-bo upo benceme nowerotes upsoycium)

$$
\begin{aligned}
& L \operatorname{Cov}(\underbrace{t^{h(L x, L y)}}_{t^{h} \simeq \tau^{J h}}, \underbrace{t^{h\left(L x^{\prime}, L y^{\prime}\right)}}_{t^{h^{\prime}} \simeq \tau^{3 t^{\prime}}})=\cdots \\
& L\left[\mathbb{E}\left(t^{h}-1\right)\left(t^{h^{\prime}}-t\right)-\mathbb{E}\left(t^{h}-1\right) \mathbb{E}\left(t^{h^{\prime}}-1\right)\right] \\
& =L(\oint \oint-\oint \cdot \oint) \\
& =\frac{L}{(2 \pi i)^{2}} \oint^{1} \oint \underbrace{\left(\frac{z_{1}-z_{2}}{z_{1}-t z_{2}}-1\right.})\left(\frac{t z_{1}-1}{z_{1}-1}\right)^{L y}\left(\frac{t z_{2}-1}{z_{2}-1}\right)^{L y^{\prime}} . \\
& \text { ใم } 0 x+2+b) \frac{z_{1}(t-1)}{z_{1}-t z_{2}} \cdot\left(\frac{1-z_{1} V}{1-t z_{1} V}\right)^{L x}\left(\frac{1-z_{2} V}{1-t z_{2} V}\right)^{L x^{\prime}} \frac{d z_{1} d z_{1}}{-z_{1} z_{2}}
\end{aligned}
$$

Tro yoke exefuras:

$$
\begin{aligned}
& \frac{\log \tau}{(2 \pi i)^{2}} \oint \oint \frac{z_{1}}{z_{1}-z_{2}} \cdot \frac{d z_{1} d z_{2}}{z_{1} z_{2}} \\
& \exp \left[y \frac{z_{1} \beta_{1}}{1-z_{1}}-x \frac{v z_{1} \beta_{1}}{1-v z_{1}}+y^{\prime} \frac{z_{2} \beta_{1}}{1-z_{2}}-x^{\prime} \frac{V z_{2} \beta_{1}}{1-v z_{2}}\right]
\end{aligned}
$$

Norwyos leorpys $0, z_{1}$ lireyppue $z_{2}$.

$$
\begin{aligned}
& \mathbb{E}\left(t^{h}-1\right)\left(t^{h^{\prime}}-t\right)-\mathbb{E}\left(t^{h}-1\right) \mathbb{E}\left(t^{h^{\prime}}-1\right) \\
& \mathbb{E}\left(t^{h+h^{\prime}}-t^{h+1}-y^{\prime \prime}+t\right)-\mathbb{E} t^{h} \mathbb{E} t^{h^{\prime}}+\mathbb{E} t^{h}+\mathbb{E} t^{\prime \prime}-1 \\
= & \operatorname{Cov}\left(t^{h}, t^{h^{\prime}}\right)+t-1+\mathbb{E} t^{h}-t \mathbb{E} t^{h} \\
= & \operatorname{Cov}\left(t^{h}, t^{h^{\prime}}\right)+(1-t)\left(\mathbb{E} t^{h}-1\right)
\end{aligned}
$$

$\Rightarrow$ monyralm popmyy gus

$$
\operatorname{Cov}\left(t^{J-L}, \tau^{H^{\prime}}\right)
$$

Stochastic Lelegraph eqn
(Berodin-Gorin)
bocomse cuacuSo!



[^0]:    $\mathbf{P} \quad P_{\lambda}$ Hall-Littlewood polynomials, functions $F_{\lambda}$
    \{ \}

