PROJECT DESCRIPTION. RANDOM SYSTEMS FROM SYMMETRIC FUNCTIONS AND VERTEX MODELS

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1. Tools: Gibbs Measures, Yang-Baxter Equation, Symmetric Functions

In this introductory section we discuss the tools used in the proposed research to obtain outcomes summarized in §4 below.

1.1. **Overview.**
A significant part of development of modern probability theory since the second half of the 20-th century was heavily influenced by physical applications. In recent decades, there haven been at least two major achievements at this interplay: (1) discovery and rigorous construction of scaling limits of conformally invariant models (including the Schramm-Loewner Evolution Gaussian Free Field, and models of Quantum Gravity); and (2) discovery and rigorous justification of the Kardar-Parisi-Zhang (KPZ) stochastic partial differential equation, and description of the KPZ universality class capturing scaling limits of many natural random interface growth models described by singular stochastic partial differential equations with white noise. Many ideas behind these achievements are purely mathematical, but some tools and principles were first developed within physics: for example, field theory heuristics for (1), and Bethe ansatz for (2). Many other physical theories, (e.g., renormalization group or string theory) keep influencing probability and other branches of pure mathematics.

The proposed research explores applications of some of the physical ideas, most importantly, the Yang-Baxter equation (also known as star-triangle, or Y-Delta move) and the Bethe Ansatz, together with mathematical ideas around symmetric functions, to stochastic models originating from two-dimensional statistical mechanics and random interface growth. When such a stochastic model is related to Yang-Baxter equation or symmetric functions, we refer to it as **integrable**.

1.2. **Gibbs measures.**
The models studied within the proposed research usually can be formulated as Gibbs probability distributions or marginals / degenerations / limits of such distributions. By a Gibbs distribution on configurations on a finite space Ω (usually a part of the two-dimensional lattice $\mathbb{Z}^2$) we mean

$$\operatorname{Prob}(\omega) = Z^{-1} \exp\{-H(\omega)\}, \qquad \omega = \{\sigma_i\}_{i\in\Omega} \in S^{\Omega}, \tag{1}$$

where S is a finite set ("set of possible spins"; for example, $S = \{0, 1\}$), and $H(\omega) \geq 0$ is the energy of a configuration ω, which may depend on global parameters (e.g., inverse temperature) and local parameters (e.g., vertex rapidities). Here Z is the *partition function* (probability normalizing constant) which also depends on all the parameters.

1.3. **Infinite volume limit.**
Besides Gibbs measures on configurations on a finite space as in (1) with fixed boundary conditions ("*boxed distributions*"), we are interested in *infinite volume* measures.

By definition, an infinite volume Gibbs measure satisfies the Gibbs property (1) in every finite subspace $\Omega \subset \Omega_\infty$ when conditioned on the configuration outside Ω, and with boundary conditions imposed by this outside configuration. Out of all possible Gibbs measures on Ω_∞ one usually selects the ones with certain special properties, like translation invariance and/or ergodicity / extremality. Classifying infinite volume Gibbs measures is a very nontrivial problem, and an explicit answer is rarely available. Let us discuss three instances:

1. The classification is only conjectural in many interesting integrable situations, such as for translation invariant ergodic Gibbs measures ("*pure states*"). While this classification (for example, for the general six vertex model) is a major open problem, a new characterization of pure states might be helped by the construction of new irreversible dynamics preserving the measure. Such dynamics could be constructed directly from the Gibbs structure and the underlying Yang-Baxter equation.

2. On the other hand, pure states of the six vertex model under a special *free fermion condition* (a codimension one condition on vertex weights at each vertex) admit a very explicit description through determinantal point processes (i.e., all correlation functions of these measures are diagonal minors of an explicit function in two variables), which follows from [She05], [KOS06]. Recently the PI and collaborators [ABPW21] obtained doubly inhomogeneous extensions of these well-known infinite volume measures. In the free fermion case, the proposed research will pursue, in particular,
   a. an explicit **particle current** of irreversible Markov dynamics (coming from the Yang-Baxter equation) preserving the pure state of the free fermion six vertex model, and an effect of inhomogeneities on it.
   b. **determinantal structure of boxed free fermion measures**, and its limit to a discretization of the Dyson Brownian Motion in inhomogeneous space with arbitrary initial data.
   c. the **effect of inhomogeneity on global fluctuations of the height function** leading to deformations of the Gaussian Free Field.

3. Certain families of (non translation invariant) Gibbs measures power the classification of irreducible representations of infinite-dimensional unitary group and other classical groups [Voi76], [VK82], [BO12], [Pet14]. This subject is closely related to symmetric functions arising as partition functions of Gibbs measures with varying parameters (rapidities) along one of the lattice coordinate direction, see §1.5 below for details.

1.4. **Connections to random growth.** By a *random growth model* we mean a Markov dynamics on one- or two-dimensional interfaces separating two regions of $\mathbb{Z}^2$ or $\mathbb{Z}^3$, respectively, started from a given initial configuration. Random growth in one dimension is essentially the same as a stochastic interacting particle system. Indeed, for example, the TASEP (Totally Asymmetric Simple Exclusion Process) is equivalent to the Corner Growth Model by passing to the height function $h(x) = \#\{\text{TASEP particles } \ge x\}$.

Let us mention three known instances when Gibbs measures are closely related to random growth models or interacting particle systems:

1. Fixed time probability distributions of a random growth model in **two dimensions** may sometimes be identified with a Gibbs measure. For example, after n steps the domino shuffling produces an exact sample of the uniformly random domino tiling of the size n Aztec Diamond [EKLP92].

2. The fixed time probability distribution of a random growth in **one dimension** may sometimes be found as a *marginal distribution* of a Gibbs random configuration in two dimensions. See §2 below for a proposed problem related to TASEP with arbitrary initial data.

3. The space-time distribution of the **whole trajectory** of a stochastic interacting particle system like TASEP or the Dyson Brownian Motion (DBM) may sometimes be an instance of an infinite volume Gibbs measure in two dimensions. Universal limits of some integrable particle systems, the Airy and KPZ line ensembles, are conjecturally characterized by their Gibbs properties. See §3 below for a group of new discrete models related to DBM.

Over the past decades, these connections have powered crucial asymptotic results related to KPZ or bulk universality in various interacting particle systems, see Tracy-Widom asymptotics in [Joh00] or discrete Dyson's conjecture in [GP19], respectively. (The term "bulk" refers to the parts of the system where the space can be rescaled to form growing regions with constant particle density.)

The **search for new interesting particle systems related to various Gibbs measures, and utilizing the latter for asymptotic analysis** is one of our main research directions.

1.5. **Symmetric functions as partition functions.** A crucial integrability structure which singles out the Gibbs measures addressed in the proposal is their connection to symmetric functions arising as partition functions. The symmetry is powered by the Yang-Baxter equation. This circle of ideas is best explained on a rather simple example. The proposed research deals with very nontrivial multiparameter generalizations of the setting explained below.

Fix integer $N \ge 1$, let $\lambda = (\lambda_1 \ge \ldots \ge \lambda_N \ge 0)$, $\lambda_i \in \mathbb{Z}$, be an integer partition with N parts. Let $x_1, \ldots, x_N \in \mathbb{C}$ be *row rapidities*. Consider a *vertex model*, i.e., a probability measure on all configurations of up-right paths in $\mathbb{Z}_{\ge 0} \times \{1, \ldots, N\}$ with the following boundary conditions. There are no paths entering from below or exiting from the right; at each horizontal $j = 1, \ldots, N$ there is exactly one path entering, and for $i = 1, \ldots, N$ a path exits through the top boundary at coordinate λ_i. Multiple paths per vertical edge are allowed, but there is at most one path per horizontal edge. Paths can meet at a vertex. Our concrete *Schur vertex model*, by definition, assigns to each path configuration the probability weight proportional to the product of the individual vertex weights $w_{(i,j)}(a_1, b_1; a_2, b_2)$ over all lattice vertices (i, j). (The Gibbs energy H in the sense of (1) is the sum of the logarithms of the vertex weights.) Here $a_1, a_2 \in \mathbb{Z}_{\ge 0}$, $b_1, b_2 \in \{0, 1\}$ are the numbers of vertical and horizontal paths at the vertex, with a_1, b_1 entering, and a_2, b_2 exiting. We require the path preservation property, i.e., $a_1 + b_1 = a_2 + b_2$, to hold at each vertex. The concrete vertex weight we take at the (i, j)-th vertex, $i \ge 0$, $1 \le j \le N$, depends on the j-th row rapidity as $w_{i,j}(a_1, b_1; a_2, b_2) := x_j^{b_2}$. Because the partition λ is fixed, there are finitely many possible configurations, so the measure is well-defined. See Figure 1 for an example (and also Figure 5 below for an equivalent lozenge tiling picture).

The *Schur symmetric polynomial*

$$s_\lambda(x_1, \ldots, x_N) = \frac{\det[x_i^{\lambda_j + N - j}]_{i,j=1}^N}{\prod_{1 \le i < j \le N}(x_i - x_j)} \tag{2}$$

serves as the as the partition function of this vertex model (recall that we fixed λ at the top boundary).

Expression (2) for the partition function readily follows by induction on N by peeling off the dependence on x_N and utilizing elementary transformations of the determinant. Let us emphasize that such a concise formula for a partition function is a rare occurrence, and partition functions of other integrable vertex models are usually much more complicated.

The presence of symmetric functions in Gibbs measures allows to employ powerful algebraic tools to study stochastic systems, including (when available) explicit formulas, difference operators acting on symmetric functions diagonally, orthogonality properties of the symmetric functions, and so on. To summarize, most of the tools worked out in the celebrated book by Macdonald [Mac95] could be applied to partition functions of various integrable vertex models.

1.6. **Yang-Baxter equation.** The symmetry of a partition function in the row rapidities $x_1, \ldots, x_N$ is a simpler fact compared to the determinantal formula for the global partition function s_λ. Namely, the symmetry follows from the *Yang-Baxter equation* which is a local property of the weights:

$$\begin{aligned}\sum_{k_1,k_2,k_3} & R_{y/x}(i_2, i_1; k_2, k_1) w_x(i_3, k_1; k_3, j_1) w_y(k_3, k_2; j_3, j_2) \\ & = \sum_{k_1,k_2,k_3} w_y(i_3, i_2; k_3, k_2) w_x(k_3, i_1; j_3, k_1) R_{y/x}(k_2, k_1; j_2, j_1),\end{aligned} \tag{3}$$

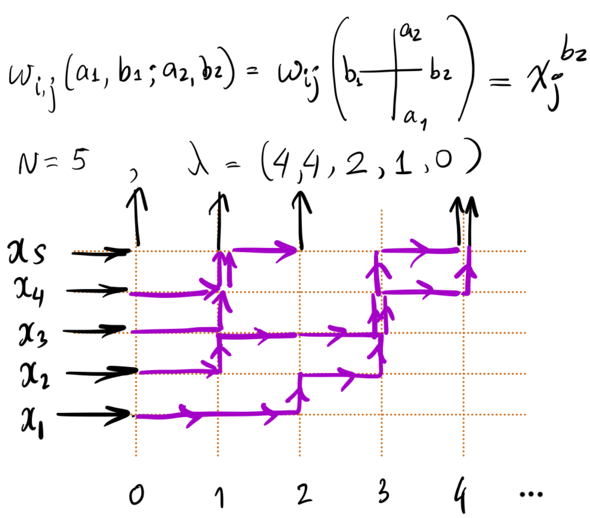

FIGURE 1. A configuration of the Schur vertex model with $N = 5$ and the given λ. The probability weight of this particular configuration is proportional to $x_1^2 x_2^2 x_3^3 x_4^2 x_5^2$.

where $i_1, i_2, j_1, j_2 \in \{0, 1\}$ and $i_3, j_3 \geq 0$ are fixed, and both summations run over $k_1, k_2 \in \{0, 1\}$ and $k_3 \geq 0$, but there are at most two nonzero terms in each side. The weights w_x are x^{j_2}, as in §1.5 above. See Figure 2 for an illustration and the definition of the cross vertex weights R_z.

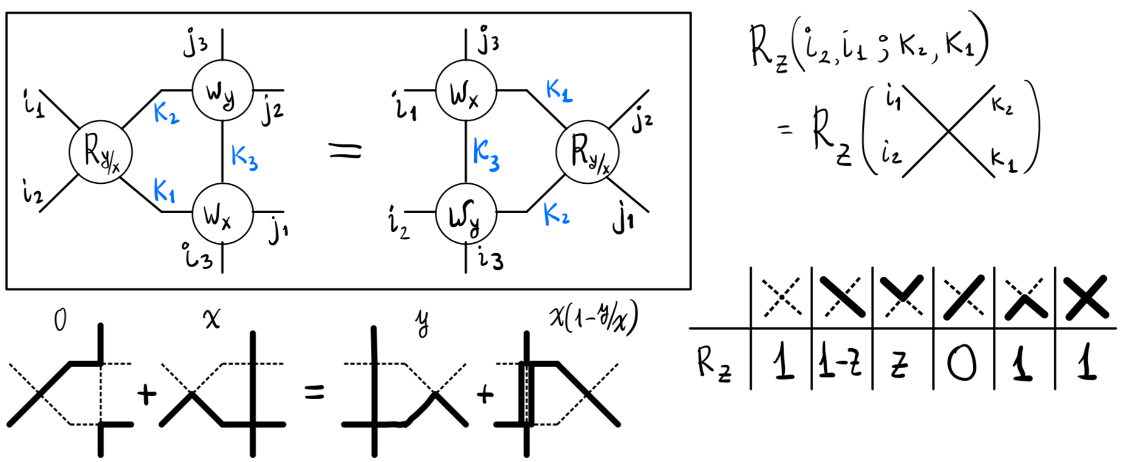

FIGURE 2. Yang-Baxter equation (3) (framed), an example for $(i_1, i_2, i_3, j_1, j_2, j_3) = (0, 1, 1, 1, 0, 1)$ (below), and the cross vertex weights R_z.

To show symmetry of the partition function in Figure 1 under any permutation $x_i \leftrightarrow x_{i+1}$, $i = 1, \ldots, N-1$, add the full cross vertex on the left between the horizontals i and $i + 1$. Since $R_{x_{i+1}/x_i}(1, 1; 1, 1) = 1$, this does not change the partition function. Then, using (3), move the cross vertex all the way to the right, which interchanges x_i and x_{i+1} along the way. Far to the right, the cross becomes empty, and thanks to $R_{x_{i+1}/x_i}(0, 0; 0, 0) = 1$, it can be removed. Thus, we get the desired symmetry.

Besides the symmetry, the Yang-Baxter equation establishes other important properties of partition functions, for example, the Cauchy summation identity for Schur polynomials:

$$\sum_{\lambda=(\lambda_1\ge\lambda_2\ge\ldots\ge\lambda_N\ge 0)} s_\lambda(x_1,\ldots,x_N)s_\lambda(y_1,\ldots,y_N)=\prod_{i,j=1}^{N}\frac{1}{1-x_iy_j},\qquad |x_iy_j|<1\ \forall i,j.$$

The Yang-Baxter equation is a hallmark of integrability in vertex models. For stochastic vertex models generalizing the Schur model, the PI and collaborators [CP16], [BP18], [OP17], [BMP21] have developed various techniques (based on Markov duality, the Yang-Baxter equation, or difference eigenoperators for symmetric functions) to obtain exact formulas for expectations of observables amenable to asymptotic analysis Moreover, in the yet more general case when paths are allowed to have different colors ([BW18], [ABW21]) the Yang-Baxter equation leads to transformation rules for partition functions defining a representation of the Iwahori-Hecke algebra. For brevity, throughout the proposal we discuss only the simplest possible uncolored models.

1.7. **Determinantal processes.** Let the rapidities x_i be positive. Fix $\lambda=(\lambda_1\ge\ldots\ge\lambda_N\ge 0)$ and consider the random path configuration as in Figure 1 with probability weights proportional to products of the vertex weights, as defined in §1.5. Encode the random configuration by an integer sequence $\lambda^k=(\lambda^k_1\ge\ldots\ge\lambda^k_k\ge 0)$, $k=1,\ldots,N$, with $\lambda^N=\lambda$ fixed. For example, in Figure 1 we have

$$\lambda^1=(2),\qquad \lambda^2=(3,1),\qquad \lambda^3=(3,3,1),\qquad \lambda^4=(4,3,1,1),\qquad \lambda^5=(4,4,2,1,0).$$

It is well-known (for example, [Ken09, Corollary 3]) that for the Schur case, the random path configuration as in Figure 1 gives rise to a *determinantal point process*. By definition, this means that for any $m\ge 1$ and $a_1,\ldots,a_m\in\mathbb{Z}$, $t_1,\ldots,t_m\in\{1,\ldots,N\}$ we have

$$\begin{aligned}\mathbb{P}\Big[\text{configuration }\{(\lambda^k_j-j,k)\colon 1\le j\le k\le N\}\text{ contains all points }(a_1,t_1),\ldots,(a_m,t_m)\Big]&\\ =\det[K(t_i,a_i;t_j,a_j)]_{i,j=1}^{m},&\end{aligned}\tag{4}$$

for a certain function K of two lattice variables. The function $K(t,a;t',a')$ is called the *correlation kernel*. In words, all multipoint correlations in the random system are expressed in a rather simple form through the two-point correlations. The determinantal property is quite special, and is not shared by all integrable vertex models. The determinantal structure relies on free (noninteracting) fermion structures behind the Schur polynomials and the vertex model in Figure 1. It is a major open problem to **understand the structure of multipoint correlations in more general integrable vertex models** in a form amenable to asymptotic analysis. In particular, more general models include many instances of interacting fermions, one example of which is the stochastic six vertex model.

1.8. **Steepest descent.** Even in the Schur vertex model the correlation kernel K is not known in a convenient enough form for general rapidities x_i. By "convenient" here we mean *double contour integral form* which can be analyzed asymptotically in most interesting regimes (bulk leading to the sine kernel, edge described by the Airy line ensemble, global fluctuations leading to the Gaussian Free Field) using steepest descent whose application to determinantal processes dates back to Okounkov [Oko02].

In more detail, a double contour integral kernel of a stochastic system usually has the form (here a,b are in $\mathbb{Z}$ or $\mathbb{Z}^2$, cf. (4))

$$K(a,b)=\iint\frac{e^{L(S(z)-S(w))}}{z-w}\,f_{a,b}(z,w)\,dz\,dw,\tag{5}$$

where L is a large parameter, $S(z)=S(\bar z)$ is a function symmetric under complex conjugation, and $f_{a,b}$ is a regular term not going to zero or infinity. To obtain its asymptotics, one looks at critical points of $S(z)$. For (a,b) in the bulk of the system, S has two complex conjugate critical points $z_c, \bar{z}_c$. Deforming the contours to intersect at them such that $\mathrm{Re}(S(z) - S(w)) < 0$ on the new contours makes the double contour integral go to zero. The remaining residue means that (5) scales to the arc integral of $f_{a,b}(z,z)$ from $\bar{z}_c$ to z_c.

For (a,b) at the edge of the system, the critical points $z_c, \bar{z}_c$ merge at a real line, and scaling S around the double critical point leads to the Airy kernel containing $e^{z^3/3 - w^3/3 + \cdots}$ under the integral.

In [Pet14] the PI has obtained the double contour integral kernel for the case of rapidities $x_i = q^{i-1}$, $i = 1, \ldots, N$, where $0 < q < 1$. The limit $q \nearrow 1$ leads to the uniformly random lozenge tilings as in Figure 5 below. The kernel's limit as $q \nearrow 1$ (which is quite nontrivial to compute from the q-dependent kernel) becomes especially amenable to asymptotics which were performed by the PI [Pet15], [GP19] and other authors [LT17], [Agg19] to obtain various universality results.

The q-dependent kernel has not yet been utilized towards asymptotics, mainly due to the presence of the q-hypergeometric function ${}_3\phi_2$ under the integral. One of the concrete problems suggested in the proposal is the **exploration of asymptotics of the q-dependent Schur vertex model and related boxed Gibbs measures**. Along this direction, preliminary computations already suggest new discretizations of the Dyson Brownian Motion, which are powered by new interesting limits of Schur and Macdonald symmetric functions. We discuss this in detail in §3 below.

1.9. **Fredholm determinants and large deviations.** Let us mention one more major open problem related to determinantal structure. The cumulative distribution of the top particle in a one-dimensional determinantal process (for example, the maximal eigenvalue of a Gaussian Hermitian random matrix) is given by a Fredholm determinant:

$$\mathbb{P}(\xi < s) = \det(Id - K|_{(s,+\infty)}) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \sum_{i_1,\ldots,i_n > s} \det[K(i_a, i_b)]_{a,b=1}^n \tag{6}$$

(the second equality may be taken as a definition of the Fredholm determinant; here the space is discrete). The Fredholm structure is even more general, as it extends to single-point distributions in many particle systems not connected to determinantal processes. A prototypical example here is ASEP (a generalization of TASEP in which particles jump in both left and right directions) for which Fredholm determinants (when ξ in (6) is a fixed particle coordinate in ASEP) were found in celebrated works of Tracy and Widom [TW08], [TW09].

The steepest descent analysis of (6) around a double critical point of K leads (in many models) to the GUE Tracy-Widom asymptotics: $\lim_{L\to+\infty} \mathbb{P}\left(\dfrac{\xi(L) - cL}{\sigma L^{1/3}} < r\right) = F_2(r)$. In addition to this asymptotics, the distribution of $\xi(L)$ typically has two different large deviation tails. For example, in ASEP, the probability $\mathbb{P}(\xi < cL - \alpha L)$ that ξ is exceedingly **slow** is of order $\exp(-\varphi^+(\alpha)L)$ (established by the PI in [DPS18]), and the exceedingly **fast** probability $\mathbb{P}(\xi > cL + \alpha L)$ is of order $\exp(-\varphi^-(\alpha)L^2)$. Heuristically, a particle may be slow just because its independent jumps are slow (a classical large deviation behavior), while to be fast, all the particles in front must also be fast. The slow large deviations can be established by the steepest descent analysis of a Fredholm determinant. Namely, in this regime the double critical point splits into two real points z_1, z_2, and the integral (5) behaves as $e^{L(S(z_1) - S(z_2))}$, which produces the order L large deviations. In (6), only the term with $n = 1$ in the sum matters in this regime. Presumably, in the fast large deviations of order L^2, all terms in the sum should contribute.

It would be beneficial to be able to prove the fast large deviation regime (or at least estimates) of order L^2 directly from the Fredholm determinant. There are several known approaches to this tail, like equilibrium measures in random matrices [AGZ10, Section 2.6], or matching of expectations for the KPZ equation tail [CG20]. Yet, most stochastic interacting particle systems or integrable directed random polymers do not possess random matrix, determinantal, or matching expectation properties, and it would be interesting to find a direct approach via Fredholm determinants.

The PI hopes that a progress in this major open problem may be achieved by considering problems with many independent parameters, like determinantal processes coming from the free fermion six vertex model. This could highlight previously missing structure.

1.10. **Outline of the rest of the proposal.** In this section we gave an overview of the tools used by the PI to attack the structural and asymptotic questions around Gibbs measures, integrable vertex models, and stochastic systems such as random configurations and interacting particle systems. In the next two sections, §§2 and 3 we discuss two concrete setups where initial progress could be achieved by a combination of these tools. When expected results are already visible and can be formulated precisely, they are called "**Conjectures**" in the text. There are several other settings in which the PI will apply the tools described here, but due to space limitations these settings can only be briefly mentioned. A brief summary of proposed research directions is given in §4. Finally, in §5 we discuss broader impacts of the PI's activities related to the project.

2. TASEP with arbitrary initial condition

2.1. **Bernoulli TASEP with sequential update.** Fix N and let $x_1, \ldots, x_N > 0$ be rapidities. Also pick the *speed parameter* $\beta > 0$. The *Bernoulli TASEP with sequential update* (in what follows we call it TASEP, for short) is an N-particle Markov process $\xi(t) = (\xi_1(t) > \ldots > \xi_N(t))$ on $\mathbb{Z}$ with discrete time $t \in \mathbb{Z}_{\geq 0}$. It evolves as follows. At each update $t \to t+1$, each particle ξ_i flips an independent coin with probability of Heads $\dfrac{\beta x_i}{1+\beta x_i}$. Then sequentially for $i = 1, \ldots, N$, update (see Figure 3 for an illustration)

$$\xi_i(t+1) = \begin{cases} \xi_i(t)+1, & \text{if the coin of } \xi_i \text{ is Heads and } \xi_{i-1}(t+1) > \xi_i(t)+1; \\ \xi_i(t), & \text{otherwise (coin is Tails, or the jump destination is occupied).} \end{cases}$$

By agreement, $\xi_0 \equiv +\infty$, so the first particle $\xi_1(t)$ performs a simple random walk. Let us denote by $\eta = (\eta_1 > \ldots > \eta_N)$ the initial state of TASEP at $t = 0$.

Remark 2.1. We consider the N-particle TASEP, but a process with infinitely many particles $\xi_1 > \xi_2 > \ldots$ is well-defined by consistency.

In the Poisson rescaling $\beta \to 0$, $t = \lfloor \tau/\beta \rfloor$, TASEP becomes a process with continuous time τ and Poisson jumps. It is well-defined started from any initial configuration $\in \{0,1\}^{\mathbb{Z}}$ by Harris graphical construction [Har78]. However, the finite-N statements we discuss in this section are sufficient for asymptotic analysis in the sense of finite-dimensional distributions.

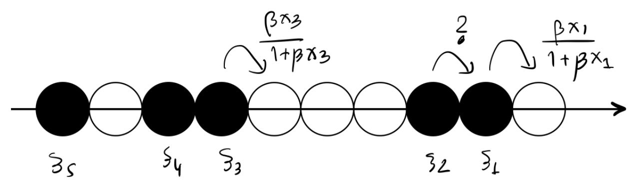

FIGURE 3. Bernoulli TASEP. The first particle jumps with probability $\beta x_1/(1+\beta x_1)$, the jump of the second particle depends on whether the first particle has jumped. The third particle jumps with probability $\beta x_3/(1+\beta x_3)$, and so on.

Let us present a vertex model representation of the joint distribution of $\xi(t)$ started from an arbitrary initial configuration η. Without loss of the generality we may assume that $\eta_N \geq -N$. This vertex model has three layers, the bottom one of size N with vertex weights w_{x_i} as in Figure 1, the middle one of size t with weights W^*_β under which the paths go up and left. In the top layer of size $N-1$, the weights are W_{-x_i}. Here the paths go up an right, and must end at points $\eta_i + i$ at consecutive horizontals in the top layer. In the middle and top layers, multiple paths per edge are allowed in both directions, but a path cannot travel more than one step on a single horizontal slice. The weights and the vertex model are given in Figure 4.

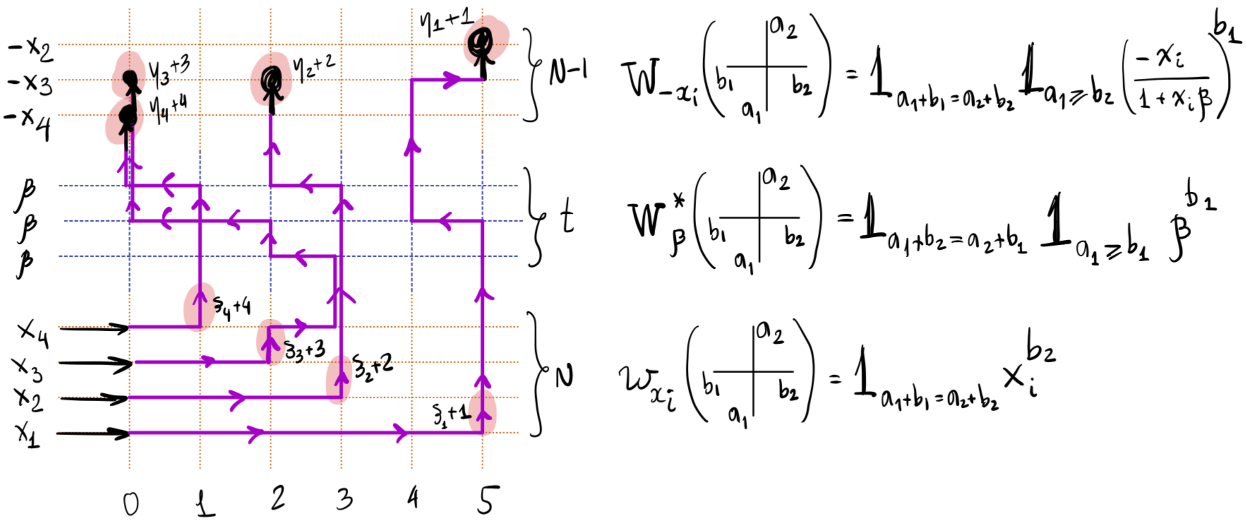

FIGURE 4. Vertex model for $\mathbb{P}[\xi(t) = \xi \mid \xi(0) = \eta]$, where $\xi_i + i$ are the coordinates of the leftmost upper arrows at the N bottom slices, and $\eta_i + i$ are the top boundary conditions located at consecutive horizontals in the top layer. The vertex weights and the rapidities are indicated, too. In the figure we have $\eta = (4, 0, -3, -4)$ and $\xi = (4, 1, -1, -3)$. Here and below $\mathbf{1}_A$ denotes the indicator of A.

Theorem 2.2. *For any $t \in \mathbb{Z}_{\ge 0}$ and any N-particle configurations ξ, η on $\mathbb{Z}_{\ge -N}$, the TASEP transition function $\mathbb{P}[\xi(t) = \xi \mid \xi(0) = \eta]$ is proportional (the normalizing constant is independent of ξ but may depend on the initial condition η) to the partition function of the vertex model as in Figure 4 with the boundary conditions determined by ξ, η.*

Note that while in the top layer the vertex weights could be negative, the partition function in Theorem 2.2 is always nonnegative, which is a part of the statement. Theorem 2.2 follows from [DW08]. The earlier case of homogeneous rapidities x_i is due to [BFPS07], and these interpretations are based on determinantal formulas of Rakos-Schutz [RS05], [RS06].

The PI will investigate whether other particle systems like q-TASEP and ASEP admit a vertex model interpretation similar to Figure 4 of their transition probabilities for arbitrary initial data. There are already vertex model candidates for the step initial data discussed by the PI in [BP19], [BMP21]. An initial question would be to see how the vertex model in Figure 4 solves the master equation (Kolmogorov equation associated with the Markov chain) for TASEP. Presumably, this fact should follow from Yang-Baxter equation only, and therefore the method would be extendable to all other integrable vertex models.

2.2. **Speed compression for step initial data.** In the case of the step initial data $\eta_j = -j$, $j = 1, \ldots, N$ and geometric rapidities $x_i = q^{i-1}$, $i = 1, \ldots, N$, where $0 < q < 1$, the PI has observed [PS21] that a certain Markov transition operator applied to the final condition $\xi(t)$ essentially preserves the TASEP distribution, but replaces β by $q\beta$. This Markov map is based on the *bijectivisation of the Yang-Baxter equation* introduced by the PI in [BP19]. In short, a bijectivisation is a mechanism of randomly updating an arrow configuration then the cross vertex moves from the left to the right in Figure 2. The probabilities of updates are determined by w_x, w_y, and $R_{y/x}$, and are chosen so that the relative weights in both sides of the Yang-Baxter equation are preserved by the random update.

Let us explain what bijectivisation brings in the concrete example of TASEP. We use the Yang-Baxter equation to take the horizontal slice with rapidity $x_1 = 1$ and exchange it with the slices with $x_2 = q, x_3 = q^2, \ldots, x_N = q^{N-1}$. The bijectivisation allows to realize each exchange as a *Markov map* (i.e., the application of a one-step Markov transition operator) which changes one row of the vertex model. When restricted to the leftmost vertical arrows at each slice, the combination of these Markov maps becomes the following random sequential update of the ξ_i's for $i = 1, \ldots, N-1$:

$$\xi_i \mapsto \tilde{\xi}_i := \xi_{i+1} + 1 + \min\left(\mathrm{Geo}_{q^{i+1}}, \xi_i - \xi_{i+1} - 1\right),$$

where $\mathrm{Geo}_q, \mathrm{Geo}_{q^2}, \ldots$ are independent geometric random variables with $\mathbb{P}\left[\mathrm{Geo}_\alpha = k\right] = (1-\alpha)\alpha^k$, $k \in \mathbb{Z}_{\geq 0}$. Let the combined Markov map $(\xi_1, \ldots, \xi_N) \mapsto (\tilde{\xi}_1, \ldots, \tilde{\xi}_{N-1})$ corresponding to the reordering of the rapidities $(1, q, q^2, \ldots, q^{N-1}) \to (q, q^2, \ldots, q^{N-1}, 1)$ be denoted by L_q. Note that the jump of ξ_{N-1} depends on ξ_N, and so L_q "forgets" the last particle.

Theorem 2.3 ([PS21])**.** *Fix $t \in \mathbb{Z}_{\geq 0}$ and let $\xi(t)$ be the N-particle TASEP configuration with speed β started from the step initial data $\eta_j = -j$, $1 \leq j \leq N$. Then the joint distribution of $(\tilde{\xi}_1, \ldots, \tilde{\xi}_{N-1}) = L_q(\xi_1(t), \ldots, \xi_N(t))$ coincides with the distribution of the $(N-1)$-particle TASEP of speed $q\beta$ started from the step initial data.*

Idea of proof. For the step initial data, the configuration in the top layer in Figure 4 (with the weights W_{-x_i}) is *frozen*. Then we use the homogeneity of the Schur polynomials (partition functions of the paths in the bottom layer, cf. Figure 1) which implies that the rapidity sequence $(q, q^2, \ldots, q^{N-1}; \beta, \ldots, \beta)$ arising after applying L_q (and forgetting the N-th particle which is possible by consistency) may be replaced by $(1, q, \ldots, q^{N-2}; q\beta, \ldots, q\beta)$ without changing the joint distribution of the path ensemble. The new speed parameter is $q\beta$, as desired. □

Remark 2.4. In the continuous time Poisson limit, the speed compression map becomes a map which turns the time in the continuous time TASEP backwards (in the sense of acting on the fixed-time distributions).

2.3. **Arbitrary initial data and KPZ fixed point.** We will establish an extension of Theorem 2.3 to general initial data η:

Conjecture 1 (Speed compression for arbitrary initial data)**.** *There exists a Markov map $M_{q,\beta}$ mapping $(\eta_1, \ldots, \eta_N) \mapsto (\tilde{\eta}_1, \ldots, \tilde{\eta}_{N-1})$ such that the following diagram of Markov maps is commutative (in the sense that the probability distribution of $\tilde{\xi}$ is the same along both paths in the diagram):*

$$\begin{array}{ccc} \eta & \xrightarrow{\mathrm{TASEP}(\beta)} & \xi \\ M_{q,\beta}\downarrow & & \downarrow L_q \\ \tilde{\eta} & \xrightarrow{\mathrm{TASEP}(q\beta)} & \tilde{\xi} \end{array} \tag{7}$$

General initial data leads to the vertex model with possibly negative vertex weights, so one has to carefully define the bijectivisation, and show that it leads to a Markov map $\eta \to \tilde{\eta}$. Presumably, this bijectivisation cannot be performed at a local level by moving the cross vertex one step at a time. However, there is hope that the resulting probability distribution after the action of L_q may be expanded with nonnegative coefficients (interpreted as probabilities) in the linear basis indexed by the possible initial configurations $\tilde{\eta}$.

The TASEP with arbitrary initial condition has gained a lot of attention recently after the identification of its space-time scaling limit as the *KPZ fixed point* process [MQR17]. The KPZ fixed point is a continuous time Markov process on one-dimensional interfaces $\mathfrak{h}(\tau, x)$, and [MQR17] expresses its transition cumulative distribution function $\mathbb{P}(\mathfrak{h}(t,x) \leq g(x) \mid \mathfrak{h}(0,x) = f(x))$ as a Fredholm determinant. The major open **strong KPZ universality conjecture** states that the KPZ fixed point is the scaling limit of any one-dimensional random growth model sufficiently similar to TASEP, that is, having symmetric slope-dependent growth and a smoothing mechanism. The corresponding scaling of the commutative diagram (7) should bring the following result:

Conjecture 2 (Compression symmetry for the KPZ fixed point)**.** *Let $\mathfrak{h}(\tau, x)$ be the KPZ fixed point starting from $f(x)$. There exists $\tau' < \tau$ and a pair of Markov maps L and M, each randomly changing a function $h(x)$ on $\mathbb{R}$, such that the distribution of $L(\mathfrak{h}(\tau, x))$ is the same as the distribution of $\mathfrak{h}'(\tau', x)$ where $\mathfrak{h}'$ is another KPZ fixed point started from $M(f(x))$.*

This conjecture, if proven, would uncover a nontrivial probabilistic symmetry of the KPZ fixed point, which would bring a better understanding of this universal limiting process. Moreover, such a symmetry might allow to characterize the KPZ fixed point by a list of its properties not including the explicit Fredholm determinantal transition probabilities. Such a characterization would be a significant step towards the strong KPZ conjecture.

3. Determinantal Processes, Dyson Brownian Motions, and Beyond

3.1. **Dyson Brownian motion.** Dyson Brownian motion (DBM), or the noncolliding Brownian motion of Coulomb repelling particles, is a fundamental dynamical model in random matrix theory. It is the Markov process on the spectrum of a random Hermitian matrix whose entries undergo independent Brownian motions. Introduced in the early 1960s [Dys62], DBM has been solved (its multipoint space-time correlations are given by a determinantal point process as in (4) with an explicit kernel, even when DBM starts from an arbitrary initial configuration) only about 35-40 years later [NF98], [Joh01].

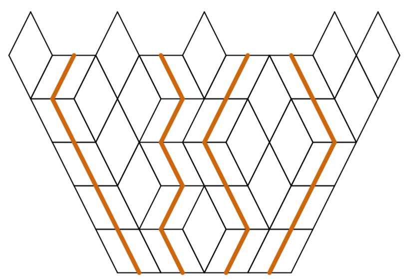

Figure 5. The lozenge tiling corresponding to the path configuration in Figure 1. The bold noncolliding paths start from an arbitrary configuration at the top (determined by λ), and in a limit turn into the q-noncolliding absorbing random walks.

The whole space-time trajectories of DBM satisfy the Brownian Gibbs property which is shared by the universal scaling limit, the Airy process [CH16]. Also DBM plays a key role in bulk universality results for random matrices [EY12] due to Dyson's conjecture (now proven). The latter states that the bulk universality under DBM is achieved at small time scales, much faster than the global shape changes. In [GP19] the PI has proven a discrete version of the Dyson's conjecture using the determinantal kernel for random tilings from [Pet14].

The proposed research deals with further discrete DBM-like stochastic systems coming from various families of symmetric functions.

3.2. **q-noncolliding absorbing random walks.** We start with the Schur vertex model (Figure 1) with rapidities $x_i = q^{i-1}$, $i = 1, \ldots, N$, forming a geometric progression, where $q \in (0, 1)$ is fixed. The path configurations are in bijection with lozenge tilings of a fixed shape (determined by the fixed top row consisting of the vertical lozenges placed at locations $\lambda_i - i$, $i = 1, \ldots, N$). Thus, we arrive at a random lozenge tiling, where the probability weight of a tiling is proportional to q^{vol}. Here vol is the volume under the 3D surface represented by a tiling, cf. Figure 5 (so q^{vol} is another natural example of a Gibbs measure). The conditional distribution of the $(N-1)$-st row having lozenges at $\mu_j - j$, $j = 1, \ldots, N-1$, is

$$\mathbb{P}[\mu \mid \lambda] = q^{(N-1)(\sum_j \mu_j - \sum_i \lambda_i)} \frac{s_\mu(1, q, q^2, \ldots, q^{N-2})}{s_\lambda(1, q, q^2, \ldots, q^{N-2}, q^{N-1})}. \tag{8}$$

Then we pass to the $N \to +\infty$ limit when the vertical lozenges at $\lambda_i - i$ are densely packed except for m gaps at $\xi = (\xi_1 > \ldots > \xi_m \geq 0)$. Reading the lozenge tiling from top to bottom brings a Markov chain of m noncolliding particles. Their trajectories are highlighted in Figure 5 (there time runs in the downward direction). Preliminary computation lead to the following results:

Conjecture 3 (q-noncolliding absorbing random walks)**.** *Let $\xi = (\xi_1 > \ldots > \xi_m \ge 0)$, $\eta = (\eta_1 > \ldots > \eta_m \ge 0)$ be such that $\eta_i - \xi_i \in \{0, -1\}$ for all i. Taking λ with gaps ξ_i and μ with gaps η_i in* (8)*, in the limit $N \to +\infty$ we obtain a discrete time Markov process with transition probabilities*

$$\mathbb{P}[\xi \to \eta] = q^{-\binom{m}{2}+(m-1)\sum_i(\xi_i-\eta_i)} \prod_{1\le i<j\le m} \frac{q^{\eta_i}-q^{\eta_j}}{q^{\xi_i}-q^{\xi_j}} \prod_{i=1}^{m} \Big(q^{\xi_i}\mathbf{1}_{\eta_i=\xi_i} + (1-q^{\xi_i})\mathbf{1}_{\eta_i=\xi_i-1} \Big). \tag{9}$$

Note that this random walk is absorbed at $(m-1, \ldots, 1, 0)$.

The fact that the probabilities (9) sum over all η to 1 is rather nontrivial and, to the best of the PI's knowledge, has not been observed before.

From the determinantal structure of the measure q^{vol} [Pet14] it follows that the noncolliding random walks also possess a determinantal structure:

Conjecture 4 (Determinantal structure)**.** *Started from an arbitrary initial configuration $(\xi_1 > \ldots > \xi_m \ge 0)$, the whole space-time trajectory $\{\xi_i(t)\colon t \ge 0,\ 1 \le i \le m\}$ is a determinantal point process with the correlation kernel*

$$\begin{aligned}\mathcal{K}(x_1, t_1; x_2, t_2) = -1_{t_2>t_1}1_{x_2>x_1} \frac{q^{x_1(t_1-t_2)}(q^{x_1-x_2+1};q)_{t_2-t_1-1}}{(q;q)_{t_2-t_1-1}} + \frac{q^{-x_1}}{(2\pi i)^2} \oint\oint \frac{dzdw}{w-z} \\ \times \frac{w^{t_1}}{z^{t_2}} \frac{(q;q)_{t_1}}{(wq^{-x_1};q)_{t_1+1}} \frac{(zq^{1-x_2};q)_{t_2-1}}{(q;q)_{t_2-1}} \frac{(w^{-1};q)_\infty}{(z^{-1};q)_\infty} \prod_{j=1}^{m} \frac{1-z^{-1}q^{\xi_j}}{1-w^{-1}q^{\xi_j}},\end{aligned} \tag{10}$$

for suitable integration contours. Here $(a;q)_k := (1-a)(1-aq)\ldots(1-aq^{k-1})$ is the q-Pochhammer symbol, which makes sense for $k = +\infty$ as well.

The kernel (10) arises as a limit of the kernel for the measure q^{vol} computed in [Pet14]. Note however that the latter contained the q-hypergeometric function ${}_3\phi_2$ under the integral, while (10) contains only products under the integral. Therefore, the kernel $\mathcal{K}$ is amenable to asymptotic analysis using steepest descent method (cf. §1.8).

Conjecture 5 (Bulk universality)**.** *Let $m \to +\infty$, $q = e^{-1/m} \to 1$, and assume that the counting measure $m^{-1}\sum_i \delta_{\xi_i}$ at the initial condition $\xi = (\xi_1 > \ldots > \xi_m \ge 0)$ converges to some fixed density profile. Then (under mild technical assumptions) at short times $t \to +\infty$, $t \ll m$, the fixed time $t_1 = t_2 = t$ local bulk (lattice) distribution of the determinantal process* (10) *coming from the noncolliding Markov chain* (9) *converges to the universal limit given by the discrete sine kernel $K(x_1, x_2) = \frac{\sin(\rho(x_1-x_2))}{\pi(x_1-x_2)}$, $x_1, x_2 \in \mathbb{Z}$. Here ρ/π is the limiting density depending on the global location in the bulk.*

The process (9) can be viewed as the Doob's h-transform of m independent space-inhomogeneous simple random walks with transitions $\mathbb{P}[k \to k] = q^k$, $\mathbb{P}[k \to k-1] = 1 - q^k$ (absorbed at $k = 0$). The harmonic function $h(\xi) = q^{-(m-1)\sum_i \xi_i} \prod_{i<j}(q^{\xi_i} - q^{\xi_j})$ has eigenvalue $q^{\binom{m}{2}}$ (instead of the more common case when the eigenvalue is equal to 1). Note that the DBM itself is the h-transform of independent Brownian motions $z_j(\tau)$, with similarly looking harmonic function $\prod_{i<j}(z_i - z_j)$.

Our q-noncolliding process (9) degenerates to DBM in a diffusive limit. We will study other fixed-m limits of (9) which could possibly lead to random matrix type diffusions at the hard edge of the Wishart ensemble. In this way the determinantal structure (10) would provide formulas for these hard edge diffusions with arbitrary initial condition.

3.3. **Macdonald generalization.** Transition probabilities (8) are defined in terms of the Schur symmetric polynomials. They admit an immediate generalization involving Macdonald symmetric polynomials depending on two parameters (q, t), and potentially further generalizations to Koornwinder polynomials related to symmetries of other Lie types. Let us present the Macdonald case:

Conjecture 6 (Macdonald noncolliding walks)**.** *The following transition probabilities from* $\xi = (\xi_1 > \ldots > \xi_m \geq 0)$ *to* $\eta = (\eta_1 > \ldots > \eta_m \geq 0)$*, with* $\eta_i - \xi_i \in \{0, -1\}$ *for all* i*, sum to one and define a discrete time Markov chain:*

$$\begin{aligned}\mathbb{P}[\xi \to \eta] &= t^{-\binom{m}{2}+\sum_{i=1}^{m}\xi_i+\sum_{i=0}^{m-1}(\xi_{m-i}-i)(\eta_{m-i}-\xi_{m-i})} \\ &\times \prod_{\substack{1\leq i<j\leq m\\ \eta_i=\xi_i,\ \eta_j=\xi_j-1}} \frac{(1-q^{j-i-1}t^{\xi_i-\xi_j-j+i+1})(1-q^{j-i+1}t^{\xi_i-\xi_j-j+i})}{(1-q^{j-i}t^{\xi_i-\xi_j-j+i})(1-q^{j-i}t^{\xi_i-\xi_j-j+i+1})} \\ &\times \prod_{1\leq i<j\leq m} \frac{\left(q^{j-i}t^{\xi_i-\xi_j-j+i+1};t\right)_\infty \left(q^{j-i-1}t^{\eta_i-\eta_j-j+i+1};t\right)_\infty}{\left(q^{j-i-1}t^{\xi_i-\xi_j-j+i+1};t\right)_\infty \left(q^{j-i}t^{\eta_i-\eta_j-j+i+1};t\right)_\infty} \prod_{i=1}^{m}\left(1-q^{m-i}t^{\xi_i-m+i}\mathbf{1}_{\eta_i=\xi_i-1}\right).\end{aligned} \tag{11}$$

This Markov chain with two Macdonald parameters turns into (9) when $q = t$. For general q, t, the chain fails to be determinantal. In a scaling limit $q = t^\theta \to 1$, where $\theta > 0$ is fixed, the dynamics (11) becomes the Beta-noncolliding Poisson process worked out in [Hua21]. A possible approach to study asymptotics of the Markov chain (11) would be a suitable discrete version of the *loop* (*Schwinger–Dyson*) *equations* [BGG17].

3.4. **Free fermion six vertex model.** Discrete determinantal processes discussed above in the proposal are associated with the integrable structure of Schur symmetric polynomials. Very recently, the PI and collaborators [ABPW21] introduced an inhomogeneous deformation of the Schur polynomials (2) coming from the free fermion six vertex model. The deformation depends on four sequences of parameters — row and column *rapidities*, and row and column *spin parameters* (previously we had only row rapidities), $\mathbf{x} = (x_1, \ldots, x_N)$, $\mathbf{r} = (r_1, \ldots, r_N)$, $\mathbf{y} = (y_1, y_2, \ldots)$, $\mathbf{s} = (s_1, s_2, \ldots)$:

$$F_\lambda(\mathbf{x};\mathbf{y};\mathbf{r};\mathbf{s}) = \left(\prod_{i=1}^{N} x_i(r_i^{-2}-1) \prod_{1\leq i<j\leq N} \frac{r_i^{-2}x_i - x_j}{x_i - x_j}\right) \det\left[\varphi_{\lambda_j+N-j}(x_i \mid \mathbf{y};\mathbf{s})\right]_{i,j=1}^{N}, \tag{12}$$

where φ_j are the inhomogeneous powers $\varphi_k(x) := \dfrac{1}{y_{k+1}-x}\prod_{j=1}^{k}\dfrac{x - s_j^{-2}y_j}{x-y_j}$.

The functions F_λ share many common properties with the Schur symmetric polynomials. Moreover, [ABPW21] a first instance of a determinantal point process based on these functions (an analogue of the Schur process [OR03]) was investigated, leading to a new bulk determinantal kernel with four bi-infinite families of inhomogeneities. The PI will continue investigating structure and asymptotics of the inhomogeneous Schur-like vertex models, by extending known results and probing new asymptotic phenomena. Here are the immediate research goals in this direction:

1. Introduce Markov dynamics on the two-dimensional ensemble coming from the free fermion six vertex model, and study its bulk limit. Via determinantal structure, this would lead to an inhomogeneous deformation of the anisotropic KPZ growth of [BF14], and the explicit particle current (as a function of the slope) should be accessible explicitly.

2. Study global fluctuations of the exiting determinantal process associated with (12). In the homogeneous case, global fluctuations are given by the Gaussian Free Field [Pet15]. The PI will probe deformations of the Gaussian Free Field in the presence of the inhomogeneities.

3. Obtain an explicit determinantal kernel for the boxed measure coming from the free fermion six vertex model. Presumably, like for the ordinary Schur case, this is limited to the geometric specialization of the rapidities $x_i = q^{i-1}$ (including the "uniform" case $q \nearrow 1$).

4. Putting the functions (12) in a combination like (8), obtain m-particle noncolliding dynamics with explicit determinantal structure for arbitrary initial configuration. Investigate possible bulk universality results, and random matrix diffusive limits.

4. Intellectual Merit: Summary of Research Directions

Overall, the project revolves around studying new and known integrable stochastic systems whose structure is accessible through Yang-Baxter equation and/or symmetric functions. Above we have outlined a number of conjectures and research directions in two settings — TASEP with general initial data (§2) and determinantal processes and Dyson Brownian motion like Markov chains (§3). Let us summarize the proposed objects of study and the proposed results:

A. New models with integrable structure: q-noncolliding absorbing random walks with determinantal structure (in the case of arbitrary initial data), their generalization with Macdonald parameters, and scaling to random matrix diffusions. Related models coming from symmetric functions associated with other Lie symmetry types. Determinantal processes with inhomogeneous parameters associated with the free fermion six vertex model.

B. New integrable structure in known models: TASEP with arbitrary initial data via a vertex model, and its distributional symmetry. Realization of distributions of q-TASEP and ASEP with arbitrary initial data through vertex models. Scaling of the vertex models to the KPZ fixed point, and distributional symmetry of the latter.

C. New asymptotic phenomena: Inhomogeneous deformations of the Gaussian Free Field; hydrodynamics and the particle current in the presence of inhomogeneities, potential SPDE limits. Besides probing new asymptotic phenomena, the project will also lead to universal results when the asymptotic distribution is predicted from previously known stochastic systems.

Initial progress along each of the directions is clearly possible, as outlined in the proposal. Let us emphasize that these directions, as well as concrete questions being asked, are informed by the four major open problems mentioned in the proposal above:

1. Classification of pure Gibbs states in the presence of interaction (for example, for the general six vertex model).

2. Structure and asymptotics of multipoint distributions in non-determinantal / interacting fermion setting.

3. Large deviations on both tails via Fredholm determinants.

4. Strong KPZ universality, in particular, characterization of the KPZ fixed point which avoids explicit formulas but may be based on distributional symmetries.

The PI believes that understanding the structure of underlying stochastic systems (by means of exact formulas, symmetric functions, and Markov maps preserving the probability distributions in question) could lead to eventual progress in at least one of these important open problems. This structure could be informed by introducing as many inhomogeneous parameters into the systems as possible.

5. Broader Impacts

5.1. **Impacts on other disciplines.** Statistical mechanical and random growth models are motivated by a wide range of real-world questions concerning the structure of ice and other condensed matter, magnetism, quantum spin systems, thermodynamics, traffic models, directed polymers. The proposed project will impact many of these questions. For example, most of the inhomogeneous space models are relevant for studying randomly growing or Coulomb repulsive systems with impurities. New irreversible Markov chains on important Gibbs measures such as the six- and eight-vertex models are expected to provide their faster sampling than the well-known reversible Glauber dynamics. This would lead to a better phenomenological and numerical understanding of possible limit regimes in these models.

5.2. **Training new researchers.** Many problems and algorithms developed as a part of the project will be accessible to undergraduate and beginning graduate students. The PI is very active at supervising undergraduate research (including numerical studies of new models), and will continue working with undergraduate and graduate students on the topics described in this proposal. In the past years, the PI has lead a number of joint projects with undergraduate and graduate students and postdocs.

5.3. **Powering research and organizational connections.** Throughout the PI's time at University of Virginia, he is actively involved in transmission of actual information relevant to research and outreach activities at Department of Mathematics. These include creation and maintenance of the Department's website, and an active role of the PI in creating online spaces for the Student Chapter of the Association of Women in Mathematics at the University of Virginia; the Math Collaborative Learning Center; the Directed Reading Program, and so on. In the COVID pandemic time, these efforts have been especially crucial, and the PI continues to play an active role in connecting colleagues with each other through online tools.

Since Fall 2021, the PI participates in teaching design team at Department of Mathematics, University of Virginia, aimed at closing the achievement gaps for under-represented students in introductory courses.

5.4. **Event organization and dissemination.** The PI is active at disseminating research results through talks at conferences; mini-courses; teaching graduate topics courses and preparing lecture notes (Fall 2012, Spring 2016, Fall 2019; Spring 2021; some of the lecture notes are posted to the AMS Open Notes); organizing reading seminars for advanced undergraduate and graduate students at University of Virginia; and organizing workshops and summer schools. Following the beginning of the COVID pandemic, in April 2020 the PI organized one of the first ever online conferences (on Statistical Mechanics, Integrable Systems and Probability) which featured six talks by leading experts in these fields, and dozens of participants from all around the world. The PI will continue organizing online and in-person meetings and conferences around integrable probability and related fields.

In particular, the PI will organize Virginia Integrable Probability Summer School 2023, which is modeled after the very successful 2019 summer school supported by **DMS–1664617** "FRG: Collaborative Research: Integrable Probability". The 2023 summer school will last 2 to 3 weeks, will feature 5 mini-courses by leading researchers in Integrable Probability and related areas, and will host 30-50 students. The students will participate in exercise sessions, networking events (such as a career development panel), and will have a possibility to present a short talk. The lectures at the summer school will be made available online.

5.5. **Availability of research outcomes.** Publications produced within the project will be posted at the arXiv preprint server and submitted to conventional journals. The PI's homepage contains a number of simulation results (including code and full raw data), and many of the colleagues have used these pictures for illustration of their talks and papers. Full results of computer simulations produced within the project will be made publicly available on the web.

Remark. This is the posted version. Some details removed, mainly section 1 on results from prior NSF support.

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