SPIN q-WHITTAKER POLYNOMIALS AND DEFORMED QUANTUM TODA

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Abstract. Spin q-Whittaker symmetric polynomials labeled by partitions λ were recently introduced by Borodin and Wheeler [BW17] in the context of integrable $\mathfrak{sl}_2$ vertex models. They are a one-parameter deformation of the $t = 0$ Macdonald polynomials. We present a new more convenient modification of spin q-Whittaker polynomials and find two Macdonald type q-difference operators acting diagonally in these polynomials with eigenvalues, respectively, $q^{-\lambda_1}$ and q^{λ_N} (where λ is the polynomial's label). We study probability measures on interlacing arrays based on spin q-Whittaker polynomials, and match their observables with known stochastic particle systems such as the q-Hahn TASEP.

In a scaling limit as $q \nearrow 1$, spin q-Whittaker polynomials turn into a new one-parameter deformation of the $\mathfrak{gl}_n$ Whittaker functions. The rescaled Pieri type rule gives rise to a one-parameter deformation of the quantum Toda Hamiltonian. The deformed Hamiltonian acts diagonally on our new spin Whittaker functions. On the stochastic side, as $q \nearrow 1$ we discover a multilevel extension of the beta polymer model of Barraquand and Corwin [BC16a], and relate it to spin Whittaker functions.

Contents

1. Introduction

1.1. **Overview.** This paper deals with new classes of symmetric functions inspired by the $U_q(\widehat{\mathfrak{sl}_2})$ Yang–Baxter equation and applications to integrable stochastic interacting particle systems and random polymer models.

Symmetric functions have been very useful in studying integrable stochastic systems in the past two decades, starting from the works on asymptotic fluctuations in longest increasing subsequences of random permutations [BDJ99] and the TASEP (totally asymmetric simple exclusion process) [Joh00], and continuing through the frameworks of Schur processes [Oko01], [OR03] and Macdonald processes [BC14]. Here and below by a *process* associated with a family of symmetric functions (like Schur or Macdonald) we mean a probability measure on sequences of partitions with probability weights expressed through these functions in a certain way (cf. Definition 4.3 in the text). See the scheme of symmetric functions in Figure 1.

More recently, quantum integrability (in the form of the Yang–Baxter equation / Bethe ansatz [Bax07]) has brought new structures allowing to extend the range of exactly solvable stochastic systems to the ASEP (partially asymmetric simple exclusion process) [TW08], [TW09] and stochastic vertex models [BCG16], [CP16], [CT15], [Lin20], and discover new asymptotic phenomena around the Kardar-Parisi-Zhang universality class [Cor16]. In the process of exploring quantum integrability from these perspectives, two new families of symmetric functions were discovered:

- Spin Hall–Littlewood symmetric functions [Bor17]. They are a one-parameter generalization of the classical Hall–Littlewood polynomials [Mac95, Ch. III], and are Bethe Ansatz eigenfunctions of a number of integrable stochastic systems, including ASEP (under a certain choice of parameters). These functions retain many properties of Hall–Littlewood polynomials including Cauchy type summation identities, Pieri type rules, torus scalar product orthogonality, and the presence of difference operators acting on them diagonally [BCPS15], [BP18], [BMP19, Section 8].
- Spin q-Whittaker polynomials [BW17]. They form a one-parameter generalization of the q-deformed $\mathfrak{gl}_n$ Whittaker functions [GLO10], and also possess Cauchy type summation identities, Pieri type rules, and certain first-order difference operators acting on them diagonally [BMP19, Section 8]. Notably, torus orthogonality relation for spin q-Whittaker polynomials is not known.

Marginals of spin Hall–Littlewood and spin q-Whittaker processes are matched in distribution to various $U_q(\widehat{\mathfrak{sl}_2})$ stochastic vertex models, including the stochastic six vertex model [GS92], [BCG16] (with its "dynamic" extension [BP19]); the stochastic higher spin six vertex model [CP16], [BP18]; and the more recent ${}_4\phi_3$ stochastic vertex model [BMP19] (which is close to the q-Hahn PushTASEP of [CMP19]).

The (undeformed) q-Whittaker polynomials admit a nontrivial scaling limit as $q \to 1$. In this limit [GLO12b], [BC14, Theorem 4.1.7], the q-Whittaker polynomials become the $\mathfrak{gl}_N$ Whittaker functions. The latter play an important role in representation theory and integrable systems [Kos80], [Giv97], [Eti99]. In particular, the Whittaker functions $\psi_{\underline{\lambda}}(\underline{u}_N)$, $\underline{\lambda} = (\lambda_1, \ldots, \lambda_N) \in \mathbb{C}^N$, $\underline{u}_N = (u_1, \ldots, u_N) \in \mathbb{R}^N$, are eigenfunctions of the quantum $\mathfrak{gl}_N$ Toda lattice Hamiltonian

$$\mathcal{H}_2^{\mathrm{Toda}} = -\frac{1}{2}\sum_{i=1}^{N} \frac{\partial^2}{\partial u_i^2} + \sum_{i=1}^{N-1} e^{u_{i+1}-u_i}, \qquad \mathcal{H}_2^{\mathrm{Toda}} \psi_{\underline{\lambda}}(\underline{u}_N) = \left(-\frac{1}{2}\sum_{i=1}^{N} \lambda_i^2\right) \psi_{\underline{\lambda}}(\underline{u}_N). \tag{1.1}$$

Probability measures based on Whittaker functions describe distributions of integrable models of directed random polymers: the semi-discrete Brownian polymer [O'C12], and fully discrete polymers in random environments with independent log-gamma distributed weights [COSZ14], [OSZ14], [OO15], [CSS15].

The contribution of this paper is two-fold. First, we present a new version of spin q-Whittaker polynomials which generalizes the ones from [BW17] and strengthen their properties. Second, in

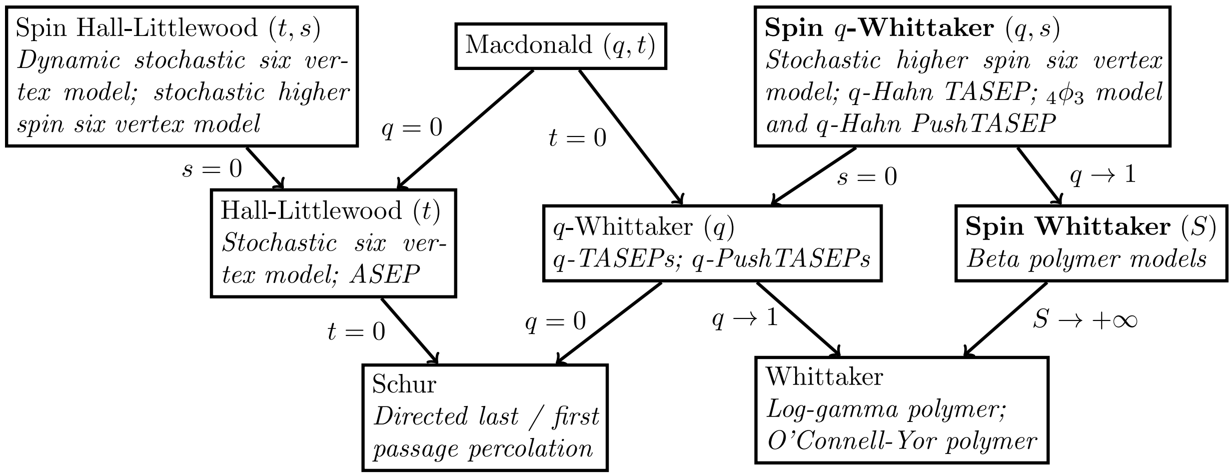

FIGURE 1. A scheme of various families of symmetric functions together with stochastic systems based on them. Arrows indicate degenerations or scaling limits. The two families which are our main focus are indicated in bold.

a $q \to 1$ limit, we discover a nontrivial one-parameter deformation of the $\mathfrak{gl}_N$ Whittaker functions. The new spin Whittaker functions are eigenfunctions of a deformed quantum Toda Hamiltonian, and are also related to random polymers with beta distributed weights. Let us briefly describe our main results.

1.2. **A new version of spin q-Whittaker polynomials.** First, we introduce *modified versions of the spin q-Whittaker symmetric polynomials* $\mathbb{F}_\lambda(x_1, \ldots, x_n)$, where $\lambda = (\lambda_1 \geq \ldots \geq \lambda_n \geq 0)$, $\lambda_i \in \mathbb{Z}$, are (nonnegative) signatures. Our polynomials are more general than the Borodin–Wheeler's version $\mathbb{F}_\lambda^{BW}$ [BW17]. More precisely, we have

$$\mathbb{F}_\lambda(x_1, x_2, \ldots, x_n)\big|_{x_1=0} = \mathbb{F}_\lambda^{BW}(x_2, \ldots, x_n).$$

Under the degeneration $s = 0$, both families $\mathbb{F}_\lambda$ and $\mathbb{F}_\lambda^{BW}$ coincide and turn into the usual q-Whittaker polynomials.

The new spin q-Whittaker polynomials $\mathbb{F}_\lambda$ share all the properties known for the $\mathbb{F}_\lambda^{BW}$'s, including symmetry, Cauchy summation identities, and Pieri type rules. Moreover, we strengthen other known properties of the spin q-Whittaker polynomials:

- (Section 3.2) We present q-difference operators $\mathfrak{D}_1$, $\overline{\mathfrak{D}}_1$ which act on our new spin q-Whittaker polynomials diagonally as $\mathfrak{D}_1\mathbb{F}_\lambda = q^{\lambda_N}\mathbb{F}_\lambda$ and $\overline{\mathfrak{D}}_1\mathbb{F}_\lambda = q^{-\lambda_1}\mathbb{F}_\lambda$. The operator $\overline{\mathfrak{D}}_1$ reduces, as $x_1 \to 0$, to the known eigenoperator $\mathfrak{E}$ [BMP19] acting on $\mathbb{F}_\lambda^{BW}$ with the same eigenvalue $q^{-\lambda_1}$. The operator $\overline{\mathfrak{D}}_1$, and the fact that the other eigenvalue q^{λ_N} can be extracted from spin q-Whittaker polynomials, are new.
- (Section 3.3) We observe that the operators $\mathfrak{D}_1$, $\overline{\mathfrak{D}}_1$ can be represented as conjugations of the first q-Whittaker q-difference operators (these are $t = 0$ degenerations of the Macdonald operators from [Mac95]). From higher q-Whittaker operators we thus get higher order q-difference operators commuting with either $\mathfrak{D}_1$ or $\overline{\mathfrak{D}}_1$ (the conjugations leading to $\mathfrak{D}_1$ and $\overline{\mathfrak{D}}_1$ are different, even though these operators commute). The higher order operators coming from the q-Whittaker operators are not diagonal in the spin q-Whittaker polynomials.

- (Section 5) For spin q-Whittaker processes on interlacing arrays of signatures, we construct sampling algorithms ("Yang–Baxter fields") based on bijectivizations of the Yang–Baxter equation, using ideas of the previous works [BP19], [BMP19]. We consider several Yang–Baxter fields, and by the very construction each of them possesses a marginally Markovian projection to a one-dimensional system: stochastic higher spin six vertex model; ${}_4\phi_3$ stochastic vertex model / q-Hahn PushTASEP; or the q-Hahn TASEP from [Pov13], [Cor14]. The former two connections already appeared in [BMP19] (for processes based on $\mathbb{F}_\lambda^{BW}$), while the matching to the q-Hahn TASEP is new.
- (Section 5.6) In a simplified "Plancherel" (or "Poisson-type") continuous time limit we construct a Markov dynamics on interlacing arrays under which the last rows marginally evolve as a continuous time version of the q-Hahn TASEP (appeared in [BC16b]). Our new two-dimensional continuous time dynamics is a one-parameter deformation of the q-Whittaker 2d-growth model introduced in [BC14, Definition 3.3.3]. The latter growth model has continuous time q-TASEP as the last row marginal dynamics.

Our modification of the spin q-Whittaker polynomials originates from computer experiments informed by the existing definition from [BW17] combined with the desire to have q-difference eigenoperators (a particular case of one of the eigenoperators appeared earlier in [BMP19]). The new spin q-Whittaker polynomials can be formulated as partition functions of up-right path ensembles (cf. Figure 3, left), where paths must stay above the diagonal, and the vertex weights at the diagonal are special. These special *corner vertex weights* turn out to satisfy a version of the Yang–Baxter equation (given in Proposition B.2 in Appendix). Combined with the Yang–Baxter equation for the spin q-Whittaker *bulk vertex weights* written down in [BW17] (which is a fusion of the most basic Yang–Baxter equation for the six vertex model), this brings most of the desired properties of the new polynomials, including their symmetry and Cauchy summation identities. We also note that for $s = 0$, corner and bulk vertex weights coincide, so the effect of the new corner weights is present only at the $s \neq 0$ level.

It would be very interesting to connect our corner vertex weights and the corresponding Yang–Baxter equation with known integrable vertex model constructions.

1.3. **Spin Whittaker functions, random polymers, and deformed quantum Toda.** Our second series of results deals with the $q \to 1$ scaling limit of the spin q-Whittaker polynomials. Stochastic systems which we have associated with the spin q-Whittaker polynomials already known to possess such limits:

- The q-Hahn TASEP becomes the strict-weak directed polymer model in an environment built from independent random variables with beta distribution [BC16a]. We recall it in Definition 7.4.
- The q-Hahn PushTASEP scales [CMP19] to another beta polymer type model — a rather complicated system determined by a random recursion with negative beta binomial random weights. We recall (a slight generalization of) this model in Definition 7.9.

Introduce the scaling

$$q \to 1, \qquad x_i = q^{X_i}, \qquad s = -q^S, \qquad q^{-\lambda_i} = L_i,$$

where $S > 0$, $|X_i| < S$, and $1 \le L_N \le \ldots \le L_1$ are fixed real numbers. We show (Theorem 6.14) that under this scaling, the spin q-Whittaker polynomial $\mathbb{F}_\lambda(x_1, \ldots, x_N)$ converges to a new object — the *spin Whittaker function* $\mathfrak{f}_{X_1,\ldots,X_N}(L_1, \ldots, L_N)$ (which also depends on S).

The functions $\mathfrak{f}_{X_1,\dots,X_N}(L_1,\dots,L_N)$ may be defined via a recursive Givental-type integral representation. Let $\underline{L}'_{N-1}=(L'_{N-1},\dots,L'_1)$ and $\underline{L}_N=(L_N,\dots,L_1)$ be interlacing sequences:

$$1\le L_N\le L'_{N-1}\le L_{N-1}\le\dots\le L'_1\le L_1$$

(notation: $\underline{L}'_{N-1}\prec\underline{L}_N$). Define

$$\mathfrak{f}_X(\underline{L}'_{N-1};\underline{L}_N)\coloneqq\frac{1}{(\mathrm{B}(S+X,S-X))^{N-1}}\left(\frac{L_N\cdots L_1}{L'_{N-1}\cdots L'_1}\right)^{-X}\\ \times\prod_{j=1}^{N-1}\left(1-\frac{L'_j}{L_j}\right)^{S-X-1}\left(1-\frac{L_{j+1}}{L'_j}\right)^{S+X-1}\left(1-\frac{L_{j+1}}{L_j}\right)^{1-2S},$$

where $\mathrm{B}(S+X,S-X)$ is the Beta function. Set $\mathfrak{f}_{X_1}(L_1)\coloneqq L_1^{-X_1}$, and, inductively,

$$f_{X_1,\dots,X_N}(\underline{L}_N)\coloneqq\int_{\underline{L}'_{N-1}\colon\underline{L}'_{N-1}\prec\underline{L}_N}\mathfrak{f}_{X_1,\dots,X_{N-1}}(\underline{L}'_{N-1})\,\mathfrak{f}_{X_N}(\underline{L}'_{N-1};\underline{L}_N)\,\frac{d\underline{L}_{N-1}}{\underline{L}'_{N-1}}.\tag{1.2}$$

Example 1.1. In the simplest nontrivial case $N=2$ we have

$$\mathfrak{f}_{X,Y}(u,z)=(z/u)^S u^{-X-Y}{}_2F_1\left(\begin{matrix}S+X\,,\,S+Y\\2S\end{matrix}\,\Big|\,1-\frac{z}{u}\right),\qquad 1\le u\le z,$$

where ${}_2F_1$ is the Gauss hypergeometric function (A.10).

Remark 1.2. Observe that in contrast with the usual Whittaker functions, the spin deformations depend on *ordered* tuples $\underline{L}_N$. This also corresponds to the fact that the integration in (1.2) is over sequences $\underline{L}'_{N-1}$ interlacing with $\underline{L}_N$.

The spin Whittaker functions $f_{X_1,\dots,X_N}(\underline{L}_N)$ defined by the recursion (1.2) are *symmetric* in the X_i's. This fact is far from being obvious from this recursive representation, and follows from the symmetry of the spin q-Whittaker polynomials (which ultimately is a consequence of the Yang–Baxter equation).

We show (Theorem 8.3) that the functions $\mathfrak{f}_{X_1,\dots,X_N}(\underline{L}_N)$ are eigenfunctions of a deformation of the $\mathfrak{gl}_N$ quantum Toda Hamiltonian

$$\mathcal{H}_2\coloneqq-\frac{1}{2}\sum_{i=1}^N\partial_{u_i}^2+\sum_{1\le i<j\le N}S^{-2(j-i)}e^{u_j-u_i}(S-\partial_{u_i})(S+\partial_{u_j}).\tag{1.3}$$

Introduce a change of variables $L_j=S^{N+1-2j}e^{u_j}$. Then in these variables we have

$$\mathcal{H}_2\mathfrak{f}_{X_1,\dots,X_N}=\left(-\frac{1}{2}\sum_{j=1}^N X_j^2\right)\mathfrak{f}_{X_1,\dots,X_N}.$$

The functions $\mathfrak{f}_{X_1,\dots,X_N}(\underline{L}_N)$ satisfy a version of the Cauchy type identity with integration over $1\le L_N\le\dots\le L_1$:

$$\int\mathfrak{f}_{X_1,\dots,X_N}(\underline{L}_N)\,\mathfrak{g}_{Y_1,\dots,Y_M}(\underline{L}_N)\,\frac{d\underline{L}_N}{\underline{L}_N}=\prod_{j=1}^M\frac{\Gamma(X_1+Y_j)}{\Gamma(S+X_1)}\left(\prod_{i=2}^N\frac{\Gamma(X_i+Y_j)\Gamma(2S)}{\Gamma(S+X_i)\Gamma(S+Y_j)}\right).\tag{1.4}$$

Here $\mathfrak{g}_{Y_1,\ldots,Y_M}(\underline{L}_N)$ are certain dual spin Whittaker functions, see Section 6.4. For the usual Whittaker functions, first Cauchy type identity with $M = N$ is due to Bump and Stade [Bum89], [Sta02], [GLO08], and was later generalized in [COSZ14, (1.2)], [BC14, Section 4.2.1].

We also define *spin Whittaker processes*. These are probability measures on interlacing sequences of reals $\underline{L}_1 \prec \underline{L}_2 \prec \ldots \prec \underline{L}_N$, $\underline{L}_k = (L_{k,k} \le L_{k,k-1} \le \ldots \le L_{k,1})$, with probability weights expressed through the spin Whittaker functions. Cauchy type identity (1.4) provides an explicit normalizing constant for the spin Whittaker process. We match the distribution of the marginal process $L_{k,k}^{-1}$ to the strict-weak beta polymer model of [BC16a] (Theorem 7.6), and the distribution of the other marginal process $L_{k,t}$ to the other beta polymer like model which appeared in [CMP19] (Theorem 7.11).

As $S \to +\infty$ and under the scaling $L_j = S^{N+1-2j}e^{u_j}$, $X_j = -\mathrm{i}\lambda_j$, the spin Whittaker functions $\mathfrak{f}_{X_1,\ldots,X_N}(\underline{L}_N)$ formally reduce to the usual Whittaker functions $\psi_{\underline{\lambda}}(\underline{u}_N)$. A similar reduction brings spin Whittaker processes to Whittaker processes from [O'C12], [COSZ14], [BC14]. We do not fully justify these limit transitions, as this requires a much finer analysis and justification of the interchange of the $S \to +\infty$ limit with Givental-type representations, which is outside the scope of this paper. However, we note that at the level of marginals, the strict-weak beta polymer becomes the strict-weak log-gamma polymer [BC16a, Remark 1.5]. We also show (Proposition 7.13) that the other beta polymer type model turns into the log-gamma polymer from [Sep12].

Finally, we note that at the level of quantum Toda Hamiltonians the limit $\lim_{S\to+\infty} \mathcal{H}_2 = \mathcal{H}_2^{\mathrm{Toda}}$ is quite straightforward. Indeed, the only terms surviving this limit have $j = i + 1$, and then $S^{-2}(S - \partial_{u_i})(S + \partial_{u_{i+1}}) \to 1$ leads to (1.1)

1.4. **Outline.** The paper has two main parts. The *discrete* part, Sections 2 to 5, discusses spin Hall–Littlewood functions and our new variant of q-Whittaker functions, Cauchy type summation identities, difference operators diagonalized by these functions, and related integrable stochastic models. The *continuous* part, Sections 6 to 8, deals with continuously labeled spin Whittaker functions and their properties. These include Givental-type integral representations for spin Whittaker functions, Cauchy type identities, connections to random polymer models with beta weights, and a new deformation of the quantum Toda Hamiltonian.

In Section 9 we discuss a number of further directions, and formulate conjectures about torus scalar product orthogonality of spin q-Whittaker and spin Whittaker functions.

Appendix A collects notation relevant to special functions used in the paper. In Appendix B we list Yang–Baxter equations used throughout the discrete part. Appendices C and D contain certain technical proofs used in the main text (in Sections 6 and 8, respectively).

1.5. **Notation.** We use the q-Pochhammer symbol notation

$$(a;q)_k := (1-a)(1-aq)\ldots(1-aq^{k-1}), \qquad (a;q)_0 := 1. \tag{1.5}$$

Occasionally we will need multiple q-Pochhammer symbols $(a_1,\ldots,a_m;q)_k := (a_1;q)_k \ldots (a_m;q)_k$. Certain special functions such as q-hypergeometric and hypergeometric functions, as well as useful probability distributions based on them are described in Appendix A.

Throughout the paper, $\mathbf{1}_A$ denotes the indicator of an event A.

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2. Spin q-Whittaker and spin Hall–Littlewood functions

Here we introduce symmetric functions we use throughout the paper which are variants of the spin Hall–Littlewood and spin q-Whittaker functions of [Bor17], [BW17].

2.1. **Signatures.** Our symmetric functions are indexed by nonnegative signatures (i.e., partitions with a specified number of parts N). We will drop the word "nonnegative", and refer to them simply as "signatures". Signatures form a set

$$\mathrm{Sign}_N := \{\lambda = (\lambda_1 \geq \cdots \geq \lambda_N \geq 0)\colon \lambda_i \in \mathbb{Z}_{\geq 0}\}, \qquad N \in \mathbb{Z}_{\geq 0}.$$

By agreement, $\mathrm{Sign}_0 = \{\varnothing\}$. The number of positive parts of a signature λ is denoted by

$$\ell(\lambda) = \#\{i\colon \lambda_i > 0\}.$$

When λ is a partition (and not a signature), the quantity $\ell(\lambda)$ takes the name of *length*. We *will not* use such terminology as it creates confusion with the number N of coordinates of the signature λ.

For notational convenience, we will also label certain symmetric functions with the transpose of a signature. To define the transposition in the context of signatures, introduce the set of *boxed signatures*

$$\mathrm{Sign}_M^{\leq N} := \{\lambda = (\lambda_1 \geq \cdots \geq \lambda_M \geq 0)\colon 0 \leq \lambda_i \leq N\} \subset \mathrm{Sign}_M.$$

Clearly, these signatures can be represented as belonging to the box $\mathrm{Box}(N, M)$, where

$$\mathrm{Box}(N, M) = \{1, \ldots, N\} \times \{1, \ldots, M\}.$$

Let $\lambda \in \mathrm{Sign}_M^{\leq N}$. By the *transposed signature* λ' we mean

$$\lambda_i' := \#\{j\colon \lambda_j \geq i\}, \qquad i = 1, \ldots, N.$$

Clearly, $\lambda' \in \mathrm{Sign}_N^{\leq M}$. See Figure 2 for an illustration.

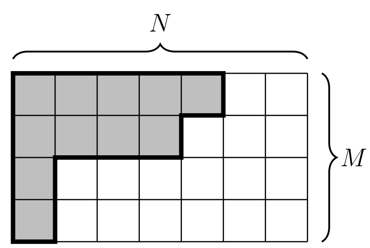


Figure 2. An example of a signature $\lambda = (5, 4, 1, 1) \in \mathrm{Sign}_4^{\leq 7}$. Its transposed signature is $\lambda' = (4, 2, 2, 2, 1, 0, 0) \in \mathrm{Sign}_7^{\leq 4}$.

We will also use multiplicative notation for signatures:

$$\lambda = 1^{m_1(\lambda)} 2^{m_2(\lambda)} \ldots, \qquad \text{where} \qquad m_i(\lambda) = \#\{j\colon \lambda_j = i\}.$$

Given two signatures $\mu \in \mathrm{Sign}_k$ and $\lambda \in \mathrm{Sign}_{k+1}$ we say that they *interlace* if

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots \geq \mu_k \geq \lambda_{k+1}. \tag{2.1}$$

We will use notation $\mu \prec \lambda$ for interlacing. Interlacing also extends to the case when λ and μ have the same number of elements by dropping the last inequality in (2.1). We will use the same notation $\mu \prec \lambda$ in this case. When λ and μ are such that $\mu' \prec \lambda'$, we say that they are *transposed interlacing*, and use the notation $\mu \prec' \lambda$.

2.2. **Directed path vertex models.** Symmetric functions introduced in this section are constructed through a vertex model formalism. That is, we define symmetric functions as partition functions (= sum of weights of allowed configurations) of ensembles of paths flowing through a planar lattice, where the *global weight* of each path configuration is the product of Boltzmann weights of local configurations around each vertex. We need two separate classes of ensembles: up-right and down-right.

Definition 2.1 (Up-right paths)**.** We consider up-right directed paths living in the half-quadrant $\{(i,j) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 1} \colon j \geq i\}$. We divide its vertices into three categories:

- left boundary vertices $\vdash$ at $(0,j)$, for $j \geq 1$;
- bulk vertices $+$ at (i,j), for $1 \leq i < j$;
- right corner vertices $\lrcorner$ at (i,i), for $i \geq 1$.

Paths we consider emanate from left boundary vertices and proceed in the up-right direction in the bulk of the lattice. Multiple paths are allowed to go along one horizontal or vertical edge. When a path meets the diagonal it gets reflected in the upward direction. The reason why we distinguish the nature of the vertices is that we will use different weighting systems for each of them. See Figure 3, left, for an illustration.

For a configuration of up-right paths, define for each $k \in \mathbb{Z}_{\geq 1}$ the signature $\lambda^k \in \mathrm{Sign}_k$ by

$$\lambda_i^k - \lambda_{i+1}^k = \#\{\text{paths occupying the edge } (i,k) \to (i,k+1)\},$$

where $i = 1, \ldots, k-1$. Let also λ_k^k will be the number of paths reflected at the right corner (k,k). In this way the up-right path ensemble is bijectively encoded by a sequence

$$\lambda^1 \subseteq \lambda^2 \subseteq \cdots, \qquad \lambda^k \in \mathrm{Sign}_k.$$

Here the relation $\lambda^k \subseteq \lambda^{k+1}$ means inclusion of the respective Young diagrams. For example, for the up-right path ensemble in Figure 3, left, we have $\lambda^2 = (1,0)$ and $\lambda^3 = (3,3,2)$.

Definition 2.2 (Down-right paths)**.** Down-right paths live inside the finite rectangle $\{0, \ldots, N\} \times \{1, \ldots, M\}$, where N, M are fixed positive integers. We divide its vertices into three categories:

- left boundary vertices $\vdash$ at $(0,j)$, for $1 \leq j \leq M$;
- bulk vertices $+$ at (i,j), for $1 \leq i < N$, $1 \leq j \leq M$;
- right boundary vertices $\dashv$ at (N,j), for $1 \leq j \leq M$.

Down-right directed paths we consider originate at left boundary vertices and terminate at the lower base of the rectangle. Once paths hit the right boundary they are automatically sent all the way down. See Figure 3, right, for an illustration.

To each configuration of down-right paths we can also associate a sequence of growing signatures

$$\lambda^1 \subseteq \lambda^2 \subseteq \cdots \subseteq \lambda^M, \qquad \lambda^k \in \mathrm{Sign}_N,$$

where

$$\lambda_i^k - \lambda_{i+1}^k = \#\{\text{paths occupying the edge } (i, M-k+1) \to (i, M-k)\},$$

with $\lambda_{N+1}^k = 0$, by agreement. For example, for the path ensemble in Figure 3, right, we have $\lambda^1 = (1,0,0,0,0)$, $\lambda^2 = (2,0,0,0,0)$, $\lambda^3 = (2,1,1,1,1)$, and $\lambda^4 = (3,2,2,1,1)$.

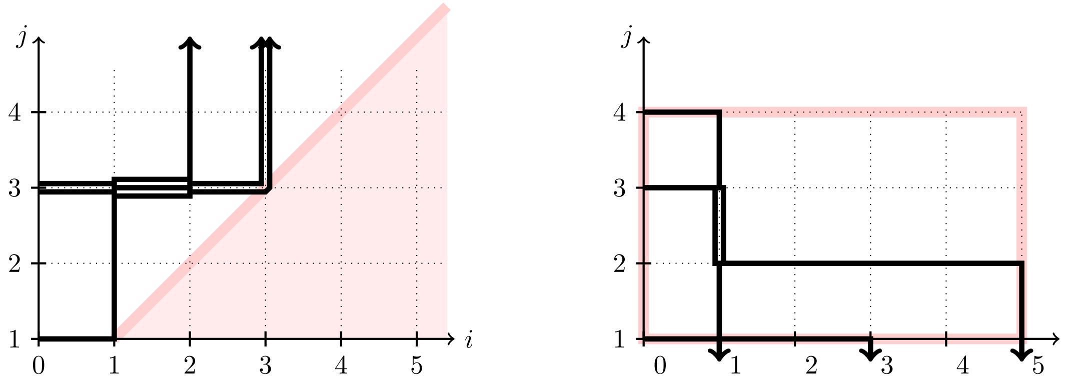


FIGURE 3. Left: Example of an up-right path ensemble. All paths must be above the main diagonal. Right: Example of a down-right path ensemble with $N = 5$ and $M = 4$. All paths must be inside the rectangle.

2.3. **Spin q-Whittaker polynomials.** The spin q-Whittaker polynomials are partition functions of up-right path ensembles. Assign the following weights to the left boundary, bulk, and right corner vertices (here and below we use the q-Pochhammer notation (1.5)):

$$W^{\vdash}_{x,s}(j) := x^j \, \frac{(-s/x;q)_j}{(q;q)_j}; \tag{2.2}$$

$$W^{+}_{x,s}(i_1,j_1;i_2,j_2) := \mathbf{1}_{i_1+j_1=i_2+j_2}\,\mathbf{1}_{i_1\ge j_2}\, x^{j_2}\, \frac{(-s/x;q)_{j_2}(-sx;q)_{i_1-j_2}(q;q)_{i_2}}{(q;q)_{j_2}(q;q)_{i_1-j_2}(s^2;q)_{i_2}}; \tag{2.3}$$

$$W^{\lrcorner}_{x,s}(j) := \frac{(q;q)_j}{(-s/x;q)_j}. \tag{2.4}$$

Here in (2.2) and (2.4), $j \in \mathbb{Z}_{\ge 0}$ denotes the number of paths going through the vertex, and in (2.3) the numbers $i_1, j_1, i_2, j_2 \in \mathbb{Z}_{\ge 0}$ denote, respectively, the numbers of entering vertical, entering horizontal, exiting vertical, and exiting horizontal paths to/from the vertex.

The weights (2.2)–(2.4) depend on the main *quantization parameter* q, on a *spectral parameter* x, a *spin parameter* s. While q, s are assumed fixed, the spectral parameter will depend on the vertical lattice coordinate.

Remark 2.3. One can readily check that the condition $i_1 \ge j_2$ in (2.3) implies that up-right path configurations with nonzero global weight are those associated with sequences of interlacing signatures $\lambda^1 \prec \lambda^2 \prec \ldots$. In particular, the configuration in Figure 3, left, has global weight zero.

Remark 2.4. When $s = 0$, the bulk and the corner weights (2.3)-(2.4) coincide. More precisely, we have $W^{\lrcorner}_{x,s}(j) = W^{+}_{x,s}(0,j;j,0) = (q;q)_j$.

Definition 2.5 (Spin q-Whittaker polynomials)**.** For given interlacing signatures $\mu \prec \lambda$ with $\mu \in \mathrm{Sign}_k$ and $\lambda \in \mathrm{Sign}_{k+1}$, the *skew spin q-Whittaker polynomial* in one variable is the weight of the unique path configuration between μ and λ at the k-th slice. It is given by

$$\mathbb{F}_{\lambda/\mu}(x) := x^{|\lambda|-|\mu|} \prod_{i=1}^{k} \frac{(-s/x;q)_{\lambda_i-\mu_i}(-sx;q)_{\mu_i-\lambda_{i+1}}(q;q)_{\lambda_i-\lambda_{i+1}}}{(q;q)_{\lambda_i-\mu_i}(q;q)_{\mu_i-\lambda_{i+i}}(s^2;q)_{\lambda_i-\lambda_{i+i}}}. \tag{2.5}$$

This is clearly a polynomial in x, even though the right corner weight (2.4) is not polynomial. We will often abbreviate the name "spin q-Whittaker" as sqW.

For $\mu \in \mathrm{Sign}_k$ and $\nu \in \mathrm{Sign}_{k+n}$, we also define n-variable polynomials in a standard way via *branching*:

$$\mathbb{F}_{\nu/\mu}(x_1, \ldots, x_n) = \sum_{\varkappa} \mathbb{F}_{\varkappa/\mu}(x_1, \ldots, x_{n-1})\, \mathbb{F}_{\nu/\varkappa}(x_n). \tag{2.6}$$

The polynomials $\mathbb{F}_{\nu/\mu}(x_1, \ldots, x_n)$ are partition functions of up-right path ensembles as in Figure 3, left, in a domain with the bottom and the top boundary conditions determined by $\mu \in \mathrm{Sign}_k$ and $\nu \in \mathrm{Sign}_{k+n}$, respectively.

We will use the shorthand notation $\mathbb{F}_\lambda(x_1, \ldots, x_n) \equiv \mathbb{F}_{\lambda/\varnothing}(x_1, \ldots, x_n)$, where $\lambda \in \mathrm{Sign}_n$.

Remark 2.6. It is important to notice that the number of variables in a sqW polynomial $\mathbb{F}_{\nu/\mu}$ is determined by the signatures ν, μ. If $\nu \in \mathrm{Sign}_{n+k}$ and $\mu \in \mathrm{Sign}_k$, then we can only evaluate $\mathbb{F}_{\nu/\mu}$ at n variables.

2.4. **Comparison with Borodin–Wheeler's spin q-Whittaker polynomials.** It is important to note that our version of the spin q-Whittaker polynomials is *different* from the original definition of Borodin and Wheeler [BW17]. Namely, the one-variable skew polynomials in [BW17] have the form

$$\mathbb{F}^{BW}_{\lambda/\mu}(x) = x^{|\lambda|-|\mu|} \prod_{i \ge 1} \frac{(-s/x;q)_{\lambda_i-\mu_i}(-sx;q)_{\mu_i-\lambda_{i+1}}(q;q)_{\lambda_i-\lambda_{i+1}}}{(q;q)_{\lambda_i-\mu_i}(q;q)_{\mu_i-\lambda_{i+i}}(s^2;q)_{\lambda_i-\lambda_{i+i}}}, \tag{2.7}$$

where $\mu \in \mathrm{Sign}_k$, $\lambda \in \mathrm{Sign}_{k+1}$, and the product over i extends to $i = k+1$ with the agreement that $\lambda_{k+2} = \mu_{k+1} = 0$. That is, our one-variable functions differ from (2.7) as

$$\mathbb{F}^{BW}_{\lambda/\mu}(x) = \frac{(-s/x;q)_{\lambda_{k+1}}}{(s^2;q)_{\lambda_{k+1}}}\, \mathbb{F}_{\lambda/\mu}(x). \tag{2.8}$$

The n-variable polynomials $\mathbb{F}^{BW}_{\nu/\mu}(x_1, \ldots, x_n)$ are defined from $\mathbb{F}^{BW}_{\lambda/\mu}(x)$ by branching as in (2.6). They admit a lattice path construction similarly to $\mathbb{F}_{\nu/\mu}$, but with the right corner weights $W^{\lrcorner}_{x,s}$ replaced by the bulk weights $W^{+}_{x,s}$.

The Borodin–Wheeler's spin q-Whittaker polynomials arise from our $\mathbb{F}_\lambda$ as a particular case:

Proposition 2.7. *For all $\lambda \in \mathrm{Sign}_n$ we have*

$$\mathbb{F}_\lambda(0, x_2, \ldots, x_n) = \mathbb{F}^{BW}_\lambda(x_2, \ldots, x_n). \tag{2.9}$$

Proof. The up-right paths that start at the left boundary at height 1 must immediately turn up at the right corner at $(1,1)$. If there are j such paths, their contribution to the global weight is $W^{\ulcorner}_{x_1,s}(j)W^{\lrcorner}_{x_1,s}(j) = x_1^j$. For $x_1 = 0$, this forces no paths to start at height 1. Next, due to the presence of $\mathbf{1}_{i_1 \ge j_2}$ in the bulk weight $W^{+}_{x,s}$, we see that paths started at height $2, \ldots, n$ cannot reach the diagonal with the special right corner weights. Therefore, the partition function for $\mathbb{F}_\lambda(x_1, x_2, \ldots, x_n)$ with $x_1 = 0$ involves only left boundary and bulk weights, and is thus the same as the partition function for $\mathbb{F}^{BW}_\lambda(x_2, \ldots, x_n)$. □

Note that $\mathbb{F}^{BW}_\lambda(x_2, \ldots, x_n)$ is well-defined for any λ, and vanishes if $\ell(\lambda)$, the number of nonzero parts in λ, exceeds $n-1$. If $\ell(\lambda) \le n-1$, then we can treat λ as an element of Sign_n with $\lambda_n = 0$, and then (2.9) holds. Moreover, one readily sees that both sides of (2.9) vanish if $\lambda_n > 0$. Therefore, any polynomial $\mathbb{F}^{BW}_\lambda$ can be obtained from our polynomial $\mathbb{F}_\lambda$ by specializing one of the variables to zero. (By symmetry, see Section 2.5 below, we can specialize to zero any variable, and not necessarily the first one.)

2.5. **Properties of the spin q-Whittaker polynomials.** The fact that the Borodin–Wheeler's sqW polynomials are symmetric in their variables follows from the Yang–Baxter equation which we reproduce in Appendix B as Proposition B.1. By looking at (2.8), it is not immediately clear why our version of the sqW polynomials should also be symmetric. We prove this next.

Proposition 2.8. *For any $\mu \in \mathrm{Sign}_k$, $\nu \in \mathrm{Sign}_{n+k}$ the polynomial $\mathbb{F}_{\nu/\mu}(x_1, \ldots, x_n)$ is symmetric with respect to permutations of its variables x_i.*

Proof. We use the Yang–Baxter equations of Propositions B.1 and B.2 and employ the standard "cross dragging" / commuting transfer matrices argument, cf. [Bor17, Theorem 3.6]. Using branching, it suffices to consider the two-variable case. The two-variable polynomial $\mathbb{F}_{\lambda/\mu}(x, y)$ is a partition function of up-right paths on two consecutive levels, with parameters x, y at the bottom and at the top, respectively, and boundary conditions determined by λ, μ.

First we use the new relation (B.3) that, as shown in Figure 13 (b), implies that swapping the spectral parameters $x \leftrightarrow y$ at the right corners makes a cross appear at their left. Then we sequentially move the cross to the left while swapping the spectral parameters using the bulk Yang–Baxter equation (B.2), as shown by Figure 13 (a). We proceed till the left boundary of the domain.

At the left boundary, we can swap the last two spectral parameters by noticing that

$$W^{\vdash}_{x,s}(j) = \frac{(s^2; q)_\infty}{(-sx; q)_\infty} W^{+}_{x,s}(\infty, l; \infty, j), \qquad \text{for any } l \in \mathbb{Z}_{\geq 0}. \tag{2.10}$$

This means that the left boundary weights $W^{\vdash}$ also satisfy the Yang–Baxter equation (B.2), and so we can take the cross out of the lattice. This completes the proof. □

Our sqW polynomials also satisfy an index shifting property which is the same as for the classical homogeneous Macdonald polynomials $P_\lambda(\cdot; q, t)$ [Mac95, VI(4.17)]:

Proposition 2.9. *For any signature $\lambda \in \mathrm{Sign}_N$ with $\lambda_N > 0$, we have*

$$\mathbb{F}_\lambda(x_1, \ldots, x_N) = x_1 \cdots x_N \, \mathbb{F}_{\lambda - 1^N}(x_1, \ldots, x_N), \qquad \lambda - 1^N = (\lambda_1 - 1, \ldots, \lambda_N - 1).$$

Proof. First, note that (2.5) implies that the one-variable skew polynomials satisfy the shifting property as

$$\mathbb{F}_{\nu/\mu}(x) = x \, \mathbb{F}_{(\nu - 1^{k+1})/(\mu - 1^k)}(x), \tag{2.11}$$

for any $\nu \in \mathrm{Sign}_{k+1}$ and $\mu \in \mathrm{Sign}_k$ with $\nu_{k+1} > 0$ (this also implies $\mu_k > 0$, since $\mu_k \geq \nu_{k+1}$). Next, we use the expansion

$$\mathbb{F}_\lambda(x_1, \ldots, x_N) = \sum_{\lambda^1 \prec \cdots \prec \lambda^{N-1} \prec \lambda} \mathbb{F}_{\lambda^1}(x_1) \, \mathbb{F}_{\lambda^2/\lambda^1}(x_2) \ldots \mathbb{F}_{\lambda^{N-1}/\lambda^{N-2}}(x_{N-1}) \, \mathbb{F}_{\lambda/\lambda^{N-1}}(x_N) \tag{2.12}$$

coming from iterating the branching rule, and apply the shifting property (2.11) to each of the terms to get the desired result. □

Remark 2.10. The polynomials $\mathbb{F}^{BW}_\lambda$ *do not* satisfy the index shifting property of Proposition 2.9, which can be seen from (2.8).

On the other hand, the polynomials $\mathbb{F}^{BW}_\lambda$ satisfy the stability property

$$\mathbb{F}^{BW}_\lambda(x_1, \ldots, x_{N-1}, -s) = \mathbb{F}^{BW}_\lambda(x_1, \ldots, x_{N-1}),$$

whereas the polynomials $\mathbb{F}_\lambda$ *do not.* More precisely, we have

$$\mathbb{F}_\lambda(x_1, \ldots, x_{N-1}, -s) = (-s)^{\lambda_N} \mathbb{F}_{\widetilde{\lambda}}(x_1, \ldots, x_{N-1}),$$

where $\widetilde{\lambda} = (\lambda_1 \geq \cdots \geq \lambda_{N-1})$ and this is easily proven since the branching coefficient (2.5) evaluates as $\mathbb{F}_{\lambda/\mu}(-s) = (-s)^{\lambda_N} \prod_{i=1}^{N-1} \mathbf{1}_{\lambda_i=\mu_i}$.

In the following proposition we use the coefficient

$$\mathsf{c}_\lambda = \prod_{i=1}^{N-1} \frac{(s^2;q)_{\lambda_i-\lambda_{i+1}}}{(q;q)_{\lambda_i-\lambda_{i+1}}}, \qquad \lambda \in \mathrm{Sign}_N.$$

Proposition 2.11. *Let $|sx_i| < 1$ for $i = 1, \dots, N$. Then we have*

$$\sum_{\substack{\lambda\in\mathrm{Sign}_N\\ \lambda_N=0}} \mathsf{c}_\lambda (-s)^{|\lambda|} \mathbb{F}_\lambda(x_1,\dots,x_N) = \frac{((-s)^N x_1 \dots x_N; q)_\infty (s^2;q)_\infty^{N-1}}{(-sx_1;q)_\infty \dots (-sx_N;q)_\infty}. \tag{2.13}$$

Proof. We will use the identity

$$\sum_{k=0}^{n} a^k \frac{(b;q)_k}{(q;q)_k} \frac{(a;q)_{n-k}}{(q;q)_{n-k}} = \frac{(ab;q)_n}{(q;q)_n}, \tag{2.14}$$

that follows from the q-Chu–Vandermonde identity (e.g., see [GR04, (II.6)]). Expand the left-hand side of (2.13) as:

$$\begin{aligned}\sum_{\substack{\lambda=\lambda^N\in\mathrm{Sign}_N\\ \lambda^N_N=0}} \sum_{\lambda^{N-1}\in\mathrm{Sign}_{N-1}} &(-sx_N)^{\lambda_1^N-\lambda_1^{N-1}} \frac{(-s/x_N;q)_{\lambda_1^N-\lambda_1^{N-1}}}{(q;q)_{\lambda_1^N-\lambda_1^{N-1}}} \\ &\times \prod_{k=2}^{N-1} \left((-sx_N)^{\lambda_k^N-\lambda_k^{N-1}} \frac{(-sx_N;q)_{\lambda_{k-1}^{N-1}-\lambda_k^N}}{(q;q)_{\lambda_{k-1}^{N-1}-\lambda_k^N}} \frac{(-s/x_N;q)_{\lambda_k^N-\lambda_k^{N-1}}}{(q;q)_{\lambda_k^N-\lambda_k^{N-1}}} \right) \\ &\times \frac{(-sx_N;q)_{\lambda_{N-1}^{N-1}}}{(q;q)_{\lambda_{N-1}^{N-1}}} \times (-s)^{|\lambda^{N-1}|} \mathbb{F}_{\lambda^{N-1}}(x_1,\dots,x_{N-1}).\end{aligned}$$

Summing over λ_1^N by means of the q-binomial theorem gives us the factor $(s^2;q)_\infty/(-sx_N;q)_\infty$. We then sum sequentially over indices $\lambda_2^N, \dots, \lambda_{N-1}^N$ and using (2.14) we are left with

$$\frac{(s^2;q)_\infty}{(-sx_N;q)_\infty} \sum_{\lambda^{N-1}\in\mathrm{Sign}_{N-1}} \frac{(-sx_N;q)_{\lambda_{N-1}^{N-1}}}{(q;q)_{\lambda_{N-1}^{N-1}}} \mathsf{c}_{\lambda^{N-1}} (-s)^{|\lambda^{N-1}|} \mathbb{F}_{\lambda^{N-1}}(x_1,\dots,x_{N-1}),$$

where $\mathsf{c}_{\lambda^{N-1}}$ is the result of applying (2.14). Repeating the same procedure we can reduce the previous expression to

$$\begin{aligned}&\frac{(s^2;q)_\infty^2}{(-sx_N;q)_\infty(-sx_{N-1};q)_\infty} \\ &\qquad \times \sum_{\lambda^{N-2}\in\mathrm{Sign}_{N-2}} \frac{((-s)^2 x_{N-1}x_N;q)_{\lambda_{N-2}^{N-2}}}{(q;q)_{\lambda_{N-2}^{N-2}}} \mathsf{c}_{\lambda^{N-2}} (-s)^{|\lambda^{N-2}|} \mathbb{F}_{\lambda^{N-2}}(x_1,\dots,x_{N-2}).\end{aligned}$$

Here the reason for the appearance of the product $(-sx_N)(-sx_{N-1})$ in the q-Pochhammer symbol is again (2.14), where we also used that λ_{N-1}^{N-1} is not necessarily zero (in contrast with the first summation over λ_N^N). Continuing inductively, we exhaust all the summations down to the bottom one over λ_1^1, from which we recover the factor $((-sx_1)\cdots(-sx_N);q)_\infty/(-sx_1;q)_\infty$. This completes the proof. ☐

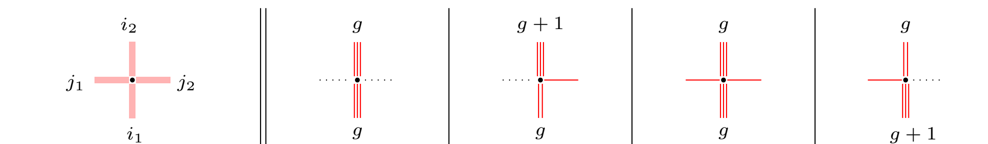
2.6. **Dual spin Hall–Littlewood rational functions.** Along with the sqW polynomials $\mathbb{F}_\lambda$ we will define two families of *dual* functions, with which the $\mathbb{F}_\lambda$'s satisfy Cauchy-type summation identities. The first are the dual spin Hall–Littlewood rational functions. For them we use down-right path ensembles as in Figure 3, right, and define the weights by

$$w_v^{*,\vdash}(j) := v^j; \tag{2.15}$$

$$w_{v,s}^{*,+} := \text{see Figure 4}; \tag{2.16}$$

$$W^{*,\dashv}(i_1,j_1;i_2) := \mathbf{1}_{i_1=j_1+i_2}. \tag{2.17}$$

These weights depend on the main parameters s,q, and on the spectral parameter v. It is easy to see that with this choice of vertex weights the only allowed configurations of down-right paths in the rectangular grid $\{0,\dots,N\}\times\{0,\dots,M\}$ are those associated with sequences of transposed interlacing signatures $\lambda^1 \prec' \cdots \prec' \lambda^M$.



$w_{v,s}^{*,+}(i_1,j_1;i_2,j_2)$	$\frac{1-svq^g}{1-sv}$	$\frac{v(1-q^{g+1})}{1-sv}$	$\frac{v-sq^g}{1-sv}$	$\frac{1-s^2q^g}{1-sv}$

FIGURE 4. Bulk vertex weights used in the construction of the dual spin Hall–Littlewood functions. Vertex configurations not listed are assigned weight zero. Note that the weights vanish unless $i_1+j_2=j_1+i_2$.

Definition 2.12. For given $\mu,\lambda\in\mathrm{Sign}_N^{\le M}$ with $\mu\prec'\lambda$, the *skew dual spin Hall–Littlewood function* in one variable $\mathsf{F}^*_{\lambda'/\mu'}(v)$ is the weight of the unique down-right path configuration between μ and λ (at the top and at the bottom, respectively) at a single row of vertices, where we take weights (2.15)–(2.17) with the spectral parameter v. More explicitly, we have

$$\mathsf{F}^*_{\lambda'/\mu'}(v) := \sum_{j_0,\dots,j_{N-1}\in\{0,1\}} v^{j_0} W^{*,\dashv}(l'_N, j_{N-1}; m'_N) \prod_{r=1}^{N-1} w^{*,+}_{v,s}(l'_r, j_{r-1}; m'_r, j_r),$$

where $\lambda' = 1^{l'_1}\cdots N^{l'_N}$ and $\mu' = 1^{m'_1}\cdots N^{m'_N}$ belong to $\mathrm{Sign}_M^{\le N}$.

Multi-variable extensions $\mathsf{F}^*_{\nu/\varkappa}(v_1,\dots,v_k)$, where $\nu,\varkappa\in\mathrm{Sign}_M^{\le N}$ and k is arbitrary, are defined using the branching rule in the same way as in (2.6). The single-index (non-skew) functions are defined by $\mathsf{F}^*_\nu(v_1,\dots,v_k)=\mathsf{F}^*_{\nu/0^M}(v_1,\dots,v_k)$, where $\nu\in\mathrm{Sign}_M^{\le N}$, and 0^M is the signature from $\mathrm{Sign}_M^{\le N}$ with all parts equal to zero.

Remark 2.13. The sHL functions $\mathsf{F}^*_{\nu/\varkappa}(v_1,\dots,v_k)$, are defined for any number of variables k, regardless of the signatures ν and $\varkappa$. This should be contrasted with the sqW polynomials, cf. Remark 2.6.

The functions $\mathsf{F}^*_{\nu/\varkappa}$ are *stable* in the sense that

$$\mathsf{F}^*_{\nu/\varkappa}(v_1,\dots,v_k,0) = \mathsf{F}^*_{\nu/\varkappa}(v_1,\dots,v_k).$$

Indeed, this readily follows from the vertex weights (2.15)–(2.17).

The version of the spin Hall–Littlewood functions of Definition 2.12 is essentially a particular case of the inhomogeneous spin Hall–Littlewood functions from [BP18], where the N-th spin parameter s_N is set to zero. This allows to derive a lot of their properties by specializing the corresponding results of [BP18]. As the functions $\mathsf{F}^*_{\nu/\varkappa}$ from Definition 2.12 are central to our discussion and we do not use other versions in the present paper, we simply refer to the $\mathsf{F}^*_{\nu/\varkappa}$'s as (dual) spin Hall–Littlewood functions. For convenience, we will omit the dependence on N in their notation. We will often abbreviate the name "spin Hall–Littlewood" as *sHL*.

The sHL functions F^*_λ admit an explicit symmetrization formula:

Proposition 2.14. *Let $\lambda \in \mathrm{Sign}^{\le N}_M$, then for all $k \ge M$ we have*

$$\mathsf{F}^*_\lambda(v_1,\ldots,v_k) = \mathcal{C}(\lambda) \sum_{\sigma\in\mathfrak{S}_k} \sigma\left\{ \prod_{1\le i<j\le k} \frac{v_i - qv_j}{v_i - v_j} \prod_{i=1}^{\ell(\lambda)} v_i \left(\frac{v_i - s}{1 - sv_i}\right)^{\lambda_i - 1} \left(\frac{1}{1-sv_i}\right)^{\mathbf{1}_{\lambda_i<N}} \right\}, \quad (2.18)$$

where the symmetric group $\mathfrak{S}_k$ acts on the variables v_i but not on elements of the signature λ_i, and the constant prefactor has the form

$$\mathcal{C}(\lambda) = \frac{(1-q)^k}{(q;q)_{k-\ell(\lambda)}} \prod_{i=1}^{N} \frac{(s^2;q)_{m_i(\lambda)}}{(q;q)_{m_i(\lambda)}}. \quad (2.19)$$

Proof. This formula follows from [BP18, Theorem 4.14, part 1] via several specializations. The latter result is a symmetrization formula for a more general vertex model partition function $\mathsf{F}^{\text{non-stab, non-dual}}_\lambda$ which involves inhomogeneity parameters ξ_j and s_j depending on the horizontal lattice coordinate $j \in \mathbb{Z}_{\ge 0}$. Let us describe the necessary specializations. In the first step, we set all the parameters ξ_j to 1.

For the second step, we take a *stable limit* described in, e.g., [BMP19, Section 3.3] (the second of those limits). Namely, put $s_0 = 0$, then by [BMP19, (3.7)] we have

$$\mathsf{F}^{\text{stab, non-dual}}_\lambda(v_1,\ldots,v_k) = \frac{1}{(q;q)_{k-\ell(\lambda)}} \mathsf{F}^{\text{non-stab, non-dual}}_{\lambda\cup 0^{k-\ell(\lambda)}}(v_1,\ldots,v_k)\big|_{s_0=0}. \quad (2.20)$$

Here $\lambda\cup 0^{k-\ell(\lambda)} \in \mathrm{Sign}_k$ is obtained by appending the partition $\lambda = (\lambda_1,\ldots,\lambda_{\ell(\lambda)})$ by $k-\ell(\lambda)$ zeroes. Then (2.20) is given by the symmetrization formula [BP18, (4.23)]

$$\frac{(1-q)^k}{(q;q)_{k-\ell(\lambda)}} \sum_{\sigma\in\mathfrak{S}_k} \sigma\left\{ \prod_{1\le i<j\le k} \frac{v_i - qv_j}{v_i - v_j} \prod_{j=1}^{\ell(\lambda)} \frac{v_j}{1 - s_{\lambda_j} v_j} \prod_{i=1}^{\lambda_j - 1} \frac{v_i - s_j}{1 - s_j v_i} \right\},$$

where we used the fact that $s_0 = 0$ to pass from the product over $1 \le j \le k$ to $1 \le j \le \ell(\lambda)$.

For the third step, we use the fact that the weights $w^{*,\mathbf{+}}_{v,s}$ we use differ from the $w_{u,s}$'s for $\mathsf{F}^{\text{non-stab, non-dual}}_\lambda$ by a conjugation factor $(s^2;q)_i/(q;q)_i$ [BP18, (2.2)], which brings the product over i in $\mathcal{C}(\lambda)$ (2.19) involved in our function F^*_λ.

For the fourth step, we add the right boundary at N to our vertex model by setting $s_N = 0$ (recall that $\lambda_1 \le N$). This turns the factor $\frac{1}{1-s_{\lambda_j}v_j}$ into $\left(\frac{1}{1-s_{\lambda_j}v_j}\right)^{\mathbf{1}_{\lambda_j<N}}$.

Finally, we set $s_1 = \ldots = s_{N-1} = s$ to recover the homogeneous parameter s, and arrive at the desired symmetrization formula. □

The Yang–Baxter equations of Propositions B.3 and B.4 translate into Cauchy identities for the functions $\mathbb{F}$ and F^*.

Proposition 2.15. *Fix $M \geq 1$. For $N > 0$, let $\mu \in \mathrm{Sign}_N^{\leq M}$ and $\lambda \in \mathrm{Sign}_{N+1}^{\leq M}$. Then, we have*

$$\sum_{\nu \in \mathrm{Sign}_{N+1}^{\leq M}} \mathbb{F}_{\nu/\mu}(x)\, \mathsf{F}^*_{\nu'/\lambda'}(v) = \frac{1+vx}{1-sv} \sum_{\varkappa \in \mathrm{Sign}_N^{\leq M}} \mathbb{F}_{\lambda/\varkappa}(x)\, \mathsf{F}^*_{\mu'/\varkappa'}(v). \tag{2.21}$$

For $N = 0$, we have

$$\sum_{\nu \in \mathrm{Sign}_1^{\leq M}} \mathbb{F}_{\nu}(x)\, \mathsf{F}^*_{\nu'/\lambda'}(v) = (1+vx)\, \mathbb{F}_{\lambda}(x). \tag{2.22}$$

Note that all the sums in this proposition are over finite sets of signatures, so there are no convergence issues.

Proof of Proposition 2.15. The proof of (2.21) is similar to that of Proposition 2.8 as it also uses a “cross dragging” argument. The summation in the left-hand side of (2.21) is the partition function of path configurations across two rows of vertices glued together:

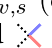
- the lower row has weights $W^{\vdash}_{x,s}, W^{+}_{x,s}, W^{\lrcorner}$ and boundary condition μ at the bottom and ν at the top;
- the upper one has weights $w^{*,\vdash}_{v,s}, w^{*,+}_{v,s}, W^{*,\urcorner}$, and boundary condition ν at the bottom and λ at the top.

Recall that the encoding of arrow configurations by signatures is described in detail in Section 2.2.

The Yang–Baxter equation (B.6) implies that the action of swapping weights at the rightmost pair of columns, makes a cross weight appear at their left, as shown in Figure 15 (b). We then push the cross to the left one vertical step at a time, each time swapping the vertex weights and using the Yang–Baxter equation (B.5) as in Figure 15 (a). This procedure sequentially turns the left-hand side of (2.21) into the right-hand side.

At the final step, we push the cross out of the lattice at the leftmost site. Using (2.10) and

$$w^{*,\vdash}_{v}(j) = (1-sv)\, w^*_{v,s}(\infty, l; \infty, j), \qquad \text{for } l = 0, 1,$$

we obtain the combined contribution of the cross vertex weights $\mathcal{R}_{x,v,s}$ (defined in Figure 14 in the Appendix) corresponding to the two cross configurations  and  . Their sum gives the factor $(1+vx)/(1-sv)$ in the right-hand side of (2.21), as desired.

The second identity (2.22) can be verified by simply using definition of functions. □

Combining the skew Cauchy identities of Proposition 2.15, we come to the following corollary for several variables:

Corollary 2.16. *For any positive integers N, M, m we have*

$$\sum_{\lambda \in \mathrm{Sign}_M^{\leq N}} \mathbb{F}_{\lambda}(x_1, \ldots, x_N)\, \mathsf{F}^*_{\lambda'}(v_1, \ldots, v_m) = \prod_{j=1}^{m} \left(\frac{1}{1-sv_j}\right)^{N-1} \prod_{i=1}^{N} \prod_{j=1}^{m} (1+v_j x_i). \tag{2.23}$$

Proof. We use the branching expansion of functions $\mathbb{F}_{\lambda}, \mathsf{F}^*_{\lambda'}$ and then apply the single-variable skew Cauchy identities (2.21) and (2.22). □

Proposition 2.17. *Let $0 < q < 1$ and $-1 < s < 0$. For any $\lambda, \mu \in \mathrm{Sign}_M^{\leq N}$ and $k \geq M$, we have*

$$\frac{1}{k!} \oint_{\gamma} \frac{dz_1}{2\pi \mathrm{i} z_1} \cdots \oint_{\gamma} \frac{dz_k}{2\pi \mathrm{i} z_k} \prod_{1 \leq i \neq j \leq k} \frac{z_i - z_j}{z_i - q z_j}\, \mathsf{F}^*_{\lambda}(z_1, \ldots, z_k) \mathsf{F}^*_{\mu}(1/z_1, \ldots, 1/z_k) = \mathcal{C}(\lambda) \mathbf{1}_{\lambda=\mu}, \tag{2.24}$$

where γ is a positively oriented contour encircling 0, $q^j s$ for all $j \geq 0$, and the contour $q\gamma$, but not the point s^{-1}.

Proof. This follows from [BP18, Corollary 7.5] after specializing the inhomogeneous spin Hall–Littlewood functions $\mathsf{F}_\lambda^{\text{non-stab, non-dual}}$ as described in the proof of Proposition 2.14. $\square$

2.7. **Dual spin q-Whittaker polynomials.** Let us also define the dual versions of the sqW weights. These dual weights correspond to down-right lattice paths, and are given by (we use the notation (2.2)–(2.3)):

$$W^{*,\vdash}_{y,s}(j) := W^{\vdash}_{y,s}(j); \tag{2.25}$$

$$W^{*,+}_{y,s}(i_1,j_1;i_2,j_2) := \frac{(s^2;q)_{i_1}}{(q;q)_{i_1}}\frac{(q;q)_{i_2}}{(s^2;q)_{i_2}}\, W^{+}_{y,s}(i_2,j_1;i_1,j_2). \tag{2.26}$$

We will also use the right boundary weights $W^{*,\dashv}(i_1,j_1;i_2)$ as in (2.17).

This choice of vertex weights implies that nonzero global weights are assigned to configurations of down-right paths in the grid $\{0,\ldots,N\}\times\{0,\ldots,k\}$ which are encoded by sequences of interlacing signatures $\lambda^1 \prec \cdots \prec \lambda^k$. (Compare this with the transposed interlacing property for the sHL functions.)

Definition 2.18. For given interlacing signatures $\lambda,\mu\in\mathrm{Sign}_N$, the *skew dual spin q-Whittaker polynomial* in one variable $\mathbb{F}^*_{\lambda/\mu}(y)$ is the weight of the unique down-right path configuration between μ and λ at a single row of vertices, with the weights (2.26), (2.25) and (2.17). Recall that the encoding of arrow configurations by signatures is described in Section 2.2.

An explicit expression for the skew dual sqW polynomial is

$$\mathbb{F}^*_{\lambda/\mu}(y) := y^{|\lambda|-|\mu|}\,\frac{(-s/y;q)_{\lambda_N-\mu_N}}{(q;q)_{\lambda_N-\mu_N}}\prod_{i=1}^{N-1}\frac{(-s/y;q)_{\lambda_i-\mu_i}(-sy;q)_{\mu_i-\lambda_{i+1}}(q;q)_{\mu_i-\mu_{i+1}}}{(q;q)_{\lambda_i-\mu_i}(q;q)_{\mu_i-\lambda_{i+i}}(s^2;q)_{\mu_i-\mu_{i+i}}}. \tag{2.27}$$

Observe that $\mathbb{F}^*_{\lambda/\mu}(y)$ is a polynomial in y.

Multi-variable extensions $\mathbb{F}^*_{\lambda/\mu}(y_1,\ldots,y_k)$, where $\lambda,\mu\in\mathrm{Sign}_N$ are arbitrary, are defined via branching in the same say as in (2.6). The non-skew functions are $\mathbb{F}^*_\nu \equiv \mathbb{F}^*_{\nu/0^N}$, where $\nu\in\mathrm{Sign}_N$, and $0^N\in\mathrm{Sign}_N$ is the signature with all parts equal to zero.

Remark 2.19. Like the dual sHL functions (cf. Remark 2.13) and unlike the usual sqW polynomials (cf. Remark 2.6), the dual sqW polynomials $\mathbb{F}^*_{\lambda/\mu}(y_1,\ldots,y_k)$ make sense for any number of variables k, regardless of the signatures λ,μ.

Proposition 2.20. *The polynomials $\mathbb{F}^*_{\lambda/\mu}(y_1,\ldots,y_k)$ are symmetric.*

Proof. This follows from the Yang–Baxter equation (B.2) and the sum-to-one property of the R-matrix R given by (B.1). It suffices to consider swapping two variables. We apply the usual "cross-dragging" argument to exchange spectral parameters of two consecutive rows of vertices. Similarly to the proof of Proposition 2.8, identity (B.2) suffices to swap spectral parameters from the leftmost column up until the rightmost one. Since the right boundary weights $W^{*,\dashv}$ differ from the bulk weights $W^{*,+}$, we have to prove that we can drag the cross one more step to the right. We have using the definition of $W^{*,\dashv}$ that the partition function near the right wall with the cross vertex is equal to

$$\sum_{k_1,k_2,k_3} R_{x,y}(i_1,i_2;k_1,k_2)W^{*,\dashv}(k_3,k_2;j_3)W^{*,\dashv}(i_3,k_1;k_3) = \sum_k R_{x,y}(i_1,i_2;i_1+i_2-j_3-k,k-j_3).$$

(We used the arrow preservation property $i_1+i_2+j_3=i_3$.) The right-hand side is equal to one. Indeed, this sum-to-one property readily follows from the q-Chu–Vandermonde identity. On the other hand, without the cross vertex, the partition function near the right wall is equal to $\sum_k W^{*,\urcorner}(i_3,i_2;k)W^{*,\urcorner}(k,i_1;j_3)$. This is also equal to 1, because only the summand with $k=i_1+j_3$ is nonzero. This completes the proof. □

We finish this subsection by describing Cauchy identities for our two sqW families $\mathbb{F},\mathbb{F}^*$.

Proposition 2.21. *For $N>0$, let $\mu\in\mathrm{Sign}_N$ and $\lambda\in\mathrm{Sign}_{N+1}$. Then, for $|xy|<1$, we have*

$$\sum_{\nu\in\mathrm{Sign}_{N+1}}\mathbb{F}_{\nu/\mu}(x)\,\mathbb{F}^*_{\nu/\lambda}(y)=\frac{(-sx;q)_\infty(-sy;q)_\infty}{(s^2;q)_\infty(xy;q)_\infty}\sum_{\varkappa\in\mathrm{Sign}_N}\mathbb{F}_{\lambda/\varkappa}(x)\,\mathbb{F}^*_{\mu/\varkappa}(y). \tag{2.28}$$

For $N=0$, we have

$$\sum_{\nu\in\mathrm{Sign}_{N+1}}\mathbb{F}_{\nu}(x)\,\mathbb{F}^*_{\nu/\lambda}(y)=\frac{(-sx;q)_\infty}{(xy;q)_\infty}\,\mathbb{F}_\lambda(x). \tag{2.29}$$

Proof. For $N>0$ this is proven using the same method explained in Proposition 2.15 with the help of identity (B.9) when extracting the cross vertex weight from the rightmost column. For $N=0$ the statement is simply the q-binomial theorem. □

Corollary 2.22. *Let $|x_iy_j|<1$ for all $i=1,\dots,N$, $j=1,\dots,k$. Then, we have*

$$\sum_{\lambda\in\mathrm{Sign}_N}\mathbb{F}_\lambda(x_1,\dots,x_N)\,\mathbb{F}^*_\lambda(y_1,\dots,y_k)=\prod_{j=1}^{k}\left(\frac{(-sy_j;q)_\infty}{(s^2;q)_\infty}\right)^{N-1}\prod_{i=1}^{N}\prod_{j=1}^{k}\frac{(-sx_i;q)_\infty}{(x_iy_j;q)_\infty}. \tag{2.30}$$

2.8. **Pieri rules.** Pieri type rules for the Borodin–Wheeler spin q-Whittaker polynomials $\mathbb{F}^{BW}_\lambda$ are given in [BW17]. These are analogs of the classical Pieri type rules for Macdonald polynomials. The Pieri type rules follow from skew Cauchy identities, and here we present these rules for our version of the spin q-Whittaker polynomials.

Proposition 2.23. *Let $|x_iy|<1$ for all $i=1,\dots,N$. Then we have*

$$\sum_{\lambda\in\mathrm{Sign}_N}\mathbb{F}_\lambda(x_1,\dots x_N)\,\mathbb{F}^*_{\lambda/\mu}(y)=\left(\sum_{i\ge0}y^i\frac{(-s/y;q)_i}{(q;q)_i}\,\mathbb{F}_{(i)}(x_1,\dots,x_N)\right)\mathbb{F}_\mu(x_1,\dots,x_N).$$

Proof. By the skew Cauchy identities of Proposition 2.21, we can write

$$\sum_{\lambda\in\mathrm{Sign}_N}\mathbb{F}_\lambda(x_1,\dots x_N)\,\mathbb{F}^*_{\lambda/\mu}(y)=\left(\frac{(-sy;q)_\infty}{(s^2;q)_\infty}\right)^{N-1}\prod_{i=1}^{N}\frac{(-sx_i;q)_\infty}{(x_iy;q)_\infty}\,\mathbb{F}_\mu(x_1,\dots,x_N), \tag{2.31}$$

and the claim follows by expanding $\left(\frac{(-sy;q)_\infty}{(s^2;q)_\infty}\right)^{N-1}\prod_{i=1}^{N}\frac{(-sx_i;q)_\infty}{(x_iy;q)_\infty}$ using (2.30). □

Proposition 2.24. *We have*

$$\sum_{\lambda}\mathbb{F}_\lambda(x_1,\dots x_N)\,\mathsf{F}^*_{\lambda'/\mu'}(v)=\left(\sum_{i=0}^{N}\mathsf{F}^*_{(i)}(v)\,\mathbb{F}_{1^i}(x_1,\dots x_N)\right)\mathbb{F}_\mu(x_1,\dots,x_N)$$

Proof. By the skew Cauchy identities of Proposition 2.15, we have

$$\sum_{\lambda\in\mathrm{Sign}_N}\mathbb{F}_\lambda(x_1,\dots x_N)\,\mathsf{F}^*_{\lambda/\mu}(v)=\left(\frac{1}{1-sv}\right)^{N-1}\prod_{i=i}^{N}(1+x_iv)\,\mathbb{F}_\mu(x_1,\dots,x_N), \tag{2.32}$$

and the claim follows by expanding $\left(\frac{1}{1-sv}\right)^{N-1}\prod_{i=i}^{N}(1+x_iv)$ using (2.23). □

Pieri type rules of Propositions 2.23 and 2.24 are eigenrelations on the spin q-Whittaker polynomials in the *label* variable. Indeed, define operators $\mathfrak{H}^{\mathrm{sqW}}, \mathfrak{H}^{\mathrm{sHL}}$ as

$$(\mathfrak{H}^{\mathrm{sqW}} f)(\mu) = \sum_{\lambda} f(\lambda)\, \mathbb{F}^*_{\lambda/\mu}(y), \qquad (\mathfrak{H}^{\mathrm{sHL}} f)(\mu) = \sum_{\lambda} f(\lambda)\, \mathsf{F}^*_{\lambda'/\mu'}(v). \tag{2.33}$$

Then these operators act diagonally on spin q-Whittaker functions $f(\lambda) = \mathbb{F}_\lambda(x_1, \ldots, x_N)$, with respective eigenvalues

$$\sum_{i \geq 0} y^i \frac{(-s/y;q)_i}{(q;q)_i}\, \mathbb{F}_{(i)}(x_1, \ldots, x_N) \qquad \text{and} \qquad \sum_{i=0}^{N} \mathsf{F}^*_{(i)}(v)\, \mathbb{F}_{1^i}(x_1, \ldots x_N).$$

3. Difference operators

Here we show that the sHL and sqW functions satisfy certain eigenrelations under operators acting the *spectral parameters* (as opposed to labels as in Section 2.8). These operators are s-deformations of the ($q = 0$ or $t = 0$) Macdonald difference operators. Half of these eigenrelations essentially appears in [BMP19], but here we obtain eigenrelations in a form which is more symmetric with respect to q, t.

We will denote the "quantization" parameter by q throughout this section, except for Section 3.1 where it will be denoted by t instead of q. This is done for consistency with classical literature (e.g., [Mac95]), where Hall–Littlewood functions (obtained from sHL functions by setting $s = 0$) are the $q = 0$ degenerations of the Macdonald polynomials $P_\lambda(\cdot; q, t)$.

Throughout the entire section we make use of the shift operator

$$T_{q,z_i} f(z_1, \ldots, z_M) = f(z_1, \ldots, z_{i-1}, q z_i, z_{i+1}, \ldots, z_M).$$

3.1. **Eigenrelations for the spin Hall–Littlewood functions.** We begin by essentially repeating the definition of a family of eigenoperators for the spin Hall–Littlewood polynomials from [BMP19].

Definition 3.1. For $r \in \{1, \ldots, M\}$, the r-th Hall–Littlewood operator is given by

$$\overline{\mathfrak{D}}^*_r \coloneqq \sum_{\substack{I \in \{1,\ldots,M\} \\ |I| = r}} \prod_{\substack{i \in I \\ k \in \{1,\ldots,M\} \setminus I}} \frac{v_k - t v_i}{v_k - v_i} \prod_{i \in I} T_{0, v_i}.$$

This is the $q = 0$ specialization of the r-th Macdonald difference operator [Mac95, Ch. VI.3]. The operators $\overline{\mathfrak{D}}^*_r$ act diagonally on the Hall–Littlewood symmetric polynomials.

It was discovered in [BMP19] that the (stable) spin Hall–Littlewood functions (first introduced in [GdGW17]), much like the classical Hall–Littlewood polynomials [Mac95, Ch. III], are eigenfunctions of the difference operators $\overline{\mathfrak{D}}^*_r$. The same result holds for our dual sHL functions F^*_ν, and it is given in the next theorem.

Theorem 3.2. *For any $\lambda \in \mathrm{Sign}_M$, we have*

$$\overline{\mathfrak{D}}^*_r \mathsf{F}^*_\lambda = e_r(1, t, \ldots, t^{M-\ell(\lambda)-1})\, \mathsf{F}^*_\lambda. \tag{3.1}$$

Here $e_r(x_1, \ldots, z_n) = \sum_{1 \leq i_1 < \cdots i_r \leq n} z_{i_1} \cdots z_{i_r}$ is the r-th elementary symmetric polynomial.

Proof. The proof is analogous to that of [BMP19, Theorem 8.2]: we get (3.1) by directly evaluating the action of $\overline{\mathfrak{D}}_r^*$ on the symmetrization formula (2.18) (with q replaced by t). We do not repeat the argument here. □

The operator we introduce next depends on the number of variables M and on an additional positive integer N. Moreover, this operator acts only on a certain subspace of rational functions. Namely, let $\mathbf{V}(M)$ be the space of symmetric rational functions in M variables $v_1, \ldots, v_M$ of degree ≤ 1 in each variable. That is, its elements are functions $f(v_1, \ldots, v_M) = a(v_1, \ldots, v_M)/b(v_1, \ldots, v_M)$, where a and b are polynomials such that $\deg_{v_i}(a) - \deg_{v_i}(b) \le 1$ for all $i = 1, \ldots, M$. One readily sees that $\mathbf{V}(M)$ is a linear space. The dual sHL functions $\mathsf{F}^*_\nu(v_1, \ldots, v_M)$ belong to $\mathbf{V}(M)$, see (2.18).

Definition 3.3. For positive integers M, N define the *dual s-deformed Macdonald operator* by

$$\mathfrak{D}^*_{1,N} \coloneqq \sum_{j=1}^{M} \prod_{\substack{l=1\\ l\neq j}}^{M} \frac{v_j - t v_l}{v_j - v_l}\, \mathfrak{C}_{j,N}, \tag{3.2}$$

where

$$\mathfrak{C}_{j,N} \coloneqq v_j \left(\frac{v_j - s}{1 - s v_j}\right)^{N-1} (-s)^{N-1} \lim_{\varepsilon \to 0} \varepsilon\, T_{\varepsilon^{-1}, v_j}.$$

The limit $\lim_{\varepsilon\to 0} \varepsilon\, T_{\varepsilon^{-1}, v_j}$ is well-defined on $\mathbf{V}(M)$, so $\mathfrak{D}^*_{1,N}$ acts in the space $\mathbf{V}(M)$.

Theorem 3.4. *For any boxed signature $\lambda \subseteq \mathrm{Box}(N, M)$ (recall that this is $\mathrm{Sign}_M^{\le N}$), we have*

$$\mathfrak{D}^*_{1,N} \mathsf{F}^*_\lambda = e_1(1, t, \ldots, t^{\lambda'_N - 1})\, \mathsf{F}^*_\lambda, \tag{3.3}$$

where λ' is the transposed signature. In particular, $\lambda'_N = \#\{i \colon \lambda_i = N\}$.

Proof. We make use of the symmetrization formula (2.18) (recall that we have replaced the parameter q by t throughout this subsection). We use the notation

$$A = \prod_{1 \le l < r \le M} \frac{v_l - t v_r}{v_l - v_r} \qquad \text{and} \qquad B = \prod_{r=1}^{\ell(\lambda)} v_r \left(\frac{v_r - s}{1 - s v_r}\right)^{\lambda_r - 1} \left(\frac{1}{1 - s v_r}\right)^{\mathbf{1}_{\lambda_r < N}},$$

so that $\mathsf{F}^*_\lambda = \mathcal{C}(\lambda) \sum_{\sigma \in \mathfrak{S}_M} \sigma\{AB\}$. The operator $\mathfrak{D}^*_{1,N}$ acts as

$$\mathfrak{D}^*_{1,N} \mathsf{F}^*_\lambda = \mathcal{C}(\lambda) \sum_{i=1}^{M} \prod_{\substack{j=1\\ j\neq i}}^{M} \frac{v_i - t v_j}{v_i - v_j} \sum_{\sigma \in \mathfrak{S}_M} \mathfrak{C}_{i,N}(\sigma\{AB\}).$$

The action of $\mathfrak{C}_{i,N}$ on the product $\sigma\{AB\}$ can be split as

$$\mathfrak{C}_{i,N}(\sigma\{AB\}) = \lim_{\varepsilon \to 0} \sigma\{A\}\Big|_{v_i = 1/\varepsilon} \times \mathfrak{C}_{i,N}(\sigma\{B\}). \tag{3.4}$$

Assume now that $\lambda'_N = L$, that is $\lambda_1 = \cdots = \lambda_L = N$ and $\lambda_{L+1} < N$, for some $L \in \{0, \ldots, M\}$. We focus on the second factor of (3.4). A simple computation shows that

$$\mathfrak{C}_{i,N}(\sigma\{B\}) = \begin{cases} \sigma\{B\} & \text{if } i \in \sigma(\{1, \ldots, L\}), \\ 0 & \text{else,} \end{cases} \tag{3.5}$$

that in particular, implies that $\mathfrak{C}_{i,N}\sigma\{B\} = 0$ when $L = 0$, confirming (3.3) in this specific case.

For $L > 0$ and a permutation σ such that $i \in \sigma(\{1,\dots,L\})$, call $\bar{k}$ the element such that $\sigma(\bar{k}) = i$. We rewrite A into a product of factors $A = A_1A_2A_3$, obtained dividing the triangular product as

$$A_1 = \prod_{1\le l<r<\bar{k}} \frac{v_l - tv_r}{v_l - v_r}, \qquad A_2 = \prod_{1\le l<\bar{k}} \frac{v_l - tv_{\bar{k}}}{v_l - v_{\bar{k}}}, \qquad A_3 = \prod_{\substack{1\le l<M \\ \max(l,\bar{k})<r\le M}} \frac{v_l - tv_r}{v_l - v_r}.$$

We can evaluate the first factor in the right-hand site of (3.4) as

$$\prod_{\substack{j=1\\j\neq i}}^{M} \frac{v_i - tv_j}{v_i - v_j} \lim_{\varepsilon\to 0} \sigma\{A\}\Big|_{v_i=1/\varepsilon} = t^{\bar{k}-1}\sigma\{A_1\widetilde{A}_2A_3\},$$

where

$$\widetilde{A}_2 \coloneqq \prod_{1\le l<\bar{k}} \frac{v_{\bar{k}} - tv_l}{v_{\bar{k}} - v_l}.$$

The action of $\mathfrak{D}^*_{1,N}$ on the sHL function can be therefore expressed (ignoring $\mathcal{C}(\lambda)$) as

$$\sum_{\bar{k}=1}^{L} t^{\bar{k}-1} \sum_{i=1}^{M} \sum_{\substack{\sigma\in\mathfrak{S}_M\\ \sigma(\bar{k})=i}} \sigma\{A_1\widetilde{A}_2A_3B\} = \sum_{\bar{k}=1}^{L} t^{\bar{k}-1} \sum_{\sigma\in\mathfrak{S}_M} \sigma\{A_1\widetilde{A}_2A_3B\}. \tag{3.6}$$

To prove relation (3.3) we show that each term $\sigma\{A_1\widetilde{A}_2A_3B\}$ is equal to one of the terms $\tau\{A_1A_2A_3B\}$ in the expansion of the original sHL function. For each permutation $\tau \in \mathfrak{S}_M$ and each $\bar{k}$ define the permutation σ as

$$\sigma(j) = \begin{cases} \tau(j+1) & \text{if } j = 1,\dots,\bar{k}-1, \\ \tau(1) & \text{if } j = \bar{k}, \\ \tau(j) & \text{if } j = \bar{k}+1,\dots M. \end{cases}$$

With this choice we can easily check that $\sigma\{A_3B\} = \tau\{A_3B\}$ and more crucially that $\sigma\{A_1\widetilde{A}_2\} = \tau\{A_1A_2\}$ since the cyclic shift in the first $\bar{k}$ terms of σ makes up for the exchange of $\widetilde{A}_2$ and A_2. This in particular shows that the symmetric sum in the right-hand side of (3.6) is independent of $\bar{k}$ and it is equal, up to a factor $\mathcal{C}(\lambda)$ that we omitted, to $\mathsf{F}^*_\lambda(v_1,\dots,v_M)$. The sum $\sum_{\bar{k}=1}^{L} t^{\bar{k}-1}$ is the desired eigenvalue $e_1(1,t,\dots,t^{\lambda'_N-1})$. This completes the proof. □

Remark 3.5 (Limit to the Hall–Littlewood case)**.** In the limit $s \to 0$, the new operator $\mathfrak{D}^*_{1,N}$ (3.2) acting on the dual sHL functions should be replaced by

$$D^*_{1,N} = \sum_{j=1}^{M} \prod_{\substack{l=1\\l\neq j}}^{M} \frac{v_j - tv_l}{v_j - v_l}\, v_j^N \lim_{\varepsilon\to 0} \varepsilon^N\, T_{\varepsilon^{-1},v_j}, \tag{3.7}$$

by mimicking the action (3.5). Similarly to Theorem 3.4, one can show that $D^*_{1,N}$ acts diagonally on the Hall–Littlewood polynomials $P_\lambda(\cdot;0,t)$.

The same operator (3.7) can be also obtained as a $q \to 0$ limit of a certain operator diagonal in the Macdonald polynomials $P_\lambda(\cdot; q, t)$. Take the first Macdonald q^{-1}-difference operator

$$M_1 = \sum_{j=1}^{M} \prod_{\substack{i=1\\ i\neq j}}^{M} \frac{tx_i - x_j}{x_i - x_j}\, T_{q^{-1},x_j}. \tag{3.8}$$

It acts on the Macdonald polynomials $P_\lambda(x_1,\ldots,x_M;q,t)$ with eigenvalues $\sum_{i=1}^{M} q^{-\lambda_i} t^{i-1}$ (this follows from, e.g., [BC14, Section 2.2.3]). Denote by $\mathbf{P}_N$ the subspace of polynomials in $x_1,\ldots,x_M$ which have degree $\le N$ in each of the variables x_i. It is spanned by the Macdonald polynomials $P_\lambda(x_1,\ldots,x_M;q,t)$ with $\lambda_1 \le N$, i.e., $\lambda \subseteq \mathrm{Box}(N,M)$. On $\mathbf{P}_N$ consider the operator $q^N M_1$. Its limit as $q \to 0$ is well-defined. By looking at eigenvalues on Hall–Littlewood polynomials $P_\lambda(x_1,\ldots,x_M;0,t)$ with $\lambda_1 \le N$, one readily sees that this limit coincides with $D^*_{1,N}$.

3.2. **Eigenrelations for the spin q-Whittaker polynomials.** The duality between sHL functions and sqW polynomials (Corollary 2.16 and Proposition 2.17) allows to pass from the eigenoperators for the sHL functions to the ones for the sqW polynomials.

Definition 3.6 (Spin q-Whittaker difference operators)**.** Fix a positive integer N, and define the *s-deformed q-Whittaker operators*

$$\mathfrak{D}_1 \coloneqq \sum_{i=1}^{N} \prod_{\substack{j=1\\ j\neq i}}^{N} \frac{(1+sx_i)}{1-x_i/x_j}\, T_{q,x_i}, \tag{3.9}$$

and

$$\overline{\mathfrak{D}}_1 \coloneqq \sum_{i=1}^{N} \prod_{\substack{j=1\\ j\neq i}}^{N} \frac{(1+s/x_i)}{1-x_j/x_i}\, T_{q^{-1},x_i}. \tag{3.10}$$

Let us make two remarks after this definition.

Remark 3.7. The operators $\mathfrak{D}_1$ and $\overline{\mathfrak{D}}_1$ reduce for $s=0$ to the $t=0$ specializations of the two Macdonald q-difference operators. The first one is the standard first order Macdonald operator $\sum_{i=1}^{N}\prod_{j\neq i}\frac{tx_i-x_j}{x_i-x_j}\,T_{q,x_i}$ (denoted by D_N^1 in [Mac95, Ch. VI]), and the second one is $\sum_{i=1}^{N}\prod_{j\neq i}\frac{x_i-tx_j}{x_i-x_j}\,T_{q^{-1},x_i}$ (denoted by $\tilde{D}_N^1$ in [BC14, Section 2.2.3]).

Remark 3.8. The operator $\mathfrak{D}_1$ is new. The other operator $\overline{\mathfrak{D}}_1$ is only a slightly more general version of the operator $\mathfrak{E}$ from [BMP19, Section 8]. The latter is diagonal in the Borodin–Wheeler's sqW polynomials $\mathbb{F}_\lambda^{BW}$. To recover $\mathfrak{E}$ from (3.10) one has to take the limit $x_1 \to 0$, which agrees with Proposition 2.7 connecting the $\mathbb{F}_\lambda^{BW}$'s with our sqW polynomials $\mathbb{F}_\lambda$.

We establish two eigenrelations for the sqW polynomials in the next two theorems.

Theorem 3.9. *For all signatures $\lambda \in \mathrm{Sign}_N$ we have*

$$\mathfrak{D}_1 \mathbb{F}_\lambda(x_1,\ldots,x_N) = q^{\lambda_N}\mathbb{F}_\lambda(x_1,\ldots,x_N). \tag{3.11}$$

Proof. We will prove the identity

$$\left(1-(1-q)\mathfrak{D}^*_{1,N}\right)\Pi(x;v) = \mathfrak{D}_1\Pi(x;v), \tag{3.12}$$

where

$$\Pi(x;v) = \prod_{j=1}^{M} \left(\frac{1}{1-sv_j} \right)^{N-1} \times \prod_{i=1}^{N} \prod_{j=1}^{M} (1+v_j x_i). \tag{3.13}$$

Indeed, modulo (3.12), the Cauchy Identity (2.23) and the eigenrelations (3.3) imply

$$\sum_{\lambda \subseteq \mathrm{Box}(N,M)} q^{\lambda_N} \mathbb{F}_\lambda(x_1,\ldots,x_N) \mathsf{F}^*_{\lambda'}(v_1,\ldots,v_M) = \sum_{\lambda \subseteq \mathrm{Box}(N,M)} \mathfrak{D}_1 \mathbb{F}_\lambda(x_1,\ldots,x_N) \mathsf{F}^*_{\lambda'}(v_1,\ldots,v_M),$$

and hence (3.11) follows by orthogonality of the sHL functions (Proposition 2.17).

It thus suffices to establish (3.12). Define

$$h(z) \coloneqq \prod_{j=1}^{M} (1+v_j z).$$

We have

$$\frac{\mathfrak{D}_1 \Pi(x;v)}{\Pi(x;v)} = \frac{\mathfrak{D}_1 h(x_1)\cdots h(x_N)}{h(x_1)\cdots h(x_N)} = -\oint_{x_1,\ldots,x_N} \prod_{i=1}^{N} \frac{x_i(1+sz)}{x_i - z} \frac{h(qz)}{h(z)} \frac{dz}{z(1+sz)},$$

where in the second equality we used the residue expansion of the complex integral and the contour encircles only the poles $x_1,\ldots,x_N$. By subtracting 1 from both sides, we can enlarge the complex contour to also include the pole at $z=0$ (note that $h(z)$ is nonsingular at $z=0$). After a change of variable $z=-1/w$, we get

$$\frac{(-1+\mathfrak{D}_1)\Pi(x;v)}{\Pi(x;v)} = -\oint_{v_1,\ldots,v_M} \prod_{k=1}^{M} \frac{w-qv_k}{w-v_k} (w-s)^{N-1} \prod_{j=1}^{N} \frac{x_j}{1+x_j w}\, dw. \tag{3.14}$$

In the right-hand side of (3.14), after the change of variable, we switched the contour to a positively oriented curve around $v_1,\ldots,v_M$, which yielded the negative sign in front. Using

$$\lim_{\varepsilon\to 0} \varepsilon \left(\frac{1}{(1-s/\varepsilon)^{N-1}} \prod_{j=1}^{N} (1+x_j/\varepsilon) \right) = \frac{x_1\cdots x_N}{(-s)^{N-1}}$$

and expanding the right-hand side of (3.14) as a sum of residues, we can rewrite it as

$$-\frac{(1-q)\mathfrak{D}^*_{1,N}\Pi(x;v)}{\Pi(x;v)}.$$

This proves (3.12), and hence the desired eigenrelation (3.11). □

Theorem 3.10. *For all signatures $\lambda \in \mathrm{Sign}_N$ we have*

$$\overline{\mathfrak{D}}_1 \mathbb{F}_\lambda = q^{-\lambda_1} \mathbb{F}_\lambda. \tag{3.15}$$

Proof. The proof of this eigenrelation is identical to that given in [BMP19] and similar to that of Theorem 3.9. It uses the fact that

$$q^{-M}\left(1-(1-q)\overline{\mathfrak{D}}^*_1\right)\Pi(x;v) = \overline{\mathfrak{D}}_1 \Pi(x;v),$$

where $\Pi(x;v)$ is given by (3.13). We will not repeat the detailed argument here. □

3.3. **Commutation and conjugation.** The q-difference operators $\mathfrak{D}_1$ (3.9) and $\overline{\mathfrak{D}}_1$ (3.10) commute. For this statement we cannot appeal to the eigenrelations of Theorems 3.9 and 3.10 since we did not prove that the sqW polynomials form a basis for the ring of symmetric polynomials in N variables. Nevertheless, the commutation can be checked independently:

Proposition 3.11. *We have $\mathfrak{D}_1\overline{\mathfrak{D}}_1 F = \overline{\mathfrak{D}}_1\mathfrak{D}_1 F$ for all symmetric polynomials F in N variables.*

Proof. By polarization, it suffices to check the action on product form functions $F(x_1,\ldots,x_N) = f(x_1)\ldots f(x_N)$, where $f(x)$ is an arbitrary polynomial.

The action of each operator can we written as a contour integral:

$$\mathfrak{D}_1 F = -\frac{1}{2\pi\mathrm{i}}\oint \prod_{i=1}^N \left(f(x_i)\,\frac{x_i(1+sz)}{x_i - z}\right) \frac{f(qz)}{f(z)}\,\frac{dz}{z(1+sz)},$$

$$\overline{\mathfrak{D}}_1 F = \frac{1}{2\pi\mathrm{i}}\oint \prod_{i=1}^N \left(f(x_i)\,\frac{w+s}{w-x_i}\right) \frac{f(q^{-1}w)}{f(w)}\,\frac{dw}{w+s},$$

where both integrals are over a contour containing $x_1,\ldots,x_N$ and no other poles of the integrand. Throughout the proof we assume that all contours exist, which might impose some restrictions on the x_i's. After checking the commutation under the restrictions, we can lift these restrictions by an analytic continuation.

We have

$$\mathfrak{D}_1\overline{\mathfrak{D}}_1 F = -\frac{1}{(2\pi\mathrm{i})^2}\oint_{\gamma_z^1}\oint_{\gamma_w^1}\prod_{i=1}^N \left(f(x_i)\,\frac{w+s}{w-x_i}\,\frac{x_i(1+sz)}{x_i-z}\right)\frac{w-z}{w-qz}\,\frac{f(qz)f(q^{-1}w)}{f(z)f(w)}\,\frac{dw}{w+s}\,\frac{dz}{z(1+sz)},$$

where γ_z^1 contains both γ_w^1 and $q^{-1}\gamma_w^1$, while γ_w^1 is around $x_1,\ldots,x_N$ and no other poles. In the other order, we have

$$\overline{\mathfrak{D}}_1\mathfrak{D}_1 F = -\frac{1}{(2\pi\mathrm{i})^2}\oint_{\gamma_w^2}\oint_{\gamma_z^2}\prod_{i=1}^N \left(f(x_i)\,\frac{x_i(1+sz)}{x_i-z}\,\frac{w+s}{w-x_i}\right)\frac{q^{-1}(w-z)}{q^{-1}w-z}\,\frac{f(q^{-1}w)f(qz)}{f(w)f(z)}\,\frac{dz}{z(1+sz)}\,\frac{dw}{w+s},$$

but now γ_w^2 contains both γ_z^2 and $q\gamma_z^2$, while γ_z^2 is around $x_1,\ldots,x_N$ and no other poles. Note that the integrands in both formulas coincide.

In the first expression, deform the integration contour γ_z^1 to coincide with γ_w^1, which picks up the residue at $z = q^{-1}w$. In the second expression, deform the contour γ_w^2 to coincide with γ_z^2, which picks up the residue at $w = qz$. The resulting double contour integrals are over the same contours and are thus equal. It remains to check the equality of the single integrals of the residues. We have

$$\operatorname*{Res}_{z=q^{-1}w}(\text{integrand in } \mathfrak{D}_1\overline{\mathfrak{D}}_1) = -(-1)^N(1-q)(s+w)^{N-1}(q+sw)^{N-1}\prod_{i=1}^N \frac{x_i f(x_i)}{(w-x_i)(w-qx_i)},$$

$$\operatorname*{Res}_{w=qz}(\text{integrand in } \overline{\mathfrak{D}_1}\mathfrak{D}_1) = (-1)^N(1-q)(1+sz)^{N-1}(s+qz)^{N-1}\prod_{i=1}^N \frac{x_i f(x_i)}{(z-x_i)(qz-x_i)}$$

We must show that the integral of the first expression over γ_w^1 is the same as the integral of the second expression over γ_z^2. Noting that both expressions have zero residue at infinity due to quadratic decay, we can compute the first integral as a sum of minus residues at $w = qx_i$. Then one readily sees that each minus residue at $w = qx_i$ is the same as the residue of the second expression at $z = x_i$. This shows the desired commutation. ☐

The discussion in the rest of this subsection aims in part to demonstrate why the result of Proposition 3.11 is a rather unexpected one.

Both operators $\mathfrak{D}_1$ (3.9) and $\overline{\mathfrak{D}}_1$ (3.10) are related via conjugation to q-Whittaker difference operators. The latter are $t = 0$ degenerations of the Macdonald q-difference operators from [Mac95]. Denote for $r = 1, \dots, N$:

$$W_N^r \coloneqq \sum_{|I|=r} \prod_{i\in I,\, j\notin I} \frac{1}{1 - x_i/x_j} \prod_{i\in I} T_{q,x_i}, \qquad \tilde{W}_N^r \coloneqq \sum_{I} \prod_{i\in I,\, j\notin I} \frac{1}{1 - x_j/x_i} \prod_{i\in I} T_{q^{-1},x_i}, \tag{3.16}$$

where the sums are over subsets of $\{1, \dots, N\}$ of cardinality r. These operators are diagonal in the usual q-Whittaker polynomials (which are $t = 0$ versions of the Macdonald polynomials). In particular, W_N^1 and $\tilde{W}_N^1$ have eigenvalues q^{λ_N} and $q^{-\lambda_1}$, respectively, on q-Whittaker polynomials. All the operators $W_N^r, \tilde{W}_N^r$, $r = 1, \dots, N$, commute. We refer to Sections 2.2.2 and 3.1.3 in [BC14] for details. Let

$$\mathbb{U}_N \coloneqq \prod_{i=1}^{N} \frac{1}{(-sx_i; q)_\infty^{N-1}}, \qquad \mathbb{V}_N \coloneqq \prod_{i=1}^{N} \frac{1}{(-s/x_i; q)_\infty^{N-1}}.$$

A straightforward computation shows:

Proposition 3.12. *The spin q-Whittaker operators* (3.9)*,* (3.10) *are conjugates of the first q-Whittaker operators* (3.16)*:*

$$\mathfrak{D}_1 = \mathbb{U}_N^{-1} W_N^1 \mathbb{U}_N, \qquad \overline{\mathfrak{D}}_1 = \mathbb{V}_N^{-1} \tilde{W}_N^1 \mathbb{V}_N,$$

where $\mathbb{U}_N$, etc., mean multiplication operators.

Because the q-Whittaker operators (3.16) commute, we get many operators commuting with either $\mathfrak{D}_1$ or $\overline{\mathfrak{D}}_1$. That is, for $r = 1, \dots, N$ we have:

$$\big[\mathfrak{D}_1, \mathbb{U}_N^{-1} W_N^r \mathbb{U}_N\big] = 0, \qquad \big[\overline{\mathfrak{D}}_1, \mathbb{V}_N^{-1} \tilde{W}_N^r \mathbb{V}_N\big] = 0. \tag{3.17}$$

For example,

$$\mathbb{U}_N^{-1} W_N^2 \mathbb{U}_N = \sum_{i,k\colon i\neq k} (1 + sx_i)^{N-1} (1 + sx_k)^{N-1} \prod_{j\colon j\neq i,k} \frac{1}{(1 - x_i/x_j)(1 - x_k/x_j)} T_{q,x_i} T_{q,x_k}. \tag{3.18}$$

However, one can directly check that the operator $\mathbb{U}_N^{-1} W_N^2 \mathbb{U}_N$ *does not commute* with $\overline{\mathfrak{D}}_1$. This suggests that the operators $\mathbb{U}_N^{-1} W_N^r \mathbb{U}_N$ or $\mathbb{V}_N^{-1} \tilde{W}_N^r \mathbb{V}_N$, $r \geq 2$, *should not be diagonal* in the spin q-Whittaker polynomials $\mathbb{F}_\lambda$. The following example shows that this is indeed the case:

Example 3.13. Take $N = 2$, then $(1 - s^2)\mathbb{F}_{(1,0)}(x_1, x_2) = s + x_1 + x_2 + sx_1x_2$. Applying (3.18) to this function, we obtain $(1 + sx_1)(1 + sx_2)(s + qx_1 + qx_2 + sq^2x_1x_2)$, which is not proportional to $\mathbb{F}_{(1,0)}(x_1, x_2)$ unless $s = 0$. Note that for $s = 0$ both $\mathbb{U}_N$ and $\mathbb{V}_N$ are the same (and are equal to the identity), and $\mathfrak{D}_1, \overline{\mathfrak{D}}_1$ are the usual q-Whittaker difference operators.

We also observe that by (3.17), polynomials of the form $\mathbb{U}_N^{-1} W_N^r \mathbb{U}_N \mathbb{F}_\lambda$, $r = 2, \dots, N$, are eigenfunctions of the operator $\mathfrak{D}_1$ with eigenvalues q^{λ_N}. Similarly, $\mathbb{V}_N^{-1} \tilde{W}_N^r \mathbb{V}_N \mathbb{F}_\lambda$ are eigenfunctions of $\overline{\mathfrak{D}}_1$ with eigenvalues $q^{-\lambda_1}$. However, one can check that $\overline{\mathfrak{D}}_1$ does not act diagonally on, say, the polynomial $\mathbb{U}_N^{-1} W_N^2 \mathbb{U}_N \mathbb{F}_{(1,0)}$.

It remains unclear how to construct higher order q-difference operators which would be diagonal in the sqW polynomials (and whether such operators exist at all).

4. Integrable stochastic dynamics on interlacing arrays

In this section we implement the general scheme of passing from symmetric functions satisfying Cauchy type summation identities to probability measures. This approach closely follows the ideas of Schur / Macdonald processes [OR03], [BC14]. We use the framework of *skew Cauchy structures* which is explained in detail in [BMP19, Section 2].

4.1. **Skew Cauchy structures and random fields.** We say that two families of functions $\mathfrak{F}, \mathfrak{G}$ form a *skew Cauchy structure* if they satisfy the following properties:

(1) $\mathfrak{F}_{\lambda/\mu}, \mathfrak{G}_{\lambda/\mu}$ are symmetric rational functions in their respective variables, parametrized by pairs of signatures λ/μ (with appropriate numbers of parts). In particular, $\mathfrak{F}_{\lambda/\mu}, \mathfrak{G}_{\lambda/\mu}$ are nonzero only if $\mu \subseteq \lambda$.

(2) Branching rules: for all μ, λ we have

$$\mathfrak{F}_{\nu/\lambda}(u_1, \dots, u_n) = \sum_{\mu} \mathfrak{F}_{\mu/\lambda}(u_1, \dots, u_{n-1}) \mathfrak{F}_{\nu/\mu}(u_n)$$

for any n and any set of variables $u_1, \dots, u_n$, and analogously for $\mathfrak{G}$.

(3) There exists a function Π and a set $\mathsf{Adm} \subseteq \mathbb{C}^2$ such that the skew Cauchy identity

$$\Pi(u; v) \sum_{\varkappa} \mathfrak{F}_{\mu/\varkappa}(u) \mathfrak{G}_{\lambda/\varkappa}(v) = \sum_{\nu} \mathfrak{F}_{\nu/\lambda}(u)\, \mathfrak{G}_{\nu/\mu}(v) \tag{4.1}$$

holds numerically for all $(u, v) \in \mathsf{Adm}$. Note that u, v stand for single variables, as in Propositions 2.15 and 2.21.

(4) There exist two sets $\mathsf{P}, \dot{\mathsf{P}} \subseteq \mathbb{C}$, with $\mathsf{P} \times \dot{\mathsf{P}} \subseteq \mathsf{Adm}$, such that for any choice of $u \in \mathsf{P}$ and $v \in \dot{\mathsf{P}}$ the functions $\mathfrak{F}_{\lambda/\mu}(u), \mathfrak{G}_{\lambda/\mu}(v)$ are non negative for all λ, μ. In this case we say that u, v are *positive specializations*. (Nonnegativity of single-variable functions together with branching implies nonnegativity of multi-variable versions of the functions.)

Consider now two sequences of signatures $\vec{\lambda} = (\lambda^1, \dots, \lambda^n)$ and $\vec{\mu} = (\mu^1, \dots, \mu^{n-1})$ with

$$\lambda^1 \supseteq \mu^1 \subseteq \lambda^2 \supseteq \mu^2 \subseteq \cdots \mu^{n-1} \subseteq \lambda^n,$$

and sequences of positive specializations $u_1, \dots, u_n$ and $v_1, \dots, v_n$ respectively of $\mathfrak{F}$ and $\mathfrak{G}$. The $\mathfrak{F}/\mathfrak{G}$ *process* is the probability measure

$$\mathrm{Prob}(\vec{\lambda}, \vec{\mu}) = \frac{1}{Z} \mathfrak{F}_{\lambda^1}(u_1) \left(\prod_{i=1}^{n-1} \mathfrak{G}_{\lambda^i/\mu^i}(v_i) \mathfrak{F}_{\lambda^{i+1}/\mu^i}(u_{i+1}) \right) \mathfrak{G}_{\lambda^n}(v_n), \tag{4.2}$$

where the normalization constant is $Z = \prod_{i,j} \Pi(u_i; v_j)$.

For applications to stochastic dynamics, it is of interest to consider *random fields* $\{\lambda^{(i,j)}\}$ of signatures indexed by $\mathbb{Z}^2_{\geq 0}$, whose marginal distributions along down-right paths are given by suitable $\mathfrak{F}/\mathfrak{G}$ processes. A *down-right path* is

$$\varpi = \{\varpi_k = (i_k, j_k) \colon 0 \leq k \leq L\}, \quad \text{where} \quad i_0 = j_L = 0 \quad \text{and} \quad \varpi_{k+1} - \varpi_k \in \{\mathbf{e}_1, -\mathbf{e}_2\}.$$

Here L is arbitrary and depends on ϖ, and $\mathbf{e}_1, \mathbf{e}_2$ are the standard basis vectors $(1, 0), (0, 1)$.

Definition 4.1. Consider positive specializations $u_1, u_2, \dots$ and $v_1, v_2, \dots$ respectively of functions $\mathfrak{F}$ and $\mathfrak{G}$. An $\mathfrak{F}/\mathfrak{G}$ *field* is a probability measure on the set $\{\lambda^{(i,j)} \colon i, j \in \mathbb{Z}_{\geq 0}\}$ that associates the probability

$$\frac{1}{Z_\varpi}\prod_{k\colon\,\varpi_{k+1}=\varpi_k+\mathbf{e}_1}\mathfrak{F}_{\lambda^{\varpi_{k+1}}/\lambda^{\varpi_k}}(u_{i_{k+1}})\prod_{k\colon\,\varpi_{k+1}=\varpi_k-\mathbf{e}_2}\mathfrak{G}_{\lambda^{\varpi_k}/\lambda^{\varpi_{k+1}}}(v_{j_k}). \tag{4.3}$$

to the event of finding signatures $\lambda^{\varpi_1},\dots,\lambda^{\varpi_{L-1}}$ along an down-right path ϖ. Here the normalization constant is $Z_\varpi=\prod_{(i,j)\text{ below }\varpi}\Pi(u_i;v_j)$, and at the boundary we fix $\lambda^{(0,j)}=\lambda^{(i,0)}=\varnothing$ with probability one.

Remark 4.2. While the $\mathfrak{F}/\mathfrak{G}$ process is defined uniquely by (4.2), an $\mathfrak{F}/\mathfrak{G}$ is not determined uniquely by Definition 4.1. Below in this section we outline two different constructions of a field in our particular cases. See also the discussion in [BMP19, Section 2.6] for more details and additional references.

To visualize an $\mathfrak{F}/\mathfrak{G}$ field, decorate edges $(i-1,j)\to(i,j)$ of the first quadrant with specializations u_i, and edges $(i,j-1)\to(i,j)$ with v_j. Then for each down-right path ϖ, the probability of finding the sequence λ^{ϖ_k} is computed by climbing down ϖ and picking up skew functions $\mathfrak{F}(u_{i_k})$ along horizontal edges, and $\mathfrak{G}(v_{j_k})$ along vertical edges. See Figure 5 for an illustration.

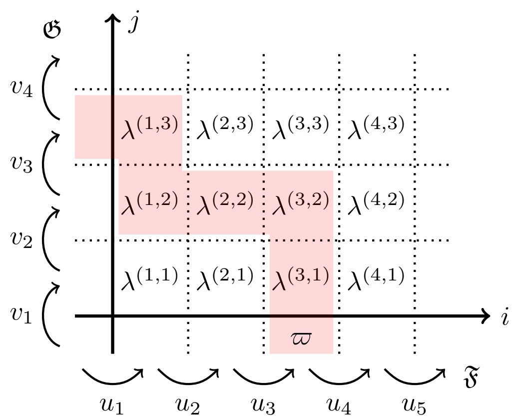


FIGURE 5. A down-right path (highlighted) in a random field, and edge decorations.

In this paper, particularly interesting instances of $\mathfrak{F}/\mathfrak{G}$ processes will be those arising when considering paths ϖ of the form $(0,T)\to(N,T)\to(N,0)$. Taking the marginal distribution of $\lambda^{(N,1)},\dots,\lambda^{(N,T)}$, we arrive at the following definition:

Definition 4.3. The *ascending* $\mathfrak{F}/\mathfrak{G}$ *process* is the probability measure on the set of signatures

$$\lambda^1\subseteq\lambda^2\subseteq\cdots\subseteq\lambda^N,$$

assigning to each such sequence the probability weight

$$\frac{1}{\prod_{i=1}^N\prod_{j=1}^T\Pi(u_i;v_j)}\mathfrak{F}_{\lambda^1}(u_1)\,\mathfrak{F}_{\lambda^2/\lambda^1}(u_2)\cdots\mathfrak{F}_{\lambda^N/\lambda^{N-1}}(u_N)\,\mathfrak{G}_{\lambda^N}(v_1,\cdots,v_T).$$

4.2. **Fields based on spin q-Whittaker polynomials.** Here we specialize skew Cauchy structures to two cases involving spin q-Whittaker and spin Hall–Littlewood functions.

Definition 4.4. Let $s \in (-1, 0)$ and take parameters $x_i \in [-s, -s^{-1}]$, $v_j \in [0, 1)$. The *sqW/sHL field* is obtained by specializing $\mathfrak{F}_{\lambda/\mu}(x_i) = \mathbb{F}_{\lambda/\mu}(x_i)$ and $\mathfrak{G}_{\lambda/\mu}(v_j) = \mathsf{F}^*_{\lambda'/\mu'}(v_j)$.

The corresponding skew Cauchy identity is Proposition 2.15. One readily verifies that the sHL and sqW functions specialized like this are nonnegative, which leads to probability distributions. Joint distributions along down-right paths in this field are given by *sqW/sHL processes* which are specializations of (4.2).

Definition 4.5. Let $s \in (-1, 0)$ and take parameters $x_i, y_j \in [-s, -s^{-1}]$. The *sqW/sqW field* is obtained by specializing $\mathfrak{F}_{\lambda/\mu}(x_i) = \mathbb{F}_{\lambda/\mu}(x_i)$ and $\mathfrak{G}_{\lambda/\mu}(y_j) = \mathbb{F}^*_{\lambda/\mu}(y_j)$.

The corresponding skew Cauchy identity is Proposition 2.21. The range of parameters here also leads to nonnegative functions $\mathbb{F}, \mathbb{F}^*$, thus producing probability measures. Joint distributions along down-right paths in the sqW/sqW field are given by *sqW/sqW processes* which are specializations of (4.2).

Remark 4.6. Both types of fields were already defined in [BMP19], though using slightly different versions of the sHL and sqW functions.

4.3. **Sampling a field via bijectivization.** As mentioned in Remark 4.2, a random field is not determined uniquely. Moreover, its properties (like marginal stochastic dynamics) heavily rely on a particular choice of the field's construction. This choice can be encoded by certain Markov transition operators. Let us return to the general formalism of skew Cauchy structures.

Suppose that we have Markov transition operators

$$\mathsf{U}^{\mathrm{fwd}}_{u,v}(\varkappa \to \nu \mid \lambda, \mu) \qquad \text{and} \qquad \mathsf{U}^{\mathrm{bwd}}_{u,v}(\nu \to \varkappa \mid \lambda, \mu),$$

that satisfy the *reversibility condition*

$$\mathsf{U}^{\mathrm{fwd}}_{u,v}(\varkappa \to \nu \mid \lambda, \mu)\Pi(u; v)\mathfrak{F}_{\mu/\varkappa}(u)\mathfrak{G}_{\lambda/\varkappa}(v) = \mathsf{U}^{\mathrm{bwd}}_{u,v}(\nu \to \varkappa \mid \lambda, \mu)\mathfrak{F}_{\nu/\lambda}(u)\,\mathfrak{G}_{\nu/\mu}(v). \tag{4.4}$$

Here $\mathsf{U}^{\mathrm{fwd}}_{u,v}(\varkappa \to \nu \mid \lambda, \mu)$ encodes the probability of a transition $\varkappa \to \nu$ conditioned on λ, μ, whereas $\mathsf{U}^{\mathrm{bwd}}_{u,v}(\nu \to \varkappa \mid \lambda, \mu)$ describes the probability of the opposite move (specializations u, v are assumed positive). See Figure 6, left, for an illustration. Summing (4.4) over both ν and $\varkappa$ and using the Markov property of $\mathsf{U}^{\mathrm{fwd}}, \mathsf{U}^{\mathrm{bwd}}$, one recovers the skew Cauchy Identity (4.1). Condition (4.4) determines $\mathsf{U}^{\mathrm{bwd}}$ once $\mathsf{U}^{\mathrm{fwd}}$ if given, and vice versa.

If $\mathsf{U}^{\mathrm{fwd}}$ is given, we can construct a random field $\{\lambda^{(i,j)} : i, j \in \mathbb{Z}_{\geq 0}\}$ as in Figure 6, right. Namely, fix empty boundary conditions. Inductively for $n \geq 2$, assuming we already sampled signatures $\lambda^{(i,j)}$ with $i + j \leq n$, pick signatures $\lambda^{(i',j')}$ for each $i' + j' = n + 1$ at random with probabilities

$$\mathsf{U}^{\mathrm{fwd}}_{u_{i'},v_{j'}}(\lambda^{(i'-1,j'-1)} \to \lambda^{(i',j')} \mid \lambda^{(i'-1,j')}, \lambda^{(i',j'-1)}),$$

independently for various pairs (i', j'). We say that the field is *generated* by $\mathsf{U}^{\mathrm{fwd}}$.

Proposition 4.7. *Assume that* $\mathsf{U}^{\mathrm{fwd}}$ *is known. Then the procedure described right above samples an* $\mathfrak{F}/\mathfrak{G}$ *field.*

Proof. One has to show that the distribution of the signatures along any down-right path is described by the corresponding $\mathfrak{F}/\mathfrak{G}$ process. This is readily verified by induction on adding one box to the area below the down-right path, and using (4.4). We omit the details. □

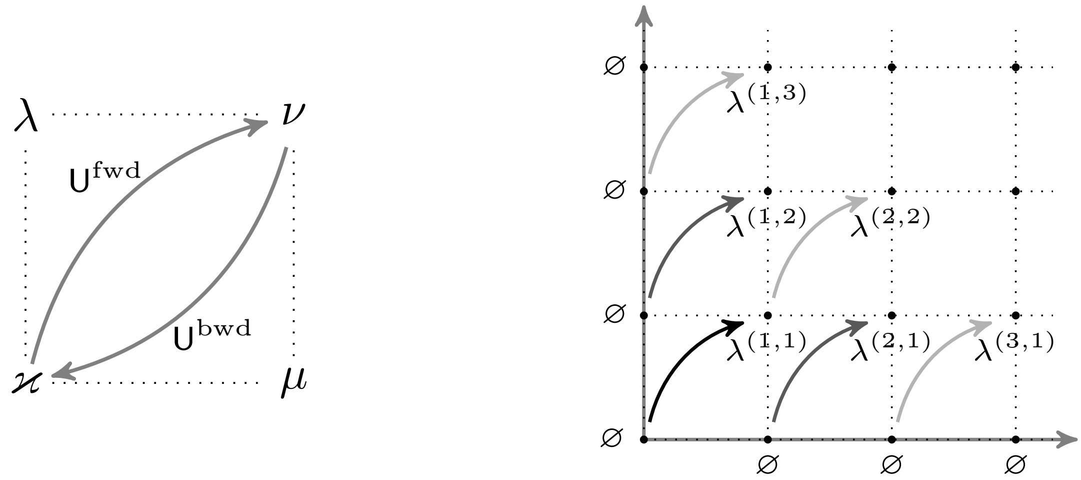


FIGURE 6. Left: Forward and backward transition operators. Right: Construction of a random field using $\mathsf{U}^{\mathrm{fwd}}$, where lighter arrows correspond to moves happening later in the update.

4.4. **Borodin–Ferrari fields.** Let us now describe a particular choice of the forward transition probabilities which guarantees the existence of a field for a skew Cauchy structure. This construction is based on [BF14] and follows an earlier coupling idea of [DF90]. Choose

$$\begin{aligned} \mathsf{U}^{\mathrm{fwd}}_{u,v}(\varkappa \to \nu \mid \lambda, \mu) &= \frac{\mathfrak{F}_{\nu/\lambda}(u)\mathfrak{G}_{\nu/\mu}(v)}{\Pi(u;v)\sum_{\varkappa}\mathfrak{F}_{\mu/\varkappa}(u)\mathfrak{G}_{\lambda/\varkappa}(v)}, \\ \mathsf{U}^{\mathrm{bwd}}_{u,v}(\nu \to \varkappa \mid \lambda, \mu) &= \frac{\Pi(u;v)\mathfrak{F}_{\mu/\varkappa}(u)\mathfrak{G}_{\lambda/\varkappa}(v)}{\sum_{\nu}\mathfrak{F}_{\nu/\lambda}(u)\mathfrak{G}_{\nu/\mu}(v)}. \end{aligned} \tag{4.5}$$

In general, although transition probabilities (4.5) are explicit, in particular examples their concrete meaning may be far from transparent.

A helpful simplification can be made if we assume that $\mathfrak{G}$ admits expansion

$$\mathfrak{G}_{\nu/\mu}(v) = (v - v^*)^{d(\nu/\mu)}(g_{\nu/\mu} + \mathsf{O}(v - v^*)), \tag{4.6}$$

for some fixed value v^* independent of ν, μ, coefficients $g_{\nu/\mu}$, and a "nice" degree function d such that $d(\nu/\nu) = 0$. Then one can consider a Poisson-type scaling limit of the field (4.5) as $v_j \to v^*$ for all j. Under this scaling, the discrete vertical axis becomes continuous, and the field turns into a Markov dynamics $\{\lambda^{(i,t)} \colon i \in \mathbb{Z}_{\ge 0}, t \in \mathbb{R}_{\ge 0}\}$, where t is the continuous time variable. The dynamics lives on sequences of signatures.

When $\mathfrak{F}, \mathfrak{G}$ are Schur functions, such continuous processes is the *push–block dynamics* introduced in [BF14].

4.5. **Bijectivization of the Yang–Baxter equation.** In many cases, skew Cauchy Identities descend directly from the Yang–Baxter equation (cf. Sections 2.6 and 2.7). This observation was used in [BP19], [BMP19] to provide an explicit construction of random fields for sHL and sqW functions, which we briefly recall here. In general, this approach produces fields which *differ* from the Borodin–Ferrari ones. On the other hand, *Yang–Baxter fields* by design possess Markovian marginals.

For any given identity with positive terms

$$\sum_{a\in A} \mathsf{w}(a) = \sum_{b\in B} \mathsf{w}(b), \tag{4.7}$$

we say that two stochastic matrices $\mathsf{p}^{\mathrm{fwd}}(a,b)$ and $\mathsf{p}^{\mathrm{bwd}}(b,a)$ (with indices $a\in A$, $b\in B$) form a (stochastic) *bijectivization of identity* (4.7) if they satisfy the reversibility condition

$$\mathsf{p}^{\mathrm{fwd}}(a\to b)\,\mathsf{w}(a) = \mathsf{p}^{\mathrm{bwd}}(b\to a)\,\mathsf{w}(b) \qquad \text{for all } a\in A,\, b\in B.$$

A bijectivization always exists since we can take $\mathsf{p}^{\mathrm{fwd}}(a\to b)\propto \mathsf{w}(b)$. A bijectivization is unique only when A or B has a single element. Another simple case is given when both A and B have only two elements.

Example 4.8. When $A=\{a_1,a_2\}$ and $B=\{b_1,b_2\}$, identity (4.7) becomes

$$\mathsf{w}(a_1)+\mathsf{w}(a_2)=\mathsf{w}(b_1)+\mathsf{w}(b_2).$$

In this case all stochastic bijectivizations $\mathbf{p}^{\mathrm{fwd}}$, $\mathbf{p}^{\mathrm{bwd}}$ are expressed as

$$\mathbf{p}^{\mathrm{fwd}}(a_1\to b_1)=\gamma, \qquad \mathbf{p}^{\mathrm{fwd}}(a_2\to b_1)=\frac{\mathsf{w}(a_2)-\mathsf{w}(b_2)+(1-\gamma)\mathsf{w}(a_1)}{\mathsf{w}(a_2)},$$
$$\mathbf{p}^{\mathrm{fwd}}(a_1\to b_2)=1-\gamma, \qquad \mathbf{p}^{\mathrm{fwd}}(a_2\to b_2)=1-\mathbf{p}^{\mathrm{fwd}}(a_2\to b_1),$$

for a parameter $\gamma\in[0,1]$.

Let now (4.7) be one of the Yang–Baxter equations (B.5), (B.6), (B.7), (B.8) from Appendix B, corresponding to Figure 15. Let us rewrite them in a unified notation as

$$\sum_{K} \mathsf{wl}(K\mid I,J) = \sum_{K'} \mathsf{wr}(K'\mid I,J), \tag{4.8}$$

where $I=\{i_1,i_2,i_3\}$, $J=\{j_1,j_2,j_3\}$, $K=\{k_1,k_2,k_3\}$ and $K'=\{k_1',k_2',k_3'\}$, and weight functions wl,wr denote the terms in the left and right-hand sides of each of (B.5)–(B.8). Equations (B.6) and (B.8) with the right boundary, by agreement, correspond to $j_1=\varnothing$.

Denote by $\mathsf{p}^{\mathrm{fwd}}_{I,J}(K\to K')$ and $\mathsf{p}^{\mathrm{bwd}}_{I,J}(K'\to K)$ a stochastic bijectivization of (4.8). Then $\mathsf{p}^{\mathrm{fwd}}_{I,J}$ is the probability of moving the cross from left to right (in the local configuration in Figure 15), while transforming the occupation numbers K into K'. The probabilities $\mathsf{p}^{\mathrm{bwd}}_{I,J}(K'\to K)$ similarly correspond to moving the cross from right to left. By the conservation of paths at each vertex, once I,J are fixed, the configuration K is completely determined specifying only one of the numbers k_1,k_2, or k_3 (and similarly for K').

Bijectivizations of the Yang–Baxter equation are building blocks of operators $\mathsf{U}^{\mathrm{fwd}},\mathsf{U}^{\mathrm{bwd}}$. Given $\varkappa,\mu\in\mathrm{Sign}_N$, $\lambda,\nu\in\mathrm{Sign}_{N+1}$ we identify path configurations through two rows of vertices as in Figure 7 (in the same way as in Section 2.2). Vertices crossed by blue paths are assigned non dual weights W (2.2)–(2.4) whereas those in red have dual weights w^* or W^*. We assume that at the leftmost column an infinite number of paths flows in the vertical direction. The transition probability $\mathsf{U}^{\mathrm{fwd}}(\varkappa\to\nu\mid\lambda,\mu)$ is the product of probabilities of sequential local moves $\mathsf{p}^{\mathrm{fwd}}$ obtained dragging the cross vertex from the leftmost column to the right. The operator $\mathsf{U}^{\mathrm{bwd}}$ is constructed using the opposite local moves with probabilities $\mathsf{p}^{\mathrm{bwd}}$, starting from the $N+1$-th column. See Figure 7 for an illustration.

Proposition 4.9. *Let* $\mathsf{p}^{\mathrm{fwd}}$ *and* $\mathsf{p}^{\mathrm{bwd}}$ *be a stochastic bijectivization of Yang–Baxter equations* (B.5), (B.6) *for the weights* W,w^*, *and* $\mathsf{U}^{\mathrm{fwd}}$ *be constructed from sequential local moves* $\mathsf{p}^{\mathrm{fwd}}$. *Then the random field generated by* $\mathsf{U}^{\mathrm{fwd}}$ *is a sqW/sHL field.*

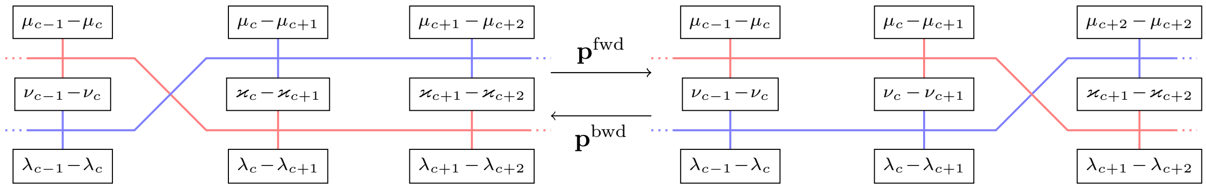


FIGURE 7. A local random move in a Yang–Baxter field. Moving the cross through the column c updates the value of $\varkappa_c - \varkappa_{c+1}$ to $\nu_c - \nu_{c+1}$.

Proof. This is analogous to [BMP19, Section 6.3]. See also [BP19, Theorem 6.3]. □

Proposition 4.10. *Let* p^{fwd} *and* p^{bwd} *be a stochastic bijectivization of Yang–Baxter equations* (B.7), (B.6) *for the weights* W, W^*, *and* U^{fwd} *be constructed from sequential local moves* $\mathbf{p}^{\text{fwd}}$. *Then the random field generated by* U^{fwd} *is a* sqW/sqW *field.*

Proof. This is again analogous to [BMP19, Section 6.3]. □

By the very construction, we see that for any fixed $c \geq 1$, the update $(\varkappa_1, \dots, \varkappa_c) \to (\nu_1, \dots, \nu_c)$ is independent of $\varkappa_i, \mu_i, \lambda_i$ for all $i \geq c+1$. Therefore, we have:

Proposition 4.11. *Let* $\{\lambda^{(i,j)} \colon i, j \in \mathbb{Z}_{j \in \geq 0}\}$ *be a Yang–Baxter random field as above. For any* $c \in \mathbb{Z}_{\geq 1}$, *the marginal process* $\{(\lambda_1^{(i,j)} \geq \dots \geq \lambda_c^{(i,j)}) : i, j \in \mathbb{Z}_{\geq 0}\}$ *is a Markov process.*

Proof. This is [BP19, Proposition 6.2]. □

In the simplest case $c = 1$, transition probabilities of the one-dimensional marginal field can be written down explicitly:

Proposition 4.12. *Let* $\{\lambda^{(i,j)} \colon i, j \in \mathbb{Z}_{\geq} 0\}$ *be a random field generated by* U^{fwd} *constructed from bijectivization of the Yang–Baxter equation. Let* $\{\lambda_1^{(i,j)} \colon i, j \in \mathbb{Z}_{\geq 0}\}$ *be the first row marginal process. Then for all* $i, j \geq 1$ *we have*

$$\operatorname{Prob}\{\lambda_1^{(i,j)} = n \mid \lambda_1^{(i,j-1)} = m, \lambda_1^{(i-1,j)} = \ell, \lambda_1^{(i-1,j-1)} = k\} = \mathbf{L}_{u_i,v_i}(m-k, \ell-k; n-\ell, n-m), \quad (4.9)$$

for all $n, m, k, n \geq 0$, *where* $\mathbf{L}$ *is the stochastic vertex weight*

$$\mathbf{L}_{u,v}(j_2, j_1; k_1', k_2') = \frac{\mathsf{wr}_{\{0,0,\infty\},\{j_1,j_2,\infty\}}(\{k_1', k_2', \infty\})}{\sum_{k_1,k_2} \mathsf{wl}_{\{0,0,\infty\},\{j_1,j_2,\infty\}}(\{k_1, k_2, \infty\})}. \quad (4.10)$$

Note that interlacing implies that $k \leq m \leq n$, $k \leq \ell \leq n$, so the arguments of $\mathbf{L}_{u_i,v_i}$ in (4.9) are all nonnegative.

Proof of Proposition 4.12. This is proven in [BMP19, Section 6.4] and we briefly reproduce the argument here. The update $\lambda_1^{(i-1,j-1)} \to \lambda_1^{(i,j)}$, once $\lambda_1^{(i-1,j)}, \lambda_1^{(i,j-1)}$ are fixed, is determined only by a single random move at the leftmost column of vertices. By construction, the vertical direction at the leftmost column has infinitely many paths. The corresponding Yang–Baxter equation is

$$\sum_{k_1,k_2} \mathsf{wl}(\{k_1, k_2, \infty\} \mid \{0,0,\infty\}, \{j_1, j_2, \infty\}) = \sum_{k_1',k_2'} \mathsf{wr}(\{k_1', k_2', \infty\} \mid \{0,0,\infty\}, \{j_1, j_2, \infty\}). \quad (4.11)$$

This implies that taking

$$\mathbf{p}^{\text{fwd}}_{\{0,0,\infty\},\{j_1,j_2,\infty\}}\left(\{k_1, k_2, \infty\} \to \{k_1', k_2', \infty\}\right) = \mathbf{L}_{u,v}(j_2, j_1; k_2', k_1')$$

indeed produces a bijectivization.[1] Here u, v denote generic spectral parameters of weights appearing in the Yang–Baxter equation. Recall that occupation numbers are related to signatures as

$$j_1 = \lambda_1^{(i-1,j)} - \lambda_1^{(i-1,j-1)}, \qquad j_2 = \lambda_1^{(i,j-1)} - \lambda_1^{(i-1,j-1)},$$
$$k_1' = \lambda_1^{(i,j)} - \lambda_1^{(i,j-1)}, \qquad k_2' = \lambda_1^{(i,j)} - \lambda_1^{(i-1,j)}.$$

This completes the proof. □

The fact that the sqW functions are parametrized by signatures with specified number of rows also allows to access the random dynamics of *last rows* of a field by writing down explicit bijectivizations. In particular, the evolution of $\{\lambda_i^{(i,j)} : i, j \geq 0\}$ is related to the Yang–Baxter equations (B.6), (B.8) corresponding to configurations depicted in Figure 15 (b).

Remark 4.13. The construction of a random field using stochastic bijectivizations *does not guarantee* that the evolution of last rows is autonomous. This contrasts with the fact that the first rows form autonomous Markov marginal processes by the very construction of Yang–Baxter fields (Proposition 4.11). In Theorems 5.7 and 5.13 below we show that the marginals $\{\lambda_i^{(i,j)} : i, j \geq 0\}$ of sqW/sHL and sqW/sqW fields, respectively, are in fact autonomous for a particular bijectivization we construct.

5. Marginals of spin q-Whittaker fields

In this section we study two random fields of signatures defined in Section 4.2 based on sqW functions. We identify their Markov marginals corresponding to the first and last coordinates $\lambda_1^{(i,j)}$ and $\lambda_i^{(i,j)}$. These are matched with stochastic vertex models or particle dynamics introduced in [Pov13], [CP16], [CMP19]. These results extend the characterization of marginals of the q-Whittaker processes given in [MP17] by adding the spin parameter s into the picture. The matchings are summarized in the table in Figure 8.

	first row $\lambda_1^{(i,j)}$	last row $\lambda_i^{(i,j)}$
sqW/sHL field	[5.3] Stochastic higher spin six vertex model [CP16], [BP18]	[5.2] Stochastic higher spin six vertex model [CP16], [BP18]
sqW/sqW field	[5.5] ${}_4\phi_3$ vertex model and q-Hahn PushTASEP [CMP19], [BMP19]	[5.4, 5.6] q-Hahn TASEP / Boson particle systems [Pov13], [Cor14]

Figure 8. A summary of matchings of Section 5, with numbers of relevant subsections.

5.1. Stochastic vertex models.

We work with two typologies of stochastic vertex models: *up-right* or *up-left*. These are probability measures on directed path ensembles (of the corresponding direction) in the integer quadrant, constructed from families of *stochastic vertex weights* $L_{(i,j)}$. By "stochastic" we mean that the weights must satisfy the sum to one condition

$$\sum_{\alpha_2, \beta_2 \geq 0} L_{(i,j)}(\alpha_1, \beta_1; \alpha_2, \beta_2) = 1 \tag{5.1}$$

for all α_1, β_1, where $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{Z}_{\geq 0}$ are the occupation numbers of edges at a vertex (i, j).

[1] This bijectivization is in fact unique for our choices of weights (this follows similarly to [BMP19]). However, we do not need this fact.

For the first type of stochastic vertex models, equip the lattice with *up-right* vertex weights $L^{\mathrm{ur}}_{(i,j)}$ subject to the arrow preservation condition

$$L^{\mathrm{ur}}_{(i,j)}(\alpha_1,\beta_1;\alpha_2,\beta_2)=0 \qquad \text{if} \qquad \alpha_1+\beta_1\neq\alpha_2+\beta_2.$$

Definition 5.1 (Up-right stochastic vertex model)**.** The *up-right stochastic vertex model* with weights $L^{\mathrm{ur}}_{(i,j)}$ and boundary conditions $B^{\mathrm{h}}=\{b^{\mathrm{h}}_1,b^{\mathrm{h}}_2,\dots\}$, $B^{\mathrm{v}}=\{b^{\mathrm{v}}_1,b^{\mathrm{v}}_2,\dots\}$, with $b^{\mathrm{h}}_i,b^{\mathrm{v}}_j\geq 0$, is the unique probability measure on the set of up-right directed paths on $\mathbb{Z}_{\geq 1}\times\mathbb{Z}_{\geq 0}$, such that:

- each vertex $(1,j)$ emanates b^{v}_j paths initially directed to the right;
- each vertex $(i,0)$ emanates b^{h}_i paths initially directed upwards;
- the probability of observing a configuration $(\alpha_1,\beta_1;\alpha_2,\beta_2)$ at vertex (i,j), conditioned on the configuration at all vertices (i',j') with $i'+j'<i+j$, is given by $L^{\mathrm{ur}}_{(i,j)}(\alpha_1,\beta_1;\alpha_2,\beta_2)$. Moreover, this event is independent of choosing arrow configurations at other vertices $\dots,(i-1,j+1),(i+1,j-1),\dots$ on the same diagonal.

Up-right directed lattice path configurations can be encoded by the *height function*:

$$\mathcal{H}^{\mathrm{ur}}(i,j)=\#\{\text{occupations at horizontal edges}\}-\#\{\text{occupations at vertical edges}\}, \qquad (5.2)$$

where occupations are counted along the path $(\frac{1}{2},\frac{1}{2})\to(i+\frac{1}{2},\frac{1}{2})\to(i+\frac{1}{2},j+\frac{1}{2})$ (equivalently, along any up-right directed path from $(\frac{1}{2},\frac{1}{2})$ to $(i+\frac{1}{2},j+\frac{1}{2})$). See Figure 9, right, for an illustration of the vertex model and the corresponding height function.

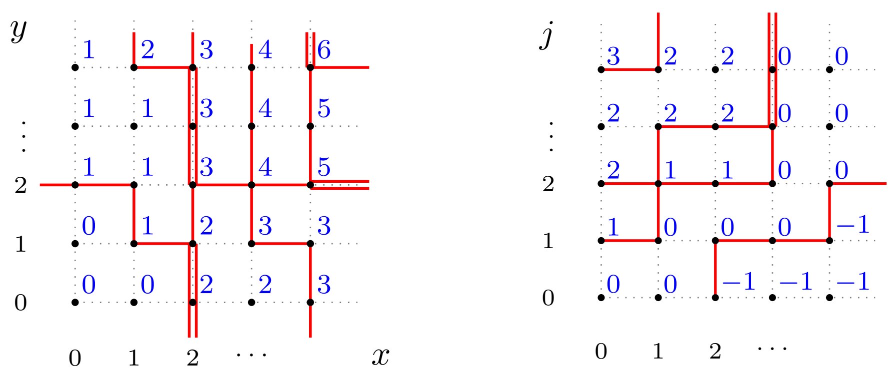


FIGURE 9. Realizations of the up-left and the up-right stochastic vertex models (left and right panels, respectively).

Remark 5.2 (Up-right model and TASEPs)**.** Path configurations can be interpreted as trajectories of particles performing totally asymmetric random walks, with time running in the upward direction. In particular, one can define a process

$$\{\mathsf{X}(t)=(\mathsf{x}_1(t)>\mathsf{x}_2(t)>\cdots)\}_{t\in\mathbb{Z}_{\geq 0}}$$

by setting $\mathsf{x}_n(t):=\mathcal{H}^{\mathrm{ur}}(n,t)-n$. Then X is a discrete time totally asymmetric simple exclusion process, in which the random jump $\mathsf{x}_n(t-1)\to\mathsf{x}_n(t)$ of the n-th particle at time t is governed by

$$L^{\mathrm{ur}}_{(n,t)}\big(\mathsf{x}_{n-1}(t-1)-\mathsf{x}_n(t-1)-1,\mathsf{x}_{n-1}(t)-\mathsf{x}_{n-1}(t-1);\mathsf{x}_{n-1}(t)-\mathsf{x}_n(t)-1,\mathsf{x}_n(t)-\mathsf{x}_n(t-1)\big).$$

Let us now turn to up-left path ensembles. The *up-left* weights $L^{\mathrm{ul}}_{(i,j)}$ satisfy the following arrow preservation property:

$$L^{\mathrm{ul}}_{(i,j)}(\alpha_1,\beta_1;\alpha_2,\beta_2)=0 \qquad \text{if} \qquad \alpha_1+\beta_2\neq\beta_1+\alpha_2.$$

Definition 5.3 (Up-left stochastic vertex model)**.** The *up-left stochastic vertex model* with weights $L^{\mathrm{ul}}_{(i,j)}$ and boundary conditions $B^{\mathrm{h}} = \{b^{\mathrm{h}}_1, b^{\mathrm{h}}_2, \dots\}$, $B^{\mathrm{v}} = \{b^{\mathrm{v}}_1, b^{\mathrm{v}}_2, \dots\}$, with $b^{\mathrm{h}}_i, b^{\mathrm{v}}_j \geq 0$, is the unique probability measure on the set of up-left directed path on $\mathbb{Z}_{\geq 1} \times \mathbb{Z}_{\geq 0}$, such that:

- each vertex $(1, j)$ has b^{v}_j paths entering from its left;
- each vertex $(i, 0)$ emanates b^{h}_i paths initially directed upwards;
- the probability of observing a configuration $(\alpha_1, \beta_1; \alpha_2, \beta_2)$ at a vertex (i, j), conditioned on the path configuration at vertices (i', j') with $i'+j' < i+j$, is given by $L^{\mathrm{ul}}_{(i,j)}(\alpha_1, \beta_1; \alpha_2, \beta_2)$. Moreover, this event is independent of choosing arrow configurations at other vertices $\dots, (i-1, j+1), (i+1, j-1), \dots$ on the same diagonal.

Up-left directed lattice path configurations can be encoded by the *height function*:

$$\mathcal{H}^{\mathrm{ul}}(i,j) = \#\{\text{occupations at horizontal edges}\} + \#\{\text{occupations at vertical edges}\}, \tag{5.3}$$

where occupations are counted along the path $(\frac{1}{2}, \frac{1}{2}) \to (i + \frac{1}{2}, \frac{1}{2}) \to (i + \frac{1}{2}, j + \frac{1}{2})$ (equivalently, along any up-right directed path from $(\frac{1}{2}, \frac{1}{2})$ to $(i + \frac{1}{2}, j + \frac{1}{2})$). Notice the difference in sign with the definition of $\mathcal{H}^{\mathrm{ur}}$ (5.2). See Figure 9, left, for an illustration of the up-left vertex model and the corresponding height function.

Remark 5.4 (Up-left model and PushTASEPs)**.** Define a process

$$\{\mathsf{Y}(t) = (\mathsf{y}_1(t) > \mathsf{y}_2(t) > \cdots)\}_{t \in \mathbb{Z}_{\geq 0}}$$

by setting $\mathsf{y}_n(t) = -\mathcal{H}^{\mathrm{ul}}(n,t) - n$. Then Y is a discrete time totally asymmetric simple exclusion process under which particles jump to the left, and a *pushing mechanism* is present. The random jump $\mathsf{y}_n(t-1) \to \mathsf{y}_n(t)$ of the n-th particle at time t is governed by

$$L^{\mathrm{ul}}_{(n,t)}\big(\mathsf{y}_{n-1}(t-1) - \mathsf{y}_n(t-1) - 1, \mathsf{y}_{n-1}(t-1) - \mathsf{y}_{n-1}(t); \mathsf{y}_{n-1}(t) - \mathsf{y}_n(t) - 1, \mathsf{y}_n(t-1) - \mathsf{y}_n(t)\big).$$

In the rest of this section we establish the matching results outlined in Figure 8.

5.2. **Last row in sqW/sHL field.** We start by defining the stochastic higher spin six vertex model:

Definition 5.5 ([CP16], [BP18])**.** Specialize the up-right stochastic vertex model of Definition 5.1 by taking $L^{\mathrm{ur}}_{(i,j)} = \mathsf{L}^{\mathrm{ur}}_{x_i, v_j}$, where the latter are given in Figure 10. We refer to this model as the *up-right stochastic higher spin six vertex model.* We consider the *step-stationary* boundary conditions:

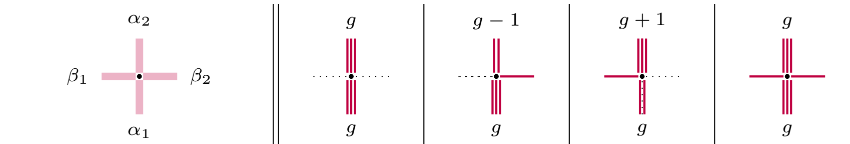
$$b^{\mathrm{v}}_j \sim \mathrm{Ber}\left(\frac{x_1 v_j}{1 + x_1 v_j}\right) \qquad \text{and} \qquad b^{\mathrm{h}}_i = 0, \tag{5.4}$$

where Ber$(\cdot)$ are independent Bernoulli random variables with the probability of success given in the parentheses.[2]

Remark 5.6. The model in Definition 5.5 is equivalent to that of [BP18] (the latter with step boundary conditions $b^{\mathrm{v}}_j = 1$, $b^{\mathrm{h}}_i = 0$), under specializations $\xi_1 \mathsf{s}_1 \to x_1$, $\mathsf{s}_1^2 \to 0$, $\mathsf{s}_\alpha \xi_\alpha \to x_\alpha$, $\mathsf{s}_\alpha^2 \to -s x_\alpha$ and $u_\beta \to -v_\beta$.

Theorem 5.7 (sqW/sHL last row)**.** *The last row marginal $\{\lambda_i^{(i,j)}\}_{i \geq 1, j \geq 0}$ of the sqW/sHL field has the same distribution as the height function $\{\mathcal{H}^{\mathrm{ur}}_{\mathrm{HS}}(i,j)\}_{i \geq 1, j \geq 0}$ of the up-right higher spin six vertex model with step-stationary boundary conditions.*

[2]A slightly broader class of boundary conditions than the step-stationary ones, where also b^{h}_i are allowed to be positive numbers, can be considered using the fusion argument introduced in [Agg18]; see also [IMS19], [BMP19].



$\mathsf{L}^{\mathrm{ur}}_{x,v}(\alpha_1,\beta_1;\alpha_2,\beta_2)$	$\frac{1+q^g xv}{1+xv}$	$\frac{xv(1-q^g)}{1+xv}$	$\frac{1+q^g sx}{1+xv}$	$\frac{xv-q^g sx}{1+xv}$

FIGURE 10. The stochastic vertex weights $\mathsf{L}^{\mathrm{ur}}_{x,v}$ for the up-right stochastic higher spin six vertex model.

Proof. We use Proposition 4.9. During the update

$$\lambda^{(n-1,t-1)} \to \lambda^{(n,t)}, \qquad \text{for fixed} \qquad \lambda^{(n-1,t)}, \lambda^{(n,t-1)}, \tag{5.5}$$

weighted by the stochastic matrix $\mathsf{U}^{\mathrm{fwd}}$, the law of the rightmost local move is given by a stochastic bijectivization of the Yang–Baxter equation (B.6). A computation shows that one such bijectivization is given by the choice

$$\mathbf{p}^{\mathrm{fwd}}_{\{i_1,i_2,i_3\},\{\varnothing,j_2,j_3\}}(\{k_1,k_2,k_3\} \to \{k_1',k_2',k_3'\}) = \mathsf{L}^{\mathrm{ur}}_{x,v}(j_3-i_2-k_1,k_1;j_3-i_2-k_1',k_1'). \tag{5.6}$$

This can be readily verified using the parametrization from Example 4.8. In terms of elements of signatures in (5.5), the integers i_2, j_3, k_1, k_1' are interpreted as

$$i_2 = \lambda^{(n,t-1)}_{n-1} - \lambda^{(n-1,t)}_{n-1}, \qquad j_3 = \lambda^{(n,t-1)}_{n-1} - \lambda^{(n,t-1)}_{n}$$
$$k_1 = \lambda^{(n-1,t)}_{n-1} - \lambda^{(n-1,t-1)}_{n-1}, \qquad k_1' = \lambda^{(n,t)}_{n} - \lambda^{(n,t-1)}_{n}.$$

Remarkably, transition weight (5.6) only depends on the difference $j_3 - i_2$ and on k_1, k_1', but not on other edge occupation numbers. Therefore, the law of $\lambda^{(n,t)}_n$ is solely determined by $\lambda^{(n-1,t)}_{n-1}$,$\lambda^{(n,t-1)}_{n}$,$\lambda^{(n-1,t-1)}_{n-1}$. This implies that the last row marginal $\{\lambda^{(i,j)}_i\}_{i\ge1,j\ge0}$ is an autonomous Markov process. Moreover, this autonomous process has the same multitime joint distribution as the height function of the up-right higher spin six vertex model because L^{ur} appears in (5.6). This completes the proof. □

In [CP16], [BP18], joint q-moments of the up-right stochastic higher spin six vertex model were expressed in terms of nested contour integrals. These moments completely determine the joint distribution of the model's height function $\mathcal{H}^{\mathrm{ur}}_{\mathrm{HS}}(\cdot, j)$ along any given horizontal line (because $q \in (0,1)$ and the random variables in question are nonnegative). Let us reproduce the q-moment formula:

Proposition 5.8 ([BP18])**.** *Consider the up-right stochastic higher spin six vertex model with step-stationary boundary conditions and assume $v_\alpha \ne q v_\beta$. For any $i_1 \ge \ldots \ge i_\ell \ge 1$ we have*

$$\begin{aligned}\mathbb{E}\prod_{k=1}^{\ell} q^{\mathcal{H}^{\mathrm{ur}}_{\mathrm{HS}}(i_k,j)} = q^{\binom{\ell}{2}} \oint_{\gamma[-\bar{\mathbf{v}}|1]} \frac{dz_1}{2\pi\mathrm{i}} \cdots \oint_{\gamma[-\bar{\mathbf{v}}|\ell]} \frac{dz_\ell}{2\pi\mathrm{i}} \prod_{1\le A<B\le\ell} \frac{z_A - z_B}{z_A - q z_B} \\ \times \prod_{k=1}^{\ell}\left(\frac{1}{z_k(1+sz_k)} \prod_{\alpha=1}^{i_k} \frac{x_\alpha(1+sz_k)}{x_\alpha - z_k} \prod_{\alpha=1}^{j} \frac{1+qv_\alpha z_k}{1+v_\alpha z_k}\right).\end{aligned} \tag{5.7}$$

Here, integration contours are $\gamma[-\bar{\mathbf{v}}|k] = \gamma[-\bar{\mathbf{v}}] \cup r^{k-1}c_0$, where $\gamma[-\bar{\mathbf{v}}]$ encircles $-1/v_1, \ldots, -1/v_j$ and no other singularity, c_0 is a small circle around 0, and $r > q^{-1}$. All curves are positively oriented, and $r^{k-1}c_0$ never intersects $\gamma[-\bar{\mathbf{v}}]$ for $k = 1, \ldots, \ell$.

Proof. This follows from Theorem 9.8 in [BP18] by identifying the parameters as in Remark 5.6 and noting that $\mathcal{H}^{\mathrm{ur}}_{\mathrm{HS}}(i,j)$ is the same as the height function $\mathfrak{h}(i)$ at the j-th horizontal slice. Note also that [BP18, Corollary 10.3] is essentially the same as our q-moments (5.7), but with contours dragged through infinity, and identification of s_1^2 with x_1. The latter follows by comparing (5.4) with [BP18, Remark 6.14]. □

Eigenrelations for sqW polynomials given in Theorem 3.9 can be employed to provide an alternative proof of the moment formula (5.7).

Alternative proof of Proposition 5.8. We express q-moments of last rows of the sqW/sHL process using the q-difference operators $\mathfrak{D}_1$ (3.9) at several levels, following the argument in [BCGS16, Proposition 4.4].

Denote by $\mathfrak{D}_1^{(i)}$ the operator $\mathfrak{D}_1$ acting on i variables $x_1,\ldots,x_i$. Then for any ℓ and any sequence $1\le i_1\le\cdots\le i_\ell$, we have

$$\mathbb{E}\prod_{k=1}^{\ell} q^{\lambda_{i_k}^{(i_k,j)}} = \frac{\mathfrak{D}_1^{(i_1)}\cdots\mathfrak{D}_1^{(i_\ell)}\Pi(x_1,\ldots,x_N;v_1,\ldots,v_j)}{\Pi(x_1,\ldots,x_N;v_1,\ldots,v_j)}, \tag{5.8}$$

where $N\ge i_\ell$ is arbitrary, and

$$\Pi(x_1,\ldots,x_N;v_1,\ldots,v_j)=\prod_{r=1}^{j}\left(\frac{1}{1-sv_r}\right)^{N-1}\prod_{i=1}^{N}\prod_{r=1}^{j}(1+v_rx_i)$$

is the partition function in the right-hand side of the sqW/sHL Cauchy identity (2.23). Equality (5.8) is a straightforward consequence of the Cauchy identities (2.21), (2.23), eigenrelation (3.11), and the branching rules for the sqW functions.

Let us now express the right-hand side of (5.8) in terms of nested contour integrals. For

$$h(z)=\prod_{r=1}^{j}(1+v_rz),$$

we have

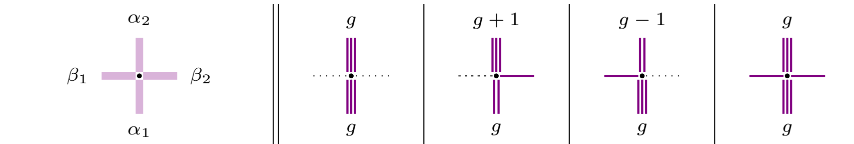
$$\text{r.h.s.}\,(5.8)=\frac{\mathfrak{D}_1^{(i_1)}\cdots\mathfrak{D}_1^{(i_\ell)}h(x_1)\cdots h(x_N)}{h(x_1)\cdots h(x_N)}.$$

Moreover, for any meromorphic function $\widetilde{h}$ we have

$$\mathfrak{D}_1^{(n)}\left(\widetilde{h}(x_1)\cdots\widetilde{h}(x_n)\right)=\frac{1}{2\pi\mathrm{i}}\oint_{\gamma_{\widetilde{h}}}\prod_{\alpha=1}^{n}\left(\widetilde{h}(x_\alpha)\,\frac{x_\alpha(1+sz)}{x_\alpha-z}\right)\frac{\widetilde{h}(qz)}{\widetilde{h}(z)}\,\frac{dz}{z(1+sz)},$$

where the curve $\gamma_{\widetilde{h}}$ encircles 0 and all poles of $\widetilde{h}(qz)/\widetilde{h}(z)$. The latter poles may include infinity, too. In other words, the integral over $\gamma_{\widetilde{h}}$ is equal to the sum of minus residues of the integrand at $x_1,\ldots,x_n$.

By iterating this integral representation, we can evaluate (5.8) and match the resulting expression with the q-moment formula (5.7). The equivalence of processes $\lambda_i^{(i,j)}$ and $\mathcal{H}^{\mathrm{ur}}_{\mathrm{HS}}(i,j)$ stated in Theorem 5.7 allows us to complete the proof. □



$\mathsf{L}^{\mathrm{ul}}_{x,v}(\alpha_1,\beta_1;\alpha_2,\beta_2)$	$\frac{1-q^g sv}{1+xv}$	$\frac{xv+q^g sv}{1+xv}$	$\frac{1-q^g}{1+xv}$	$\frac{xv+q^g}{1+xv}$

FIGURE 11. The stochastic vertex weights $\mathsf{L}^{\mathrm{ul}}_{x,v}$ for the up-left stochastic higher spin six vertex model.

5.3. **First row in sqW/sHL field.** Let us define an *up-left version of the stochastic higher spin six vertex model.* Take an up-left model in the sense of Definition 5.3, with the weights $L^{\mathrm{ul}}_{(i,j)} = \mathsf{L}^{\mathrm{ul}}_{x_i,v_j}$, given in Figure 11. We take this model with the same step-stationary boundary conditions (5.4). In fact, this model is essentially the same as the one from Definition 5.5:

Remark 5.9. When at most one path occupies each horizontal edge (as in our case), swapping the horizontal occupation numbers $0 \leftrightarrow 1$ is a bijection between up-left and up-right models. Their height functions are related as $\mathcal{H}^{\mathrm{ur}}_{\mathrm{HS}}(i,j) = j - \mathcal{H}^{\mathrm{ul}}_{\mathrm{HS}}(i,j)$. Moreover, the weights $\mathsf{L}^{\mathrm{ul}}_{x,v}$ become the weights $\mathsf{L}^{\mathrm{ur}}_{x,v}$ from Figure 10 after this swapping of horizontal occupations, and the inversion of the parameters $(x,v) \mapsto (x^{-1}, v^{-1})$.

However, it is convenient to work with the up-right and the up-left models separately, as in the sqW/sqW case they are genuinely different.

Theorem 5.10 (sqW/sHL first row)**.** *The first row marginal* $\{\lambda_1^{(i,j)}\}_{i\ge 1, j\ge 0}$ *of the sqW/sHL field has the same distribution as the height function* $\{\mathcal{H}^{\mathrm{ul}}_{\mathrm{HS}}(i,j)\}_{i\ge1,j\ge0}$ *of the up-left stochastic higher spin six vertex model with step-stationary boundary conditions.*

Proof. We use Yang–Baxter fields similarly to the approach taken in [BMP19, Section 7.3]. Let us specialize the general notation of Proposition 4.12. We need to match the stochastic vertex weight $\mathbf{L}$ of (4.10) with L^{ul}, and verify boundary conditions.

The random move $\lambda_1^{(i-1,j-1)} \to \lambda_1^{(i,j)}$, conditioned on $\lambda_1^{(i,j-1)}, \lambda_1^{(i-1,j)}$ is determined by the bijectivization of the Yang–Baxter equation (B.5) for $i > 1, j \ge 1$ and by the bijectivization of (B.6), if $i = 1$. We start with the first case, where in (4.10) we get (after canceling common factors)

$$\mathsf{wl}_{\{0,0,\infty\},\{j_1,j_2,\infty\}}(\{k_1,k_2,\infty\}) = \mathcal{R}_{x,v,s}(0,0;k_1,k_2)\,\frac{v^{j_1}}{1-sv}\,x^{j_2}\,\frac{(-s/x;q)_{j_2}}{(q;q)_{j_2}},$$

$$\mathsf{wr}_{\{0,0,\infty\},\{j_1,j_2,\infty\}}(\{k_1',k_2',\infty\}) = \mathcal{R}_{x,v,s}(k_2',k_1';j_2,j_1)\frac{v^{k_1'}}{1-sv}\,x^{k_2'}\,\frac{(-s/x;q)_{k_2'}}{(q;q)_{k_2'}}.$$

One readily sees that then (4.10) gives the stochastic weight $\mathsf{L}^{\mathrm{ul}}_{x,v}$.

For the boundary signature $\lambda_1^{(1,j)}$ case, configuration weights wl, wr become (after canceling common factors)

$$\mathsf{wl}_{\{0,0,\infty\},\{\varnothing,j_2,\infty\}}(\{k_1,k_2,\infty\}) = \mathcal{R}_{x,v,s}(0,0;k_1,k_2)\,x^{j_2},$$

$$\mathsf{wr}_{\{0,0,\infty\},\{\varnothing,j_2,\infty\}}(\{k_1',k_2',\infty\}) = \frac{v^{k_1'}}{1-sv}\,x^{k_2'},$$

which leads to the step-stationary boundary conditions (5.4) since $\lambda^{(i,0)} = \varnothing$ for all i. □

5.4. **Last row in sqW/sqW field.** Define the up-right stochastic weight by

$$\mathbb{L}^{\mathrm{ur}}_{x,y}(\alpha_1,\beta_1;\alpha_2,\beta_2) := \mathbf{1}_{\alpha_1+\beta_1=\alpha_2+\beta_2}\,\varphi_{q,xy,-sx}(\beta_2 \mid \alpha_1), \tag{5.9}$$

where φ is the q-beta-binomial distribution (A.1)–(A.2).

Definition 5.11 ([Pov13])**.** The *q-Hahn vertex model* is the up-right stochastic vertex model, in the sense of Definition 5.1, with weights $L^{\mathrm{ur}}_{i,j} = \mathbb{L}^{\mathrm{ur}}_{x_i,y_j}$. We consider *step-stationary* boundary conditions:

$$b^{\mathrm{v}}_j \sim \varphi_{q,x_1y_j,-sx_1}(\bullet \mid \infty) \qquad \text{and} \qquad b^{\mathrm{h}}_i = 0, \tag{5.10}$$

where the random variables for b^{v}_j are independent. Denote the corresponding height function by $\mathcal{H}^{\mathrm{ur}}_{q\text{-Hahn}}$.

Remark 5.12. The model of Definition 5.11 is equivalent to that of [BP18, Section 6.6.2], where parameters have been specialized as $\mathsf{s}^2_\alpha \to -sx_\alpha$ and $q^{J_\alpha} \to -y_\alpha/s$.

Theorem 5.13 (sqW/sqW last row)**.** *The last row marginal $\{\lambda^{(i,j)}_i\}_{i\ge1,j\ge0}$ of the sqW/sqW field has the same distribution as the height function $\{\mathcal{H}^{\mathrm{ur}}_{q\text{-Hahn}}(i,j)\}_{i\ge1,j\ge0}$ of the up-right q-Hahn vertex model.*

Proof. This follows from Theorem 5.7 which established an analogous result matching the last row of the sqW/sHL field and the height function of the up-right higher spin six vertex model. By fusion, the dual sHL functions turn into the dual sqW functions (cf. [BW17]). Therefore, the sqW/sHL field under fusion turns into the sqW/sqW field.

On the other hand, the same fusion procedure turns the up-right higher spin six vertex model into the up-right q-Hahn vertex model[3]. This completes the proof. ☐

In [BP18, Corollary 10.4] the multi-point q-moments of the up-right q-Hahn vertex model were expressed in terms of nested contour integrals:

Proposition 5.14 ([BP18], Corollary 10.4)**.** *Assume* $\min_\alpha |sx_\alpha| > q\max_\alpha |sx_\alpha|$*. For any* $i_1 \ge \ldots \ge i_\ell \ge 1$ *we have*

$$\mathbb{E}\prod_{k=1}^{\ell} q^{\mathcal{H}^{\mathrm{ur}}_{q\text{-Hahn}}(i_k,j)} = (-1)^\ell q^{\binom{\ell}{2}} \oint_{\gamma_1^+[-s\mathbf{x}]}\frac{dw_1}{2\pi\mathrm{i}}\cdots\oint_{\gamma_\ell^+[-s\mathbf{x}]}\frac{dw_\ell}{2\pi\mathrm{i}}\prod_{1\le A<B\le\ell}\frac{w_A-w_B}{w_A-qw_B} \times\prod_{k=1}^{\ell}\left(\frac{1}{w_k(1-w_k)}\prod_{\alpha=1}^{i_k}\frac{1-w_k}{1+w_k/(sx_\alpha)}\prod_{\alpha=1}^{j}\frac{1+w_ky_\alpha/s}{1-w_k}\right). \tag{5.11}$$

Integration contours encircle $-sx_1,-sx_2,\ldots$, and leave out 0,1 and are q-nested in the sense that $q\gamma^+_{k+1}[-s\mathbf{x}]$ is inside $\gamma^+_k[-s\mathbf{x}]$ for all $k=1,\ldots,\ell-1$.

Proposition 5.14 was obtained in [BP18] as a corollary (under fusion) of the multi-point q-moment formula (5.7) for the up-right higher spin six vertex model. Both of these q-moment formulas have several different proofs: via duality [CP16], manipulations with symmetric functions using Bethe Ansatz [BP18], or distributional matchings and difference operators [OP17]. Eigenrelations for the sqW polynomials provide yet another independent proof:

[3]For a practical explanation of fusion in the context of $\mathfrak{sl}_2$ stochastic vertex models see [BW17] and [CP16], [BP18] and references therein.

Alternative proof of Proposition 5.14. Similarly to the alternative proof of Proposition 5.8 given in Section 5.2, we will use eigenrelations of the sqW polynomials to compute q-moments. To express q-moments of the sqW/sqW field, we use formula (5.8), after replacing the function Π with the right-hand side of (2.30). The action of the difference operator $\mathfrak{D}_1$ (3.9) (in n variables) on a meromorphic function $\widetilde{h}$ can be written as

$$\mathfrak{D}_1\left(\widetilde{h}(x_1)\cdots\widetilde{h}(x_n)\right) = -\frac{1}{2\pi\mathrm{i}}\oint_{x_1,\dots,x_n}\prod_{\alpha=1}^{n}\left(\widetilde{h}(x_\alpha)\,\frac{x_\alpha(1+sz)}{x_\alpha - z}\right)\frac{\widetilde{h}(qz)}{\widetilde{h}(z)}\frac{dz}{z(1+sz)},$$

where the integration contour contains $x_1,\dots,x_n$, but doesn't contain 0 or any pole of $\widetilde{h}(qz)/\widetilde{h}(z)$. Using this formula repeatedly, we can match the q-moments of the marginal $\lambda_i^{(i,j)}$ to expression (5.11). The equivalence of processes between last row of the sqW/sqW field and height function of the q-Hahn vertex model stated in Theorem 5.13 yields the proof. $\square$

5.5. **First row in sqW/sqW field.** For our fourth and final vertex model, define the up-left stochastic weight by

$$\begin{aligned}\mathbb{L}^{\mathrm{ul}}_{x,y}(\alpha_1,\beta_1;\alpha_2,\beta_2) := \mathbf{1}_{\alpha_1+\beta_2=\alpha_2+\beta_1}\,&\frac{y^{\alpha_2}s^{\alpha_1}x^{\alpha_2-\alpha_1}\,q^{\beta_1\beta_2+\frac{1}{2}\alpha_1(\alpha_1-1)}\,(-s/x;q)_{\alpha_2}(-s/y;q)_{\beta_2}}{(-s/x;q)_{\alpha_1}(-s/y;q)_{\beta_1}(q;q)_{\beta_2}(-q/(sy);q)_{\beta_2-\alpha_2}}\\ &\times\frac{(s^2q^{\alpha_1+\beta_2};q)_\infty(xy;q)_\infty}{(-sy;q)_\infty(-sx;q)_\infty}\,{}_4\overline{\phi}_3\left(\begin{matrix}q^{-\beta_1};q^{-\beta_2},-sx,-q/(sy)\\ -s/y,q^{1+\alpha_1-\beta_1},-xq^{1-\beta_2-\alpha_1}/s\end{matrix}\,\Big|\,q,q\right),\end{aligned}\tag{5.12}$$

where ${}_4\overline{\phi}_3$ is the regularized q-hypergeometric function (A.4).

Remark 5.15. An expression equivalent to (5.12) for the stochastic weight $\mathbb{L}^{\mathrm{ul}}_{x,y}$ is given by

$$\mathbb{L}^{\mathrm{ul}}_{x,y}(g,\ell;g+L-\ell,L) = \sum_{k=0}^{\min(\ell,L)}\varphi_{q^{-1},q^g,-syq^{g-1}}(k\mid\ell)\,\psi_{q,-q^ks/y,-q^gs/x,s^2q^{g+k}}(L-k),\tag{5.13}$$

where we used the q-beta-binomial and the q-hypergeometric distributions (A.1), (A.6). This can be proved through simple manipulations of the q-Pochhammer terms. From (5.13) it is immediate to see that $\mathbb{L}^{\mathrm{ul}}_{x,y}$ possesses the sum to one property (5.1). The positivity of the weights (under certain restrictions on the parameters) follows from Proposition B.8.

Definition 5.16. The ${}_4\phi_3$ *vertex model* is the up-left stochastic vertex model, in the sense of Definition 5.3, with weights $L^{\mathrm{ur}}_{i,j} = \mathbb{L}^{\mathrm{ul}}_{x_i,y_j}$. We consider the same *step-stationary* boundary conditions as in (5.10). The height function of this model is denoted by $\mathcal{H}^{\mathrm{ul}}_\phi$.

Theorem 5.17 (sqW/sqW first row)**.** *Let $s\in(-\sqrt{q},0)$. The first row marginal $\{\lambda_1^{(i,j)}\}_{i,j\in\mathbb{Z}_{\geq 0}}$ of the sqW/sqW field has the same distribution as the height function $\{\mathcal{H}^{\mathrm{ul}}_\phi(i,j)\}_{i,j\in\mathbb{Z}_{\geq 0}}$ of the ${}_4\phi_3$ stochastic vertex model.*

Proof. The proof of this matching is similar to that of Theorem 5.10, and follows from Proposition 4.12. Namely, we specialize formula (4.10) using the Yang–Baxter equations (B.7), (B.8). For updates of "bulk" transition $\lambda_1^{(i-1,j-1)}\to\lambda_1^{(i,j)}$, for $i>1, j\geq 1$, conditioned on $\lambda_1^{(i,j-1)},\lambda_1^{(i-1,j)}$, the stochastic weight (4.10) uses

$$\mathsf{wl}_{\{0,0,\infty\},\{j_1,j_2,\infty\}}(\{k_1,k_2,\infty\}) = \mathbb{R}_{x,y,s}(0,0;k_1,k_2)\, y^{j_1} \frac{(-s/y;q)_{j_1}}{(q;q)_{j_1}}\, x^{j_2} \frac{(-s/x;q)_{j_2}}{(q;q)_{j_2}},$$
$$\mathsf{wr}_{\{0,0,\infty\},\{j_1,j_2,\infty\}}(\{k'_1,k'_2,\infty\}) = \mathbb{R}_{x,y,s}(k'_2,k'_1;j_2,j_1)\, y^{k'_1} \frac{(-s/y;q)_{k'_1}}{(q;q)_{k'_1}}\, x^{k'_2} \frac{(-s/x;q)_{k'_2}}{(q;q)_{k'_2}}.$$

Using the expression of the R-matrix $\mathbb{R}_{x,y,s}$ and summation identity (B.9) one can match $\mathbf{L}_{x,y}$ with $\mathbb{L}_{x,y}$. At the boundary $\lambda_1^{(1,j)}$, we use a stochastic bijectivization of (B.8) and therefore in this case we have

$$\mathsf{wl}_{\{0,0,\infty\},\{\varnothing,j_2,\infty\}}(\{k_1,k_2,\infty\}) = \mathbb{R}_{x,y,s}(0,0;k_1,k_2)\, x^{j_2},$$
$$\mathsf{wr}_{\{0,0,\infty\},\{\varnothing,j_2,\infty\}}(\{k'_1,k'_2,\infty\}) = y^{k'_1} \frac{(-s/y;q)_{k'_1}}{(q;q)_{k'_1}} \frac{(-sy;q)_\infty}{(s^2;q)_\infty}\, x^{k'_2},$$

that yields boundary conditions (5.10) after using again summation identity (B.9). ☐

5.6. **Push–block dynamics for sqW/sqW process.** Let us now present another, more explicit matching of last rows of the sqW/sqW field in a "Plancherel" (or "Poisson-type") continuous time limit. Here the dynamics of the last rows is matched to the corresponding continuous time limit of the q-Hahn TASEP. This construction is very similar to how the continuous time q-TASEP emerges from q-Whittaker processes in [BC14].

Consider the Borodin–Ferrari forward transition map (cf. Section 4.4)

$$\mathsf{U}^{\mathrm{fwd}}_{x,y}(\varkappa \to \nu \mid \lambda,\mu) = \frac{\mathbb{F}_{\nu/\lambda}(x)\mathbb{F}^*_{\nu/\mu}(y)}{\Pi(x;y)\sum_{\varkappa}\mathbb{F}_{\mu/\varkappa}(x)\mathbb{F}^*_{\lambda/\varkappa}(y)}, \tag{5.14}$$

where $\Pi(x;y) = \frac{(-sx;q)_\infty(-sy;q)_\infty}{(xy;q)_\infty(s^2;q)_\infty}$. In the limit as $y = -s + \varepsilon(1-q)$, $\varepsilon \to 0$, the dual sqW function at a single variable becomes (we use the notation $[r]_q = (1-q^r)/(1-q)$)

$$\mathbb{F}^*_{\lambda/\mu}(-s+\varepsilon(1-q)) = \begin{cases} 1 + \mathsf{O}(\varepsilon), & \lambda = \mu; \\ \varepsilon\, \dfrac{(-s)^{r-1}}{[r]_q} \dfrac{(q^{\mu_{i-1}-\lambda_i+1};q)_r}{(q^{\mu_{i-1}-\lambda_i}s^2;q)_r} + \mathsf{O}(\varepsilon^2), & \lambda = \mu + r\mathbf{e}_i \text{ for some } i, r > 0. \end{cases}$$

see (2.27). Take $y_j = -s + \varepsilon(1-q)$ for all j and rescale $M = \lfloor t/\varepsilon \rfloor$, $t \in \mathbb{R}_{\ge 0}$, in the sqW/sqW field. Thus, we get a continuous time dynamics on interlacing arrays $\lambda^1(t) \prec \lambda^2(t) \prec \cdots$, where at time t, each λ_i^k jumps to $\lambda_i^k + r$, $r \ge 1$, according to an exponential clock with rate (see (5.14))

$$\mathrm{rate}(\lambda^k \to \lambda^k + r\mathbf{e}_i \mid \lambda^{k-1}) = x_k^r \frac{(-s)^{r-1}}{[r]_q} \frac{(-q^{\lambda_i^k-\lambda_i^{k-1}}s/x_k\,,\, q^{\lambda_i^k-\lambda_{i+1}^k+1}\,,\, q^{\lambda_{i-1}^{k-1}-\lambda_i^k+1-r}\,;\,q)_r}{(q^{\lambda_i^k-\lambda_i^{k-1}+1}\,,\, q^{\lambda_i^k-\lambda_{i+1}^k}s^2\,,\, -q^{\lambda_{i-1}^{k-1}-\lambda_i^k-r}sx_k\,;\,q)_r}. \tag{5.15}$$

When an update occurs at level j bringing $\lambda^j \to \widetilde{\lambda}^j = \lambda^j + r\mathbf{e}_i$, the signature λ^{j+1} is instantaneously updated to $\widetilde{\lambda}^{j+1}$ in the following way:

- if $\widetilde{\lambda}_i^j \le \lambda_i^{j+1}$, then $\widetilde{\lambda}^{j+1} = \lambda^{j+1}$
- if $\widetilde{\lambda}_i^j > \lambda_i^{j+1}$, then assume $\widetilde{\lambda}_i^j - \lambda_i^{j+1} = m$ and set $\widetilde{\lambda}^{j+1} = \lambda^{j+1} + (m+\ell)\mathbf{e}_i$ with probability

$$\mathrm{prob}(\lambda^{j+1} \to \widetilde{\lambda}^{j+1} \mid \lambda^j \to \widetilde{\lambda}^j) = \lim_{\varepsilon\to 0} \frac{\mathbb{F}_{\widetilde{\lambda}^{j+1}/\widetilde{\lambda}^j}(x)\mathbb{F}^*_{\widetilde{\lambda}^{j+1}/\lambda^{j+1}}(y)}{\sum_{\eta=\lambda^{j+1}+(m+\ell')\mathbf{e}_i}\mathbb{F}_{\eta/\widetilde{\lambda}^j}(x)\mathbb{F}^*_{\eta/\lambda^{j+1}}(y)}\Bigg|_{y=-s+\varepsilon(1-q)}.$$

for any $\ell \geq 0$ (for ℓ large enough this probability vanishes). See Figure 12 for an illustration.

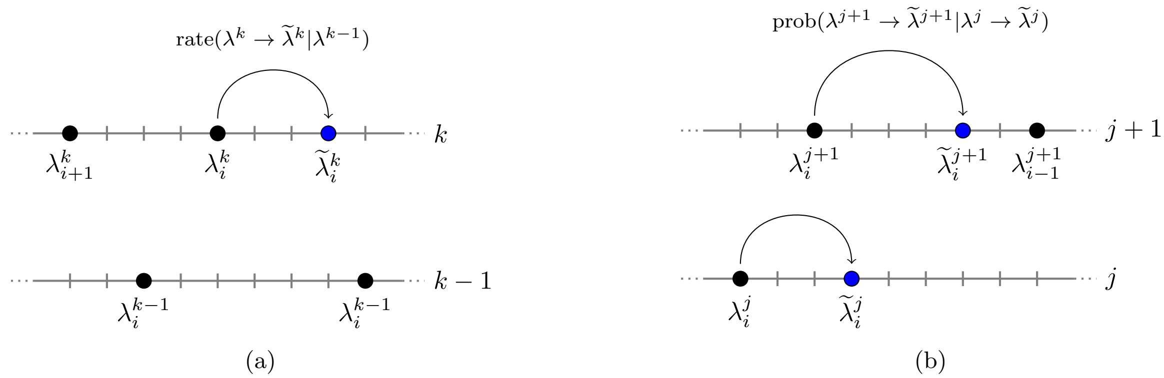


FIGURE 12. Push–block mechanism in the half-continuous sqW/sqW field. Each λ_i^k jumps to $\widetilde{\lambda}_i^k = \lambda_i^k + r$ at rate (5.15), which only depends on λ^{k-1}; see left panel. When a jump happens at level k and breaks interlacing, it triggers an instantaneous push at levels above to re-establish interlacing; see right panel.

When $s = 0$ and $q \in (0, 1)$ in our dynamics, we recover the continuous time q-Whittaker 2d-growth model introduced in [BC14, Definition 3.3.3]. Further setting $q = 0$ brings the original Borodin–Ferrari's push–block process corresponding to Schur measures [BF14]. Note that in our case, in contrast with the Schur and q-Whittaker situations, jumps are long range.

Restricting attention to the last rows (leftmost diagonal) of the array and setting $i = k$ in (5.15), we see that the rate only depends on λ_k^k and λ_{k-1}^{k-1}. Moreover, the pushing mechanism does not affect the leftmost diagonal of the array. Thus, the marginal evolution of the particles in the leftmost diagonal is an autonomous Markov process. Its jump rates are

$$\mathrm{rate}(\lambda_k^k \to \lambda_k^k + r \mid \lambda_{k-1}^{k-1}) = x_k^r \, \frac{(-s)^{r-1}}{[r]_q} \frac{(q^{\lambda_{k-1}^{k-1} - \lambda_k^k + 1 - r}; q)_r}{(-q^{\lambda_{k-1}^{k-1} - \lambda_k^k - r} s x_k; q)_r}.$$

These rates correspond to an inhomogeneous version of the continuous time q-Hahn TASEP studied in [BC16b], which is also a continuous time degeneration of the q-Hahn TASEP of [Cor14]. Thus, we see that the continuous time push–block dynamics in the sqW case agrees with the last row marginal evolution.

6. Spin Whittaker Functions from $q \to 1$ limit

In this section we introduce new one-parameter deformations of the $\mathfrak{gl}_n$ Whittaker functions [Jac67], [Kos78]. These deformations arise from our version of spin q-Whittaker polynomials in a scaling limit as $q \to 1$. The deformation parameter is denoted by $S > 0$.

6.1. **Whittaker functions.** Before proceeding with deformations of Whittaker functions, let us recall the usual $\mathfrak{gl}_N$ Whittaker functions. These functions play a central role in representation theory and integrable systems [Kos80], [Eti99], [Giv97] as well as are related to several models of random polymers [O'C12], [COSZ14], [OSZ14], [BC14].

The $\mathfrak{gl}_N$ Whittaker functions $\psi_{\lambda_1,\ldots,\lambda_N}(\underline{u}_N)$ are indexed[4] by N-tuples $\underline{u}_N = (u_{N,1},\ldots,u_{N,N}) \in \mathbb{R}^N$, depend on $\underline{\lambda} = (\lambda_1,\ldots,\lambda_N) \in \mathbb{C}^N$, and may be defined through the recursion (following from the Givental integral representation [Giv97], cf. [GKLO06]):

$$\psi_{\lambda_1,\ldots,\lambda_N}(\underline{u}_N) = \int_{\mathbb{R}^{N-1}} \psi_{\lambda_1,\ldots,\lambda_{N-1}}(\underline{u}_{N-1})\, Q^{N\to N-1}_{\lambda_N}(\underline{u}_N,\underline{u}_{N-1}) \prod_{k=1}^{N-1} du_{N-1,k}, \tag{6.1}$$

where

$$Q^{N\to N-1}_{\lambda}(\underline{u}_N,\underline{u}_{N-1}) = e^{\mathrm{i}\lambda\left(\sum_{i=1}^{N} u_{N,i} - \sum_{i=1}^{N-1} u_{N-1,i}\right)} \prod_{i=1}^{N-1} \exp\left\{-e^{u_{N-1,i}-u_{N,i}} - e^{u_{N,i+1}-u_{N-1,i}}\right\} \tag{6.2}$$

is known as the Baxter Q-operator. The function $\underline{\lambda} \mapsto \psi_{\underline{\lambda}}(\underline{u}_N)$ is an entire function of $\underline{\lambda} \in \mathbb{C}^N$ for all $\underline{u}_N \in \mathbb{R}^N$. For $N = 1$, we have $\psi_\lambda(u) = e^{\mathrm{i}\lambda u}$. For $N = 2$, the Whittaker functions can be expressed through the (single-variable) Bessel K function $K_v(z) = \frac{1}{2}\int_{-\infty}^{\infty} e^{xv} \exp\left(-\frac{z}{2}(e^x + e^{-x})\right) dx$.

For the Whittaker functions, $Q^{N\to N-1}_{\lambda_N}(\underline{u}_N,\underline{u}_{N-1})$ plays the role of a branching function like the single-variable sqW function $\mathbb{F}_{\nu/\mu}(x)$ (2.5) (here x plays the same role as λ_N, and ν,μ correspond to $\underline{u}_N,\underline{u}_{N-1}$). Note that the Whittaker functions are not indexed by ordered sequences of numbers $\underline{u}_N$. Rather, in the Baxter Q-operator, the interlacing condition among arrays $\underline{u}_{N-1},\underline{u}_N$ is replaced by the "mild interlacing". Namely, $Q^{N\to N-1}$ (6.2) decays doubly exponentially whenever $u_{N,i+1} > u_{N-1,i}$ or $u_{N-1,i} > u_{N,i}$.

The Whittaker functions satisfy the following analogue of the Cauchy identity due to Bump and Stade [Bum89], [Sta02], [GLO08]:

$$\int_{\mathbb{R}^N} e^{-e^{-u_{N,N}}}\,\overline{\psi_{\underline{\lambda}_N}(\underline{u}_N)}\,\psi_{\underline{\nu}_N}(\underline{u}_N) \prod_{j=1}^{N} du_{N,j} = \prod_{j,k=1}^{N} \Gamma(\mathrm{i}\nu_j - \mathrm{i}\lambda_k). \tag{6.3}$$

See also [COSZ14, (1.2)], [BC14, Section 4.2.1] for a generalization when one of the Whittaker functions is replaced by a certain integral coming from the limit of the torus product representation of Macdonald polynomials:

$$\theta_{\underline{Y}}(\underline{u}_N) := \int_{\mathbb{R}^N} \overline{\psi_{\underline{\nu}}(\underline{u}_N)} \prod_{i=1}^{T}\prod_{k=1}^{N} \Gamma(Y_i - \mathrm{i}\nu_k) \cdot \frac{1}{(2\pi)^N N!} \prod_{1\le A\ne B\le N} \frac{1}{\Gamma(\mathrm{i}\nu_A - \mathrm{i}\nu_B)}\, d\underline{\nu}, \tag{6.4}$$

where $\underline{Y} = (Y_1,\ldots,Y_T) \in \mathbb{R}^T$. We refer to $\theta_{\underline{Y}}(\underline{u}_N)$ as the *dual Whittaker function.*

Similar integral representations for dual spin Hall–Littlewood functions are found in [Bor17, Proposition 7.3], [BP18, Section 7.3].

The Whittaker functions are eigenfunctions of the $\mathfrak{gl}_N$ quantum Toda Hamiltonian $\mathcal{H}_2^{\mathrm{Toda}}$, see formula (1.1) in the Introduction.

Convention on multiplicative notation. The papers [COSZ14], [OSZ14] use *multiplicative parameters* $U_{N,i} = e^{u_{N,i}} \in \mathbb{R}_{>0}$ instead of the additive ones. In multiplicative notation, the integration in (6.1) and (6.3) is over the product measures of the form $\prod \frac{dU_{m,i}}{U_{m,i}}$. It is convenient for us to adopt multiplicative notation throughout most of the discussion of the spin Whittaker functions. We will often denote multiplicative variables and parameters by capital letters.

[4]To match the historical notation for Whittaker functions, here and in the discussion of the spin Whittaker functions we place the "variables" into the subscript of a Whittaker function, and the "index" in the parentheses.

6.2. **Signatures in continuous space.** In contrast with the usual Whittaker functions indexed by unordered N-tuples of reals, the spin Whittaker functions will be indexed by *nondecreasing* sequences of real numbers. Introduce the *Weyl chamber* of $\mathbb{R}^N_{\geq 1}$ by

$$\mathcal{W}_N := \{\underline{L}_N = (L_{N,i})_{1\leq i\leq N} \in \mathbb{R}^N_{\geq 1} \colon L_{N,N} \leq L_{N,N-1} \leq \ldots \leq L_{N,1}\}. \tag{6.5}$$

By $\mathring{\mathcal{W}}_N$ denote the interior of the Weyl chamber with strict inequalities in (6.5).

Given two sequences $\underline{L}_{N-1} \in \mathcal{W}_{N-1}$ and $\underline{L}_N \in \mathcal{W}_N$, we say that they *interlace* if

$$L_{N,i+1} \leq L_{N-1,i} \leq L_{N,i}, \qquad \text{for } 1 \leq i \leq N-1. \tag{6.6}$$

As in discrete setting, we denote interlacing by $\underline{L}_{N-1} \prec \underline{L}_N$. The interlacing relation is naturally extended to sequences of the same length by dropping the last inequality in (2.1).

We endow the Weyl chamber $\mathcal{W}_N$ with the measure $\frac{d\underline{L}_N}{\underline{L}_N} = \prod_{k=1}^N \frac{dL_{N,k}}{L_{N,k}}$. In most cases we do not explicitly indicate the integration domain $\mathcal{W}_N$ when the measure $\frac{d\underline{L}_N}{\underline{L}_N}$ is used.

Define the *continuous Gelfand-Tsetlin cone* as

$$\mathcal{GT}_N := \{\underline{\underline{L}}_N = (L_{k,i})_{1\leq i\leq k\leq N} \in \mathbb{R}^{N(N+1)/2}_{\geq 1} \colon L_{k+1,i+1} \leq L_{k,i} \leq L_{k+1,i}\}, \tag{6.7}$$

which is the set of interlacing sequences $\underline{L}_1 \prec \cdots \prec \underline{L}_N$. The set $\mathcal{GT}_N$ is endowed with the measure $\frac{d\underline{\underline{L}}_N}{\underline{\underline{L}}_N} = \prod_{1\leq i\leq j\leq N} \frac{dL_{j,i}}{L_{j,i}}$.

6.3. **Spin Whittaker functions.** We begin with a branching function from which we can recursively build spin Whittaker functions. The branching function is an analogue of the skew polynomial evaluated at a single variable.

Fix a deformation parameter $S > 0$ throughout the section. Let us denote

$$\mathcal{A}_{S,X}(u,v,z) := \frac{1}{\mathrm{B}(S+X,S-X)} \left(1-\frac{v}{z}\right)^{S-X-1} \left(1-\frac{u}{v}\right)^{S+X-1} \left(1-\frac{u}{z}\right)^{1-2S}, \tag{6.8}$$

where $1 \leq u < v < z$ are real, and $|X| < S$. Here $\mathrm{B}(\cdot,\cdot)$ is the beta function (A.7).

Definition 6.1. Let $|X| < S$ and $k \geq 1$. The *spin Whittaker branching functions* are given by

$$\mathfrak{f}_X(\underline{L}_k; \underline{L}_{k+1}) := \mathbf{1}_{\underline{L}_k \prec \underline{L}_{k+1}} \left(\frac{L_{k+1,k+1}\cdots L_{k+1,1}}{L_{k,k}\cdots L_{k,1}}\right)^{-X} \prod_{i=1}^k \mathcal{A}_{S,X}(L_{k+1,i+1}, L_{k,i}, L_{k+1,i}).$$

We now introduce the main object of the present section.

Definition 6.2 (Spin Whittaker functions)**.** For $N \geq 1$, consider parameters $X_1, \ldots, X_N$ and S such that $|X_i| < S$ for all i. The *spin Whittaker functions* $\mathfrak{f}_{X_1,\ldots,X_N}(\underline{L}_N)$, $\underline{L}_N \in \mathcal{W}_N$, are defined recursively by

$$\mathfrak{f}_{X_1}(L_{1,1}) := L_{1,1}^{-X_1} \tag{6.9}$$

for $N = 1$, and via the branching rule

$$\mathfrak{f}_{X_1,\ldots,X_N}(\underline{L}_N) := \int_{\underline{L}_{N-1}\prec\underline{L}_N} \mathfrak{f}_{X_1,\ldots,X_{N-1}}(\underline{L}_{N-1})\, \mathfrak{f}_{X_N}(\underline{L}_{N-1}; \underline{L}_N) \frac{d\underline{L}_{N-1}}{\underline{L}_{N-1}} \tag{6.10}$$

for $N \geq 2$.

Example 6.3 (Two-variable spin Whittaker function)**.** Let us compute the integral (6.10) for $N = 2$. Denote $\underline{X}_2 = (X, Y)$, $\underline{L}_2 = (u, u + \alpha)$, where $u \geq 1$, $\alpha > 0$. Then

$$\begin{aligned}&\mathfrak{f}_{X,Y}(u, u+\alpha)\\&= \frac{(u(u+\alpha))^{-Y}}{\mathrm{B}(S+Y, S-Y)} \left(1 - \frac{u}{u+\alpha}\right)^{1-2S} \int_u^{u+\alpha} v^{Y-X-1} \left(1 - \frac{v}{u+\alpha}\right)^{S-Y-1} \left(1 - \frac{u}{v}\right)^{S+Y-1} dv\\&= \frac{u^{-Y}(u+\alpha)^S}{\mathrm{B}(S+Y, S-Y)} \int_0^1 (u + t\alpha)^{-X-S} (1-t)^{S-Y-1} t^{S+Y-1} dt,\end{aligned}$$

where we changed the variable as $v = u + \alpha t$, $t \in [0, 1]$. The integral can now be evaluated using Euler's representation of the Gauss hypergeometric function ${}_2F_1$ (A.10). Let us also rename back $z = u + \alpha$. We have

$$\mathfrak{f}_{X,Y}(u, z) = (z/u)^S u^{-X-Y} {}_2F_1 \left(\begin{matrix} S+X\ ,\ S+Y \\ 2S \end{matrix} \middle| \, 1 - \frac{z}{u} \right). \tag{6.11}$$

When $|1 - z/u| \geq 1$, the hypergeometric function in (6.11) should be understood in the sense of analytic continuation.

We remark that most of the properties of the spin Whittaker functions given below in this section can be directly derived for $N = 2$ from known properties of the Gauss hypergeometric function ${}_2F_1$.

Proposition 6.4. *For $\underline{X}_N = (X_1, \ldots, X_N)$ with $|X_i| < S$, the spin Whittaker function $\mathfrak{f}_{\underline{X}_N}(\underline{L}_N)$ is well-defined and continuous in $\underline{L}_N \in \mathcal{W}_N$.*

In particular, we can first define $\mathfrak{f}_{\underline{X}_N}(\underline{L}_N)$ for $\underline{L}_N \in \mathring{\mathcal{W}}_N$, and then extend to the whole Weyl chamber by continuity. (Note that $\mathcal{A}_{S,X}(u, v, z)$ (6.8) might have a singularity at $u = z$.) The proof of Proposition 6.4 is based on the next two lemmas.

Lemma 6.5. *Let $\ell_1 > 0$ and let $f(\cdot)$ be a left continuous function on $\mathbb{R}_{\geq 1}$. Then, we have*

$$\lim_{\ell_3 \to \ell_1^-} \int_{\ell_3}^{\ell_1} \frac{d\ell_2}{\ell_2} \mathcal{A}_{S,X}(\ell_3, \ell_2, \ell_1) f(\ell_2) = f(\ell_1). \tag{6.12}$$

Proof. To compute the limit set $\ell_3 = \ell_1 - \delta$ for a small positive δ. After a change of variable $\ell_2 = \ell_1 - \delta(1 - \ell_2')$, the integral in (6.12) becomes

$$\frac{1}{\mathrm{B}(S+X, S-X)} \int_0^1 d\ell_2' \left(\frac{\ell_1}{\ell_1 - \delta(1 - \ell_2')} \right)^{S+X} (1 - \ell_2')^{S-X-1} \ell_2'^{S+X-1} f(\ell_1 - \delta(1 - \ell_2')).$$

Using the left continuity of f, we see that the integrand converges to $(1 - \ell_2')^{S-X-1} \ell_2'^{S+X-1} f(\ell_1)$ as $\delta \to 0$. The limiting integrand integrates to $\mathrm{B}(S + X, S - X)$, and so by the Dominated Convergence Theorem the lemma follows. □

Lemma 6.6. *Let $f : \mathcal{W}_{N-1} \to \mathbb{C}$ be left continuous in each of $L_{N-1,i}$. Define $F : \mathring{\mathcal{W}}_N \to \mathbb{C}$ as*

$$F(\underline{L}_N) = \int f(\underline{L}_{N-1})\, \mathfrak{f}_X(\underline{L}_{N-1}; \underline{L}_N) \frac{d\underline{L}_{N-1}}{\underline{L}_{N-1}}. \tag{6.13}$$

Then F is continuous and can be extended by continuity to $\mathcal{W}_N$.

Proof. For $\underline{L}_N \in \mathring{\mathcal{W}}_N$, the singularities of the integrand in (6.13) come only from the branching function $\mathfrak{f}_X(\underline{L}_{N-1}, \underline{L}_N)$ and they are of the form

$$\left(1 - \frac{L_{N-1,i}}{L_{N,i}}\right)^{S-X-1}, \qquad \text{or} \qquad \left(1 - \frac{L_{N,i+1}}{L_{N-1,i}}\right)^{S+X-1}$$

for some i. Because $|X| < S$ these singularities are summable. Therefore F, is continuous inside the interior $\mathring{\mathcal{W}}_N$ of the Weyl chamber.

To prove that F can be extended by continuity to $\mathcal{W}_N$ we first define, from small positive increments $\delta_1, \dots, \delta_{N-1}$, the quantities $d_i = \delta_i + \cdots + \delta_{N-1}$ for each $i = 1, \dots, N-1$. We aim to compute the limit

$$\lim_{\delta_1, \dots, \delta_{N-1} \to 0} F(L_{N,N}, L_{N,N-1} + d_{N-1}, \dots, L_{N,1} + d_1),$$

when some of the $L_{N,i}$'s are equal to each other. Before the limit, this function is equal to

$$\int_{L_{N,N}}^{L_{N,N-1}+\delta_{N-1}} \mathcal{A}_{S,X}(L_{N,N}, L_{N-1,N-1}, L_{N,N-1} + \delta_{N-1}) \frac{dL_{N-1,N-1}}{L_{N-1,N-1}}$$

$$\cdots \int_{L_{N,2}+d_2}^{L_{N,1}+d_2+\delta_1} \mathcal{A}_{S,X}(L_{N,2} + d_2, L_{N-1,1}, L_{N,1} + d_2 + \delta_1) \frac{dL_{N-1,1}}{L_{N-1,1}} f(\underline{L}_{N-1}) \left(\frac{\prod_{i=1}^{N-1} L_{N-1,i}}{\prod_{i=1}^{N} (L_{N,i} + d_i)} \right)^X .$$

For any i such that $L_{N,i} = L_{N,i+1}$, make the change of variables $L_{N-1,i} = L_{N,i} + d_{i+1} - \delta_i(1 - \ell_{N-1,i})$. As in the proof of Lemma 6.5, this removes all the corresponding singularities. Therefore, the limit as $\delta_1, \dots, \delta_{N-1} \to 0$ exists, is finite, and can be computed using (6.12). □

Proof of Proposition 6.4. For $N = 1$ the spin Whittaker function (6.9) is clearly continuous. Therefore, by Lemma 6.6, $\mathfrak{f}_{X_1,X_2}(\underline{L}_2)$ is well defined and continuous on $\mathcal{W}_2$. Proceeding by induction on N, we get the result of Proposition 6.4. □

The next corollary gives a Givental type representation of the spin Whittaker functions, obtained by writing down explicitly the recursive definition (6.10).

Corollary 6.7. *We have*

$$\mathfrak{f}_{X_1,\dots,X_N}(\underline{L}_N) = \int \prod_{1 \le k \le N} \frac{\prod_{i=1}^{k-1} L_{k-1,i}^{X_k}}{\prod_{i=1}^{k} L_{k,i}^{X_k}} \prod_{1 \le i \le k \le N-1} \mathcal{A}_{S,X_{k+1}}(L_{k+1,i+1}, L_{k,i}, L_{k+1,i}) \frac{d\underline{\underline{L}}_{N-1}}{\underline{\underline{L}}_{N-1}}. \quad (6.14)$$

Proof. Because the sequence of integrations as in (6.10) leading to $\mathfrak{f}_{\underline{X}_N}(\underline{L}_N)$ is (absolutely) convergent, so is the integration over the Gelfand-Tsetlin array $\mathcal{GT}_{N-1}$. The two integration procedures give the same result by the Fubini–Tonelli theorem. □

6.4. **Dual Spin Whittaker functions.** In this section we define a dual family of functions. Given interlacing sequences $\widetilde{\underline{L}}_k \prec \underline{L}_k$ of the same length k, introduce the *dual spin Whittaker branching functions*

$$\mathfrak{g}_Y(\widetilde{\underline{L}}_k; \underline{L}_k) := \mathbf{1}_{\widetilde{\underline{L}}_k \prec \underline{L}_k} \frac{1}{\Gamma(S-Y)} \left(\frac{\widetilde{L}_{k,k} \cdots \widetilde{L}_{k,1}}{L_{k,k} \cdots L_{k,1}} \right)^Y \left(1 - \frac{\widetilde{L}_{k,1}}{L_{k,1}} \right)^{S-Y-1} \times \prod_{i=2}^{k} \mathcal{A}_{S,-Y}(\widetilde{L}_{k,i}, L_{k,i}, \widetilde{L}_{k,i-1}). \quad (6.15)$$

For pairs of interlacing sequences $\underline{L}_{k-1} \prec \underline{L}_k$, $k \geq 1$, of different lengths, set

$$\mathfrak{g}_Y(\underline{L}_{k-1}; \underline{L}_k) \coloneqq \mathfrak{g}_Y((1, \underline{L}_{k-1}); \underline{L}_k).$$

Remark 6.8. One can also write $\mathfrak{g}_Y$ as

$$\mathfrak{g}_Y(\underline{\widetilde{L}}_k; \underline{L}_k) = \frac{L_{k,1}^{-Y}}{\Gamma(S-Y)} \left(1 - \frac{\widetilde{L}_{k,1}}{L_{k,1}}\right)^{S-Y-1} \mathfrak{f}_{-Y}(\underline{\ell}_{k-1}; \underline{\widetilde{L}}_k), \tag{6.16}$$

where $\underline{L}_k = (\underline{\ell}_{k-1}, L_{k,1})$.

Definition 6.9. Let $N \leq M$ and consider parameters $Y_1, \ldots, Y_M$ such that $|Y_i| < S$ for all i. The *dual spin Whittaker functions* are defined recursively by

$$\mathfrak{g}_{Y_1,\ldots,Y_M}(\underline{L}_N) = \begin{cases} \displaystyle\int \mathfrak{g}_{Y_1,\ldots,Y_{M-1}}(\underline{\widetilde{L}}_N) \mathfrak{g}_{Y_M}(\underline{\widetilde{L}}_N; \underline{L}_N) \frac{d\underline{\widetilde{L}}_N}{\underline{\widetilde{L}}_N} & \text{if } N < M, \\ \displaystyle\int \mathfrak{g}_{Y_1,\ldots,Y_{N-1}}(\underline{\widetilde{L}}_{N-1}) \mathfrak{g}_{Y_N}(\underline{\widetilde{L}}_{N-1}; \underline{L}_N) \frac{d\underline{\widetilde{L}}_{N-1}}{\underline{\widetilde{L}}_{N-1}} & \text{if } N = M. \end{cases} \tag{6.17}$$

In particular, for $M = N = 1$ we have

$$\mathfrak{g}_Y(L) = \mathfrak{g}_Y(1; L) = \frac{L^{-Y}(1 - L^{-1})^{S-Y-1}}{\Gamma(S-Y)}.$$

The next two propositions explain that $\mathfrak{g}_{Y_1,\ldots,Y_M}$ are well-defined as elements of the "dual" space of compactly supported continuous functions on the Weyl chamber $\mathcal{W}_N$.

Proposition 6.10. *Let $f(\underline{L}_N)$ be a compactly supported continuous function on $\mathcal{W}_N$. Then the function*

$$\underline{\widetilde{L}}_N \mapsto \int \mathfrak{g}_Y(\underline{\widetilde{L}}_N; \underline{L}_N) f(\underline{L}_N) \frac{d\underline{L}_N}{\underline{L}_N}, \tag{6.18}$$

is also compactly supported and continuous.

Proof. We evaluate the integral (6.18) using expression (6.16) for $\mathfrak{g}_Y$ as

$$\int \frac{dL_{N,1}}{L_{N,1}^{1+Y}} \frac{1}{\Gamma(S-Y)} \left(1 - \frac{\widetilde{L}_{N,1}}{L_{N,1}}\right)^{S-Y-1} \int f(\underline{\ell}_{N-1}, L_{N,1}) \frac{d\underline{\ell}_{N-1}}{\underline{\ell}_{N-1}} \mathfrak{f}_{-Y}(\underline{\ell}_{N-1}; \underline{\widetilde{L}}_k).$$

By Lemma 6.6, the integral in the variables $\underline{\ell}_{N-1}$ defines a family of continuous bounded functions in $\underline{\widetilde{L}}_N$, depending on $L_{N,1}$. The (improper) integral in $L_{N,1}$ is convergent both at $\widetilde{L}_{N,1}$ and ∞ (the latter because f vanishes for $L_{N,1}$ large enough). This proves the claim. □

Proposition 6.11. *Let $f(\underline{L}_N)$ be a compactly supported continuous function. Then the integral*

$$\int \mathfrak{g}_{Y_1,\ldots,Y_M}(\underline{L}_N) f(\underline{L}_N) \frac{d\underline{L}_N}{\underline{L}_N}$$

is absolutely convergent.

Proof. This follows from Proposition 6.10 applied recursively after expanding $\mathfrak{g}_{Y_1,\ldots,Y_M}$ using the branching rules (6.17). □

6.5. **Convergence of the sqW functions as $q \to 1$.** Here and in the following subsection we establish that the spin Whittaker functions $\mathfrak{f}_{\underline{X}}(\underline{L}_N)$ and $\mathfrak{g}_{\underline{Y}}(\underline{L}_N)$ are scaling limits, as $q \to 1$, of the spin q-Whittaker functions $\mathbb{F}_\lambda(x_1, \ldots, x_N)$ and $\mathbb{F}^*_\mu(y_1, \ldots, y_k)$, respectively. Recall that they also depend on two parameters, $q \in (0, 1)$ and $s \in (-1, 0)$.

First, this subsection we deal with the non-dual functions. Let us fix a scaling of all parameters.

Definition 6.12 (Scaling)**.** We consider the following renormalization of parameters:

$$x_i = q^{X_i}, \qquad s = -q^S, \qquad \lambda^i_j = \left\lfloor \log_q(1/L_{i,j}) \right\rfloor. \tag{6.19}$$

We will assume throughout that

$$S > 0, \qquad |X_i| < S, \qquad \text{and} \qquad 1 \le L_{i+1,j+1} \le L_{i,j} \le L_{i+1,j}$$

for all i, j. Therefore, the pre-limit quantities in (6.19) satisfy $s \in (0, 1)$, $x_i \in (-s, -s^{-1})$, and $0 \le \lambda^{i+1}_{j+1} \le \lambda^i_j \le \lambda^{i+1}_j$.

For any triple of real numbers $1 \le \ell_3 \le \ell_2 \le \ell_1$, set $n_i := \left\lfloor \log_q(1/\ell_i) \right\rfloor$ (so $0 \le n_3 \le n_2 \le n_1$).

Lemma 6.13. *With the above notation, for any function $f : \mathbb{Z} \to \mathbb{R}$ we have*

$$\sum_{n_2=n_3}^{n_1} f(n_2) = \int_{\ell_3}^{\ell_1} \frac{1}{\Delta_q(\ell_3, \ell_2, \ell_1)} f(\left\lfloor \log_q(1/\ell_2) \right\rfloor) \frac{d\ell_2}{\ell_2}, \tag{6.20}$$

where

$$\Delta_q(\ell_3, \ell_2, \ell_1) := \int_{\max(\ell_3, q^{-n_2})}^{\min(\ell_1, q^{-n_2-1})} \frac{d\ell_2'}{\ell_2'} = \begin{cases} -\log q & \text{if } n_3 < n_2 < n_1; \\ \log(q^{n_1}\ell_1) & \text{if } n_3 < n_2 = n_1; \\ -\log(q^{n_3+1}\ell_3) & \text{if } n_3 = n_2 < n_1; \\ \log(\ell_1/\ell_3) & \text{if } n_3 = n_2 = n_1. \end{cases} \tag{6.21}$$

When $\ell_3 = \ell_1$, the integral in (6.20) *is understood in the limiting sense.*

Proof. This follows by observing that Δ_q is the measure of intervals where the function $\ell_2 \mapsto \left\lfloor \log_q(1/\ell_2) \right\rfloor$ is constant, and simultaneously ℓ_2 lies in the interval $[\ell_3, \ell_1]$. □

The rescaled spin q-Whittaker functions are defined recursively as

$$\mathfrak{f}^{(q)}_{X_N}(\underline{L}_{N-1}; \underline{L}_N) = \prod_{k=1}^{N-1} \frac{1}{\Delta_q(L_{N,k+1}, L_{N-1,k}, L_{N,k})} \mathbb{F}_{\lambda^N/\lambda^{N-1}}(x_N) \Big|_{\text{scaling (6.19)}};$$

$$\mathfrak{f}^{(q)}_{X_1}(L_{1,1}) = x_1^{\lambda^1_1} \Big|_{\text{scaling (6.19)}} = q^{X_1 \left\lfloor \log_q(1/L_{1,1}) \right\rfloor};$$

$$\mathfrak{f}^{(q)}_{X_1,\ldots,X_N}(\underline{L}_N) = \int \mathfrak{f}^{(q)}_{X_1,\ldots,X_{N-1}}(\underline{L}_{N-1}) \, \mathfrak{f}^{(q)}_{X_N}(\underline{L}_{N-1}; \underline{L}_N) \frac{d\underline{L}_{N-1}}{\underline{L}_{N-1}},$$

The next theorem is the main result of this subsection:

Theorem 6.14. *We have*

$$\lim_{q \to 1} \mathfrak{f}^{(q)}_{X_1,\ldots,X_N} = \mathfrak{f}_{X_1,\ldots,X_N}, \tag{6.22}$$

uniformly on any compact subset of $\mathcal{W}_N$.

Pointwise convergence in (6.22) is a consequence of a simpler result stated in Lemma 2.2 of [BC16a] (reproduced as Lemma C.1 in Appendix C):

$$\lim_{q\to 1} \frac{(\ell q^A;q)_\infty}{(\ell q^B;q)_\infty} = (1-\ell)^{B-A}, \tag{6.23}$$

for any $\ell \in (0,1)$ and $A, B > 0$.

By (6.23) and through a repeated use of the identity

$$\frac{(q^a;q)_n}{(q^b;q)_n} = \mathbf{1}_{n=0} + \mathbf{1}_{n\geq 1} \frac{\Gamma_q(b)}{\Gamma_q(a)} (1-q)^{b-a} \frac{(q^{b+n};q)_\infty}{(q^{a+n};q)_\infty}, \tag{6.24}$$

where Γ_q is the q-Gamma function (A.5), one readily gets the pointwise convergence of the branching function $\mathfrak{f}_X^{(q)}(\underline{L}_{N-1};\underline{L}_N)$ to $\mathfrak{f}_X(\underline{L}_{N-1};\underline{L}_N)$. Nevertheless, for the finer uniform convergence result of Theorem 6.14, a slightly more accurate analysis of ratios of q-Pochhammer symbols appearing in the sqW functions is required. We postpone this technical discussion to Appendix C. Let us summarize the main technical result proven in Appendix C:

Proposition 6.15. *Let $f(\underline{L}_{N-1})$ be a continuous function on $\mathcal{W}_{N-1}$. Then for any $\underline{L}_N \in \mathcal{W}_N$ we have*

$$\lim_{q\to 1} \int f(\underline{L}_{N-1})\, \mathfrak{f}_X^{(q)}(\underline{L}_{N-1};\underline{L}_N) \frac{d\underline{L}_{N-1}}{\underline{L}_{N-1}} = \int f(\underline{L}_{N-1})\, \mathfrak{f}_X(\underline{L}_{N-1};\underline{L}_N) \frac{d\underline{L}_{N-1}}{\underline{L}_{N-1}}, \tag{6.25}$$

and the convergence is uniform on compact subsets of $\mathcal{W}_N$.

The continuous function f in Proposition 6.15 can also be replaced by a uniformly converging sequence:

Corollary 6.16. *Let $f^{(q)}(\underline{L}_{N-1})$ be a sequence uniformly convergent as $q \to 1$ on compact subsets of $\mathcal{W}_{N-1}$ to a continuous function $f(\underline{L}_{N-1})$. Then*

$$\lim_{q\to 1} \int f^{(q)}(\underline{L}_{N-1})\, \mathfrak{f}_X^{(q)}(\underline{L}_{N-1};\underline{L}_N)\, \frac{d\underline{L}_{N-1}}{\underline{L}_{N-1}} = \int f(\underline{L}_{N-1})\, \mathfrak{f}_X(\underline{L}_{N-1};\underline{L}_N)\, \frac{d\underline{L}_{N-1}}{\underline{L}_{N-1}}$$

and the convergence is uniform on compact subsets of $\mathcal{W}_N$.

Proof. This follows from Proposition 6.15 and the fact that for fixed $\underline{L}_N \in \mathcal{W}_N$, the functions $\underline{L}_{N-1} \mapsto \mathfrak{f}_X^{(q)}(\underline{L}_{N-1};\underline{L}_N)$ and $\underline{L}_{N-1} \mapsto \mathfrak{f}_X(\underline{L}_{N-1};\underline{L}_N)$ are compactly supported on $\mathcal{W}_{N-1}$. □

Proof of Theorem 6.14. For $N = 1$ we have

$$\mathfrak{f}_{X_1}^{(q)}(L) = q^{X_1 \lfloor \log_q(1/L) \rfloor} \xrightarrow[q\to 1]{} L^{-X_1} = \mathfrak{f}_{X_1}(L),$$

uniformly with respect to $L \geq 1$ varying in any compact domain. Corollary 6.16 then implies Theorem 6.14 by induction on N. □

6.6. **Convergence of the dual sqW functions as $q \to 1$.** We now establish the convergence of functions $\mathbb{F}^*$ to the dual spin Whittaker functions $\mathfrak{g}$. The scaling of parameters we adopt is that of Definition 6.12. For consistency with the previous sections, dual functions will depend on y variables for which the scaling is

$$y_i = q^{Y_i}, \qquad |Y_i| < S. \tag{6.26}$$

For two interlacing arrays $\underline{\widetilde{L}}_k \prec \underline{L}_k$ define the rescaled dual spin Whittaker branching functions

$$\mathfrak{g}^{(q)}_{Y_k}(\underline{\widetilde{L}}_k; \underline{L}_k) = (1-q)^{S-Y_k} \left(\prod_{j=1}^{k} \frac{1}{\Delta_q(\widetilde{L}_{k,j}, L_{k,j}, \widetilde{L}_{k,j-1})} \right) \mathbb{F}^*_{\lambda^k/\widetilde{\lambda}^k}(y_k) \Big|_{\text{scaling (6.19),(6.26)}}, \tag{6.27}$$

where, by agreement, $\widetilde{L}_{k,0} = \infty$, and Δ_q is given by (6.21). In particular, the rescaled one-variable function is (assuming $L > 1$ and q close enough to 1)

$$\mathfrak{g}^{(q)}_Y(L) = \mathfrak{g}^{(q)}_Y(1; L) = (1-q)^{S-Y} \frac{1}{(-\log q)} \frac{(q^{S-Y};q)_{\lfloor \log_q(1/L) \rfloor}}{(q;q)_{\lfloor \log_q(1/L) \rfloor}} q^{Y \lfloor \log_q(1/L) \rfloor}.$$

For interlacing arrays of different lengths $\underline{L}_{k-1} \prec \underline{L}_k$, we set $\mathfrak{g}^{(q)}_Y(\underline{L}_{k-1}; \underline{L}_k) = \mathfrak{g}^{(q)}_Y((1, \underline{L}_{k-1}); \underline{L}_k)$, as before. Define the rescaled dual spin q-Whittaker functions recursively as

$$\mathfrak{g}^{(q)}_{Y_1,\dots,Y_M}(\underline{L}_N) = \begin{cases} \displaystyle\int \mathfrak{g}^{(q)}_{Y_1,\dots,Y_{M-1}}(\underline{\widetilde{L}}_N) \mathfrak{g}^{(q)}_{Y_M}(\underline{\widetilde{L}}_N; \underline{L}_N) \frac{d\underline{\widetilde{L}}_N}{\underline{\widetilde{L}}_N} & \text{if } N < M, \\ \displaystyle\int \mathfrak{g}^{(q)}_{Y_1,\dots,Y_{N-1}}(\underline{\widetilde{L}}_{N-1}) \mathfrak{g}^{(q)}_{Y_N}(\underline{\widetilde{L}}_{N-1}; \underline{L}_N) \frac{d\underline{\widetilde{L}}_{N-1}}{\underline{\widetilde{L}}_{N-1}} & \text{if } N = M. \end{cases}$$

The next result establishes a weak convergence of rescaled branching functions $\mathfrak{g}^{(q)}$.

Theorem 6.17. *Let $f(\underline{L}_N)$ be a compactly supported continuous function on $\mathcal{W}_N$. Then*

$$\lim_{q\to 1} \int \mathfrak{g}^{(q)}_Y(\underline{\widetilde{L}}_N; \underline{L}_N) f(\underline{L}_N) \frac{d\underline{L}_N}{\underline{L}_N} = \int \mathfrak{g}_Y(\underline{\widetilde{L}}_N; \underline{L}_N) f(\underline{L}_N) \frac{d\underline{L}_N}{\underline{L}_N}, \tag{6.28}$$

and the convergence is uniform with respect to $\underline{\widetilde{L}}_N$.

Proof. We start by rewriting the branching function $\mathfrak{g}^{(q)}_Y(\underline{\widetilde{L}}_N; \underline{L}_N)$ as (this follows from straightforward algebraic manipulations with (6.27))

$$q^{Y\lambda_1^k} \frac{(q^{S-Y};q)_{\lambda_1^k - \widetilde{\lambda}_1^k}}{(q;q)_{\lambda_1^k - \widetilde{\lambda}_1^k}} \frac{(1-q)^{S-Y}}{\Delta_q(\widetilde{L}_{k,1}, L_{k,1}, \infty)} \mathfrak{f}^{(q)}_{-Y}(\underline{\ell}_{k-1}; \underline{\widetilde{L}}_k).$$

The integral in the left-hand side of (6.28) becomes

$$\int \frac{dL_{k,1}}{L^1_{k,1}} \left(q^{Y\lambda_1^k} \frac{(q^{S-Y};q)_{\lambda_1^k - \widetilde{\lambda}_1^k}}{(q;q)_{\lambda_1^k - \widetilde{\lambda}_1^k}} \frac{(1-q)^{S-Y}}{\Delta_q(\widetilde{L}_{k,1}, L_{k,1}, \infty)} \right) \int \frac{d\underline{\ell}_{k-1}}{\underline{\ell}_{k-1}} \mathfrak{f}^{(q)}_{-Y}(\underline{\ell}_{k-1}; \underline{\widetilde{L}}_k) f(\underline{\ell}_{k-1}, L_{k,1}). \tag{6.29}$$

The inner integral involving the function $\mathfrak{f}^{(q)}_{-Y}$ is uniformly (with respect to $\underline{\widetilde{L}}_k$) convergent to

$$\int \frac{d\underline{\ell}_{k-1}}{\underline{\ell}_{k-1}} \mathfrak{f}_{-Y}(\underline{\ell}_{k-1}; \underline{\widetilde{L}}_k) f(\underline{\ell}_{k-1}, L_{k,1})$$

by virtue of Proposition 6.15. On the other hand, the term inside the parentheses in (6.29) is uniformly convergent to

$$\frac{L_{k,1}^{-Y}}{\Gamma(S-Y)} \left(1 - \widetilde{L}_{k,1}/L_{k,1}\right)^{S-Y-1},$$

when $L_{k,1}$ is kept away from $\widetilde{L}_{k,1}$. Moreover, the term inside the parentheses is absolutely bounded by const $\times (1 - \widetilde{L}_{k,1}/L_{k,1})^{S-Y-1}$ when $L_{k,1}$ approaches $\widetilde{L}_{k,1}$, thanks to Lemma C.2. Since the resulting term after the $q \to 1$ limit coincides with the expression (6.16) for the dual branching function $\mathfrak{g}_Y(\underline{\widetilde{L}}_N; \underline{L}_N)$, we are done. □

Similarly to Corollary 6.16, we can let the test function f depend on q:

Corollary 6.18. *Let $f^{(q)}(\underline{L}_N)$ converge, as $q \to 1$, to a compactly supported continuous function $f(\underline{L}_N)$, uniformly on $\mathcal{W}_N$. Then*

$$\lim_{q\to 1} \int \mathfrak{g}_Y^{(q)}(\underline{\widetilde{L}}_N; \underline{L}_N) f^{(q)}(\underline{L}_N) \frac{d\underline{L}_N}{\underline{L}_N} = \int \mathfrak{g}_Y(\underline{\widetilde{L}}_N; \underline{L}_N) f(\underline{L}_N) \frac{d\underline{L}_N}{\underline{L}_N}$$

and the convergence is uniform with respect to $\underline{\widetilde{L}}_N$.

6.7. **Properties of the spin Whittaker functions.** In this subsection we describe the properties of the spin Whittaker functions which follow in the $q \to 1$ limit from the corresponding properties of the spin q-Whittaker functions.

Proposition 6.19 (Symmetry and shifting)**.** *The spin Whittaker function $\mathfrak{f}_{X_1,\ldots,X_N}(\underline{L}_N)$ is symmetric in the X_i's for all $\underline{L}_N \in \mathcal{W}_N$. They also satisfy the shifting property:*

$$\mathfrak{f}_{X_1,\ldots,X_N}(a\underline{L}_N) = a^{-X_1-\ldots-X_N} \mathfrak{f}_{X_1,\ldots,X_N}(\underline{L}_N), \qquad a > 1.$$

Proof. The symmetry follows from the corresponding symmetry of the sqW polynomial $\mathbb{F}_\lambda(x_1, \ldots, x_N)$, which ultimately is a consequence of the Yang–Baxter equation. The shifting property can either be deduced from Proposition 2.9, or obtained in a similar way by noting that the branching spin Whittaker functions themselves satisfy $\mathfrak{f}_X(a\underline{L}_k; a\underline{L}_k) = a^{-X} \mathfrak{f}_X(\underline{L}_k; \underline{L}_k)$. □

We now turn to Cauchy type identities for the spin Whittaker functions.

Theorem 6.20 (Skew Cauchy type identity)**.** *Assume $|X|, |Y| < S$ and $X + Y > 0$. Then, for any $\underline{L}_{N-1}, \underline{\widetilde{L}}_N$ we have*

$$\int \mathfrak{f}_X(\underline{L}_{N-1}; \underline{L}_N) \mathfrak{g}_Y(\underline{\widetilde{L}}_N; \underline{L}_N) \frac{d\underline{L}_N}{\underline{L}_N} = \frac{\Gamma(X+Y)\Gamma(2S)}{\Gamma(S+X)\Gamma(S+Y)} \int \mathfrak{f}_X(\underline{\widetilde{L}}_{N-1}; \underline{\widetilde{L}}_N) \mathfrak{g}_Y(\underline{\widetilde{L}}_{N-1}; \underline{L}_{N-1}) \frac{d\underline{\widetilde{L}}_{N-1}}{\underline{\widetilde{L}}_{N-1}} \quad (6.30)$$

and, when $N = 1$ we have

$$\int \mathfrak{f}_X(L_{1,1}) \mathfrak{g}_Y(\widetilde{L}_{1,1}; L_{1,1}) \frac{dL_{1,1}}{L_{1,1}} = \frac{\Gamma(X+Y)}{\Gamma(S+X)} \mathfrak{f}_X(\widetilde{L}_{1,1}). \quad (6.31)$$

Proof. We first observe that (6.31) is equivalent to the integral representation of $\mathrm{B}(S-Y, X+Y)$.

In order to prove the general case (6.30) we use Corollaries 6.16 and 6.18. Take a compactly supported continuous test function $\phi(\underline{L}_{N-1})$, and set

$$\Phi_{\mathfrak{f}}(\underline{L}_N) := \int \phi(\underline{L}_{N-1}) \mathfrak{f}_X(\underline{L}_{N-1}; \underline{L}_N) \frac{d\underline{L}_{N-1}}{\underline{L}_{N-1}}, \quad \Phi_{\mathfrak{g}}(\underline{\widetilde{L}}_{N-1}) := \int \mathfrak{g}_Y(\underline{\widetilde{L}}_{N-1}; \underline{L}_{N-1}) \phi(\underline{L}_{N-1}) \frac{d\underline{L}_{N-1}}{\underline{L}_{N-1}}.$$

Analogously define $\Phi_{\mathfrak{f}}^{(q)}$ and $\Phi_{\mathfrak{g}}^{(q)}$ by substituting respectively $\mathfrak{f}_X$ and $\mathfrak{g}_Y$ with $\mathfrak{f}_X^{(q)}$ and $\mathfrak{g}_Y^{(q)}$ in the above formulas. It follows from the skew Cauchy Identity for sqW functions (Proposition 2.21) that

$$\int \Phi_{\mathfrak{f}}^{(q)}(\underline{L}_N) \mathfrak{g}_Y^{(q)}(\underline{\widetilde{L}}_N; \underline{L}_N) \frac{d\underline{L}_N}{\underline{L}_N} = \frac{\Gamma_q(X+Y)\Gamma_q(2S)}{\Gamma_q(S+X)\Gamma_q(S+Y)} \int \mathfrak{f}_X^{(q)}(\underline{L}_{N-1}; \underline{\widetilde{L}}_N) \Phi_{\mathfrak{g}}^{(q)}(\underline{L}_{N-1}) \frac{d\underline{\widetilde{L}}_{N-1}}{\underline{\widetilde{L}}_{N-1}}. \quad (6.32)$$

By Corollary 6.18 we have $\Phi_{\mathfrak{g}}^{(q)} \to \Phi_{\mathfrak{g}}$ uniformly, and further $\Phi_{\mathfrak{g}}$ is compactly supported and continuous by Proposition 6.10. This implies, by Corollary 6.16, that the right-hand side of (6.32) converges to

$$\frac{\Gamma(X+Y)\Gamma(2S)}{\Gamma(S+X)\Gamma(S+Y)} \int \mathfrak{f}_X(\underline{\widetilde{L}}_{N-1}; \underline{\widetilde{L}}_N) \Phi_{\mathfrak{g}}(\underline{\widetilde{L}}_{N-1}) \frac{d\underline{\widetilde{L}}_{N-1}}{\underline{\widetilde{L}}_{N-1}}.$$

The integral in the left-hand side of (6.32) is absolutely convergent when $X+Y>0$. Since $\Phi_{\mathfrak{f}}^{(q)} \to \Phi_{\mathfrak{f}}$ uniformly by Proposition 6.15, Corollary 6.18 implies that the left-hand side of (6.32) converges to

$$\int \Phi_{\mathfrak{f}}(\underline{L}_N) \mathfrak{g}_Y(\underline{\widetilde{L}}_N; \underline{L}_N) \frac{d\underline{L}_N}{\underline{L}_N}.$$

Since the function ϕ was arbitrary, equality (6.30) follows. □

Corollary 6.21 (Full Cauchy type identity)**.** *Let $N \le M$ and $|X_i|, |Y_j| < S$, $X_i + Y_j > 0$ for all i, j. We have*

$$\int \mathfrak{f}_{X_1,\ldots,X_N}(\underline{L}_N)\, \mathfrak{g}_{Y_1,\ldots,Y_M}(\underline{L}_N)\, \frac{d\underline{L}_N}{\underline{L}_N} = \prod_{j=1}^{M} \frac{\Gamma(X_1+Y_j)}{\Gamma(S+X_1)} \left(\prod_{i=2}^{N} \frac{\Gamma(X_i+Y_j)\Gamma(2S)}{\Gamma(S+X_i)\Gamma(S+Y_j)} \right). \tag{6.33}$$

Proof. Immediately follows from Theorem 6.20 and the branching rules for the functions $\mathfrak{f}, \mathfrak{g}$. □

We also have an identity involving a single spin Whittaker function:

Proposition 6.22. *Let $|X_i| < S$. Then we have*

$$\int_{L_{N,N}=1} \mathfrak{f}_{X_1,\ldots,X_N}(\underline{L}_N) \prod_{j=1}^{N-1} \left(1 - \frac{L_{N,j+1}}{L_{N,j}}\right)^{2S-1} \frac{dL_{N,j}}{L_{N,j}^{1+S}} = \frac{\Gamma(S+X_1)\cdots\Gamma(S+X_N)}{\Gamma(SN+X_1+\cdots+X_N)}.$$

Proof. This is a scaling limit of Proposition 2.11. □

We now consider the scaling limits of eigenrelations for the sqW functions stated in Theorems 3.9 and 3.10. This produces two operators acting in the X_i variables which are diagonal in the spin Whittaker functions. For the next definition we use the shift operator

$$\mathcal{T}_X f(X) \coloneqq f(X+1). \tag{6.34}$$

Definition 6.23. For any $N \ge 1$ set

$$\mathcal{D}_1 \coloneqq \sum_{i=1}^{N} \prod_{\substack{j=1 \\ j \ne i}}^{N} \frac{X_i + S}{X_i - X_j}\, \mathcal{T}_{X_i}, \qquad \overline{\mathcal{D}}_1 \coloneqq \sum_{i=1}^{N} \prod_{\substack{j=1 \\ j \ne i}}^{N} \frac{X_i - S}{X_i - X_j}\, \mathcal{T}_{X_i}^{-1}.$$

The next proposition represents a partial generalization of eigenrelations satisfied by Whittaker functions (e.g., see [KL01]).

Proposition 6.24 (Eigenrelations for spin Whittaker functions)**.** *We have*

$$\mathcal{D}_1 \mathfrak{f}_{X_1,\ldots,X_N}(\underline{L}_N) = L_{N,N}^{-1}\, \mathfrak{f}_{X_1,\ldots,X_N}(\underline{L}_N),$$
$$\overline{\mathcal{D}}_1 \mathfrak{f}_{X_1,\ldots,X_N}(\underline{L}_N) = L_{N,1}\, \mathfrak{f}_{X_1,\ldots,X_N}(\underline{L}_N).$$

Proof. We easily see that operators $\mathcal{D}_1, \overline{\mathcal{D}}_1$ are limiting forms of $\mathfrak{D}_1, \overline{\mathfrak{D}}_1$ (Definition 3.6) under the scaling (6.19). At the same time we have $q^{\lambda_N} \to L_{N,N}^{-1}$ and $q^{-\lambda_1} \to L_{N,1}$ under the same scaling. Therefore, (3.11), (3.15) and convergence (6.22) imply the claimed eigenrelation. □

6.8. **Formal reduction to the usual Whittaker functions.** Just like the sqW polynomials reduce to the q-Whittaker polynomials setting $s = 0$, it should be possible to prove that, under the correct scaling, our spin Whittaker functions converge to the Whittaker functions. An evidence for this is suggested by the following computation.

Set

$$L_{k,i} = S^{k+1-2i} e^{u_{k,i}}, \qquad X_k = -\mathrm{i}\lambda_k, \tag{6.35}$$

then, in the limit $S \to \infty$ we have

$$\begin{aligned}
\left(\frac{L_{k,k} \cdots L_{k,1}}{L_{k+1,k+1} \cdots L_{k+1,1}} \right)^{X_{k+1}} &\longrightarrow \exp\left\{ \mathrm{i}\lambda_k \left(\sum_{i=1}^{k+1} u_{k+1,i} - \sum_{i=1}^{k} u_{k,i} \right) \right\}; \\
\left(1 - \frac{L_{k,i}}{L_{k+1,i}} \right)^{S - X_{k+1} - 1} &\longrightarrow \exp\left\{ -e^{u_{k,i} - u_{k+1,i}} \right\}; \\
\left(1 - \frac{L_{k+1,i+1}}{L_{k,i}} \right)^{S + X_{k+1} - 1} &\longrightarrow \exp\left\{ -e^{u_{k+1,i+1} - u_{k,i}} \right\}; \\
\left(1 - \frac{L_{k+1,i+1}}{L_{k+1,i}} \right)^{1 - 2S} &\longrightarrow 1; \\
4^S S^{\frac{1}{2}} \, \mathrm{B}(S + X, S - X) &\longrightarrow 2\sqrt{\pi}.
\end{aligned} \tag{6.36}$$

All the limits in (6.36) are straightforward (note that the last one requires the Stirling approximation). Thus, the branching function $\mathfrak{f}$, rescaled by a factor depending solely on S, converges locally uniformly to the Baxter Q-operator $Q^{N \to N-1}$ (6.2) for the usual Whittaker functions:

$$\left(\frac{4\pi}{S 16^S} \right)^{\frac{N-1}{2}} \mathfrak{f}_{X_N}(\underline{L}_{N-1}; \underline{L}_N) \xrightarrow[S \to \infty]{\text{scaling (6.35)}} Q^{N \to N-1}_{\lambda_N}(\underline{u}_N, \underline{u}_{N-1}).$$

These computations suggest that the same type of convergence should hold for the full functions. Namely, under (6.35) and as $S \to +\infty$, the spin Whittaker functions $\mathfrak{f}_{\underline{X}_N}(\underline{L}_N)$ rescaled by $(4S^{-1}\pi/16^S)^{\frac{N(N-1)}{4}}$ should converge to the usual Whittaker functions $\psi_{\underline{\lambda}_N}(\underline{u}_N)$. A proof of this convergence would require a finer analysis to justify the exchange of the $S \to +\infty$ limit and integration, and goes beyond the scope of this paper.

7. Spin Whittaker Processes and beta polymers

In this section define *spin Whittaker processes*, and establish their connection with two beta polymer type models introduced in [BC16a] and [CMP19], respectively.

7.1. **Spin Whittaker processes.** The definition of spin Whittaker processes is a straightforward analogy of the discrete level $\mathfrak{F}/\mathfrak{G}$ processes (Definition 4.3). The key role is played by the Cauchy type identities (established for spin Whittaker functions in Section 6.7).

Definition 7.1. Set $\mathbf{X} = (X_1, \dots, X_N)$ and $\mathbf{Y} = (Y_1, \dots, Y_T)$, with $|X_i|, |Y_j| < S$ and $X_i + Y_j > 0$ for all i, j. The (ascending) *spin Whittaker process* is the probability measure on interlacing sequences $\underline{\underline{L}}_N(T) = (L_{k,i}(T))_{1 \le i \le k \le N}$ (that is, on the Gelfand-Tsetlin cone $\mathcal{GT}_N$ (6.7)) with the following density with respect to the measure $\frac{d\underline{\underline{L}}_N}{\underline{\underline{L}}_N} = \prod_{1 \le i \le j \le N} \frac{dL_{j,i}}{L_{j,i}}$:

$$\mathfrak{P}_{\mathbf{X};\mathbf{Y}}(\underline{\underline{L}}_N) = \frac{\mathfrak{f}_{X_1}(\underline{L}_1)\mathfrak{f}_{X_2}(\underline{L}_1; \underline{L}_2) \cdots \mathfrak{f}_{X_N}(\underline{L}_{N-1}; \underline{L}_N) \mathfrak{g}_{\mathbf{Y}}(\underline{L}_N)}{\Pi(\mathbf{X}; \mathbf{Y})}. \tag{7.1}$$

The normalizing constant in (7.1) follows from the Cauchy identity of Corollary 6.21:

$$\Pi(\mathbf{X};\mathbf{Y}) = \prod_{j=1}^{T} \frac{\Gamma(X_1+Y_j)}{\Gamma(S+X_1)} \left(\prod_{i=2}^{N} \frac{\Gamma(X_i+Y_j)\Gamma(2S)}{\Gamma(S+X_i)\Gamma(S+Y_j)} \right).$$

For the next result we denote the ascending sqW/sqW process, subject to the rescaling (6.19), (6.26), by

$$\mathfrak{P}^{(q)}_{\mathbf{X};\mathbf{Y}}(\underline{\underline{L}}_N) = \frac{\mathfrak{f}^{(q)}_{X_1}(\underline{L}_1)\mathfrak{f}^{(q)}_{X_2}(\underline{L}_1;\underline{L}_2)\cdots\mathfrak{f}^{(q)}_{X_N}(\underline{L}_{N-1};\underline{L}_N)\mathfrak{g}^{(q)}_{\mathbf{Y}}(\underline{L}_N)}{\Pi^{(q)}(\mathbf{X};\mathbf{Y})},$$

where the normalization constant is (cf. (6.32))

$$\Pi^{(q)}(\mathbf{X};\mathbf{Y}) = \prod_{j=1}^{T} \frac{\Gamma_q(X_1+Y_j)}{\Gamma_q(S+X_1)} \left(\prod_{i=2}^{N} \frac{\Gamma_q(X_i+Y_j)\Gamma_q(2S)}{\Gamma_q(S+X_i)\Gamma_q(S+Y_j)} \right).$$

Theorem 7.2. *Under the scaling* (6.19), (6.26), *the ascending sqW/sqW process converges weakly to the spin Whittaker process*

$$\mathfrak{P}^{(q)}_{\mathbf{X};\mathbf{Y}} \xrightarrow[q\to 1]{} \mathfrak{P}_{\mathbf{X};\mathbf{Y}}. \tag{7.2}$$

Proof. For any continuous bounded test function $\phi(\underline{\underline{L}}_N)$ on $\mathcal{GT}_N$ we have

$$\mathbb{E}^{(q)}(\phi) = \frac{1}{\Pi^{(q)}(\mathbf{X};\mathbf{Y})} \int \frac{d\underline{L}_N}{\underline{L}_N} \mathfrak{g}^{(q)}_{\mathbf{Y}}(\underline{L}_N) \int \frac{d\underline{\underline{L}}_{N-1}}{\underline{\underline{L}}_{N-1}} \mathfrak{f}^{(q)}_{X_1}(\underline{L}_1)\mathfrak{f}^{(q)}_{X_2}(\underline{L}_1;\underline{L}_2)\cdots\mathfrak{f}^{(q)}_{X_N}(\underline{L}_{N-1};\underline{L}_N)\phi(\underline{\underline{L}}_N). \tag{7.3}$$

The integral is absolutely convergent and as a consequence of Corollaries 6.16 and 6.18 it converges to the average $\mathbb{E}(\phi)$ with respect to the spin Whittaker process. □

Remark 7.3. Developing the argument sketched in Section 6.8 it should be possible to show that the spin Whittaker process of Definition 7.1 converges to the α-Whittaker process from [COSZ14], [BC14]. In this case the correct way to rescale the random variables $L_{k,i}(T)$ is

$$L_{k,i}(T) = S^{T+k+1-2i} e^{u_{k,i}(T)}. \tag{7.4}$$

In the limit $S \to \infty$ the process $\{u_{k,i}(T) \colon 1 \le i \le k \le N\}$ should be then described by the density

$$\frac{Q^{1\to 0}_{\mathrm{i}X_1}(\underline{u}_1, 0) Q^{2\to 1}_{\mathrm{i}X_2}(\underline{u}_2, \underline{u}_1) \cdots Q^{N\to N-1}_{\mathrm{i}X_N}(\underline{u}_N, \underline{u}_{N-1}) \theta_{\mathbf{Y}}(\underline{u}_N)}{\prod_{i=1}^{N}\prod_{j=1}^{T} \Gamma(X_i+Y_j)}, \tag{7.5}$$

where $Q^{k+1\to k}$ and $\theta_{\mathbf{Y}}$ are given in (6.2) and (6.4), respectively.

7.2. **Strict-weak beta polymer model.** We will now recall the strict-weak beta polymer introduced in [BC16a].

Definition 7.4. Let $B_{i,j} \sim \mathcal{B}(X_i+Y_j, S-Y_j)$ be a family of independent beta random variables. The *strict-weak beta polymer model* partition function $Z(i,j)$, $i \ge 1$, $j \ge 0$, is the random function satisfying the recurrence

$$\begin{cases} Z(i,j) = Z(i,j-1)B_{i,j} + Z(i-1,j-1)(1-B_{i,j}) & \text{for } 1 < i \le j; \\ Z(1,j) = Z(1,j-1)B_{1,j} & \text{for } j > 0; \\ Z(i,0) = 1 & \text{for } i > 0. \end{cases}$$

Note that all the partition functions $Z(i,j)$ belong to $(0,1]$. In particular, the probability distribution of the strict-weak beta polymer is completely determined by the joint moments.

Proposition 7.5. *Recall the q-Hahn vertex model height function $\mathcal{H}^{\mathrm{ur}}_{q\text{-Hahn}}$ (Section 5.4). Define $Z^{(q)}(i,j) = q^{\mathcal{H}^{\mathrm{ur}}_{q\text{-Hahn}}(i,j)}$. Then, under the scaling* (6.19)*, $Z^{(q)}$ converges weakly to the strict-weak beta polymer partition function:*

$$Z^{(q)}(i,j) \xrightarrow[q\to 1]{} Z(i,j).$$

Proof. This result is equivalent to Proposition 2.1 of [BC16a] in the homogeneous case $X_i = X$, $Y_j = Y$ for all i,j. One can easily check that the proof given there also works in our inhomogeneous setting. □

Theorem 7.6. *The marginal process $\{L_{k,k}(T)^{-1} \colon k = 1, \dots N,\, T \geq N\}$ of the spin Whittaker process $\mathfrak{P}_{\mathbf{X},\mathbf{Y}}$ is equivalent in distribution to the strict-weak beta polymer partition functions model $\{Z(k,T) \colon k = 1, \dots, N,\, T \geq N\}$.*

Note that since $L_{k,k}(T) \in \mathbb{R}_{\geq 1}$, we have $L_{k,k}(T)^{-1} \in (0,1]$, which agrees with the range of the beta polymer partition functions.

Proof of Theorem 7.6. This is a direct consequence of Theorems 5.13 and 7.2 and Proposition 7.5. Indeed, the last row marginal of the sqW/sqW process is matched to the q-Hahn vertex model height function, and in the limit $q \to 1$ this implies that the last coordinate marginal of the spin Whittaker process is matched to the beta polymer $Z(i,j)$. □

Let us make two remarks on this result.

Remark 7.7. A weaker version of Theorem 7.6 that matches $L_{k_i,k_i}(T)^{-1}$ and $Z(k_i,T)$ for each single time T can alternatively be proved using moment formulas. Namely, the eigenoperators of Definition 6.23 may be used to extract multiple integral formulas for the joint moments of $L_{k_i,k_i}(T)^{-1}$ under the spin Whittaker processes. These formulas can then be matched to the ones for the joint moments of the beta polymer. The latter in the homogeneous case are obtained in [BC16a, Proposition 3.4], and their inhomogeneous generalization is rather straightforward, cf. [Pet19, Proposition 6.1].

Remark 7.8. It was noticed in [BC16a, Remark 1.5] that under the scaling $Z(i,T) = S^{T-i+1}z(i,T)$ the process $z(i,T)$ converges, when $S \to \infty$, to the strict-weak gamma polymer model introduced by Seppäläinen in an unpublished note and studied in [OO15], [CSS15]. This scaling of polymers corresponds to the scaling (7.4) for the full spin Whittaker process. As S goes to infinity, Theorem 7.6 turns into the matching between strict weak gamma polymer model and α-Whittaker process that was originally discovered in [OO15]. This observation is another piece of evidence supporting the formal $S \to +\infty$ scaling described in Section 6.8.

7.3. **Another beta polymer type model.** Let us now recall the beta polymer type model which was introduced in [CMP19]. We employ notation from Appendix A.3.

Definition 7.9. The random function $\widetilde{Z}(i,j)$, for $i,j \in \mathbb{Z}_{\geq 0}$ is defined by the recurrence

$$\widetilde{Z}(i,j) = \begin{cases} 1 & \text{for } j = 0, \\ \widetilde{Z}(1,j-1)\widetilde{B}_{1,j} & \text{for } i = 1, \\ W^{>}_{i,j}\widetilde{Z}(i,j-1) + (1 - W^{>}_{i,j})\widetilde{Z}(i-1,j) & \text{if } \widetilde{Z}(i,j-1) > \widetilde{Z}(i-1,j), \\ (1 - W^{<}_{i,j})\widetilde{Z}(i,j-1) + W^{<}_{i,j}\widetilde{Z}(i-1,j) & \text{if } \widetilde{Z}(i,j-1) < \widetilde{Z}(i-1,j), \end{cases} \tag{7.6}$$

where $\widetilde{B}_{1,j} \sim \mathcal{B}^{-1}(X_1+Y_j, S-Y_j)$ are independent inverse beta random variables, and

$$\begin{aligned} W^{>}_{i,j} &\sim \mathcal{NBB}^{-1}\left(2S-1, \frac{\widetilde{Z}(i-1,j)-\widetilde{Z}(i-1,j-1)}{\widetilde{Z}(i,j-1)-\widetilde{Z}(i-1,j-1)}, X_i+Y_j, S-Y_j\right), \\ W^{<}_{i,j} &\sim \mathcal{NBB}^{-1}\left(2S-1, \frac{\widetilde{Z}(i,j-1)-\widetilde{Z}(i-1,j-1)}{\widetilde{Z}(i-1,j)-\widetilde{Z}(i-1,j-1)}, X_i+Y_j, S-X_i\right). \end{aligned} \tag{7.7}$$

For $i>1$, $j>0$, we have $\widetilde{Z}(i,j-1) \neq \widetilde{Z}(i-1,j)$ with probability one.

Proposition 7.10. *Recall the ${}_4\phi_3$ vertex model height function $\mathcal{H}^{\mathrm{ul}}_{\phi}(i,j)$ (Section 5.5). Define $\widetilde{Z}^{(q)}(i,j) = q^{-\mathcal{H}^{\mathrm{ul}}_{\phi}(i,j)}$. Then, under the scaling* (6.19)*, $Z^{(q)}$ converges weakly to the process $\widetilde{Z}$:*

$$\widetilde{Z}^{(q)}(i,j) \xrightarrow[q\to 1]{} \widetilde{Z}(i,j).$$

Proof. In the homogeneous case $X_i = 0$, $Y_j = Y$ for all i,j, this was proven in [CMP19]. The same argument also essentially applies to the inhomogeneous case, and we will not repeat the computations here. The non-trivial part of the proof is to understand how the X_i parameters appear in the definition of $W^{<}_{i,j}, W^{>}_{i,j}$. The interested reader can check the validity of our statements starting from (5.13) and reproducing the computations of Section 4.3 of [CMP19]. ☐

Theorem 7.11. *The marginal process $\{L_{k,1}(T) \colon k = 1 \dots N, T \geq N\}$ of the spin Whittaker process $\mathfrak{P}_{\mathbf{X},\mathbf{Y}}$ is equivalent in distribution to the process $\{\widetilde{Z}(k,T) \colon k=1,\dots N, T \geq N\}$.*

Proof. This is established similarly to Theorem 7.6 by combining the matching of Theorem 5.17 with the $q \to 1$ scaling limits. ☐

7.4. **Reduction to log-gamma polymer.** Here we show that the model $\widetilde{Z}(i,j)$ of Definition 7.9 reduces, as $S \to +\infty$, to the well-known log-gamma polymer model introduced in [Sep12]. This proof is more involved than the rather straightforward observation for the strict-weak beta polymer (Remark 7.8).

Definition 7.12 (Log-gamma polymer)**.** Let $\{g_{i,j} \colon i,j \in \mathbb{Z}_{\geq 1}\}$ be a sequence of independent inverse gamma random variables, $g_{i,j} \sim \mathrm{Gamma}^{-1}(X_i+Y_j)$ with density (A.8). The random function $\widetilde{z}$ defined by the recurrence

$$\widetilde{z}(i,j) = \begin{cases} g_{i,j}\left(\widetilde{z}(i-1,j)+\widetilde{z}(i,j-1)\right) & \text{if } i,j \geq 1 \text{ and } i+j \geq 3; \\ g_{1,1} & \text{if } i=j=1; \\ 0 & \text{else,} \end{cases} \tag{7.8}$$

is the *point-to-point log-gamma polymer partition function.*

The log-gamma polymer model was introduced (with a proof of its exact solvability) by Seppäläinen in [Sep12]. One can view $\widetilde{z}(i,j)$ as a partition function of up-right directed paths from $(1,1)$ to (i,j), where the weight of each path equals the product of the quantities $g_{i',j'}$ along the path. In [COSZ14] the log-gamma polymer model was given a powerful combinatorial interpretation using Kirillov's geometric RSK (Robinson-Schensted-Knuth) algorithm. This showed the distributional matching of the log-gamma polymer with a marginal of the Whittaker process (7.5).

The next statement shows that the log-gamma polymer model can be obtained in a $S \to +\infty$ scaling limit from the beta polymer like model of Definition 7.9. Modulo Remark 7.3, this together with Theorem 7.11 produces an alternative derivation of the results of [COSZ14].

Proposition 7.13. *Consider the scaling* $\widetilde{Z}(i,j) = S^{j+i-1}\widetilde{z}^{(S)}(i,j)$ *of the process from Definition 7.9. Then the rescaled process* $\widetilde{z}^{(S)}$ *converges weakly to the log-gamma polymer:*

$$\widetilde{z}^{(S)}(i,j) \xrightarrow[S\to\infty]{} \widetilde{z}(i,j).$$

Proof. We argue by induction. When $i=j=1$, then $\widetilde{z}^{(S)}(1,1) = S^{-1}\mathcal{B}^{-1}(X_1+Y_1, S-Y_1)$. In the large S limit this converges to $\mathrm{Gamma}^{-1}(X_1+Y_1)$, which is precisely $\widetilde{z}(1,1)$.

Fix i,j and assume that for all i',j' such that $i'+j' < i+j$ the convergence $\widetilde{z}^{(S)}(i',j') \to \widetilde{z}(i',j')$ holds. Let us compute the densities of random variables $S^{-1}W^{>}_{i,j}$ and $S^{-1}W^{<}_{i,j}$, that are rescalings of (7.7), in the large S limit. We show the computations only for $W^{>}_{i,j}$ since the other case is very similar. The density of $S^{-1}W^{>}_{i,j}$ (depending on the variable $x \in (0,1)$) is, from (7.7) and (A.9), equal to

$$\frac{\left(\frac{1}{x}\right)^{X_i+Y_j+1}\left(1-\frac{1}{Sx}\right)^{S-Y_j-1}}{\Gamma(X_i+Y_j)} \frac{\Gamma(S+X_i)}{S^{X_i+Y_j}\Gamma(S-Y_j)} \times (1-p_S)^{2S-1}{}_2F_1\left(\begin{matrix} 2S-1, S+X_i \\ S-Y_j \end{matrix} \Big| \, p_S\left(1-\frac{1}{Sx}\right)\right), \quad (7.9)$$

where

$$p_S = \frac{\widetilde{Z}(i-1,j)-\widetilde{Z}(i-1,j-1)}{\widetilde{Z}(i,j-1)-\widetilde{Z}(i-1,j-1)} = \frac{S\widetilde{z}^{(S)}(i-1,j)-\widetilde{z}^{(S)}(i-1,j-1)}{S\widetilde{z}^{(S)}(i,j-1)-\widetilde{z}^{(S)}(i-1,j-1)} \sim p := \frac{\widetilde{z}(i-1,j)}{\widetilde{z}(i,j-1)}$$

is smaller than 1.

The limit of the first few factors in (7.9) is straightforward:

$$\frac{\left(\frac{1}{x}\right)^{X_i+Y_j+1}\left(1-\frac{1}{Sx}\right)^{S-Y_j-1}}{\Gamma(X_i+Y_j)} \frac{\Gamma(S+X_i)}{S^{X_i+Y_j}\Gamma(S-Y_j)} \xrightarrow[S\to\infty]{} \frac{\left(\frac{1}{x}\right)^{X_i+Y_j+1}e^{-\frac{1}{x}}}{\Gamma(X_i+Y_j)}.$$

To compute the limit of the Gaussian hypergeometric function, we use the Euler transformation

$${}_2F_1\left(\begin{matrix} a\,,\,b \\ c \end{matrix} \Big| \, z\right) = (1-z)^{c-a-b}{}_2F_1\left(\begin{matrix} c-a\,,\,c-b \\ c \end{matrix} \Big| \, z\right),$$

so that the remaining terms in (7.9) become

$$\left(\frac{1-p_S}{1-p_S+\frac{p_S}{Sx}}\right)^{2S-1}\left(1-p_S+\frac{p_S}{Sx}\right)^{-X_i-Y_j}{}_2F_1\left(\begin{matrix} -S-Y_j+1, -X_i-Y_j \\ S-Y_j \end{matrix} \Big| \, p_S\left(1-\frac{1}{Sx}\right)\right).$$

We have

$$\left(\frac{1-p_S}{1-p_S+\frac{p_S}{Sx}}\right)^{2S-1}\left(1-p_S+\frac{p_S}{Sx}\right)^{-X_i-Y_j} \xrightarrow[S\to\infty]{} e^{-\frac{2p}{(1-p)x}}(1-p)^{-X_i-Y_j},$$

whereas

$${}_2F_1\left(\begin{matrix} -S-Y_j+1, -X_i-Y_j \\ S-Y_j \end{matrix} \Big| \, p_S\left(1-\frac{1}{Sx}\right)\right) \xrightarrow[S\to\infty]{} {}_1F_0\left(\begin{matrix} -X_i-Y_j \\ - \end{matrix} \Big| -p\right) = (1+p)^{X_i+Y_j}.$$

Our computations imply that

$$S^{-1}W^{>}_{i,j} \xrightarrow[S\to\infty]{} \frac{\widetilde{z}(i-1,j)+\widetilde{z}(i,j-1)}{\widetilde{z}(i,j-1)-\widetilde{z}(i-1,j)}\,\mathrm{Gamma}^{-1}(X_i+Y_j).$$

Essentially repeating the computations for $S^{-1}W^{<}_{i,j}$, we obtain

$$S^{-1}W^{<}_{i,j} \xrightarrow[S\to\infty]{} \frac{\widetilde{z}(i-1,j)+\widetilde{z}(i,j-1)}{\widetilde{z}(i-1,j)-\widetilde{z}(i,j-1)}\,\mathrm{Gamma}^{-1}(X_i+Y_j).$$

Thus, we see that in the scaling limit as $S \to +\infty$, the beta polymer like model recurrence relation (7.6) becomes (7.8), the recurrence for the log-gamma polymer partition functions. This completes the proof. □

8. Deformed quantum Toda system

In this section we consider the scaling limit of the Pieri rule (2.32) which states that the spin q-Whittaker polynomials $\mathbb{F}_\lambda(x_1,\dots,x_N)$ are eigenfunctions of an operator acting on the *label* λ. This scaling limit leads to an eigenoperator for the spin Whittaker functions. This operator acts as a second order differential operator in the (additive versions of the) variables $\underline{L}_N$. We call is the *S-deformed quantum Toda Hamiltonian.* Our scaling of the Pieri rules are inspired by [GLO12b] where the Pieri rule for the q-Whittaker polynomials was understood as a discretization of the (undeformed) quantum Toda Hamiltonian.

8.1. Refined Pieri operators.

We start by refining the Pieri operator $(\mathfrak{H}^{\mathrm{sHL}}f)(\mu)=\sum_\lambda f(\lambda)\,\mathsf{F}^*_{\lambda'/\mu'}(v)$ introduced in (2.33), by considering its expansion in powers of v. Recall identity (2.32) which states that

$$(\mathfrak{H}^{\mathrm{sHL}}f)(\lambda)=\left(\left(\frac{1}{1-sv}\right)^{N-1}\prod_{i=i}^{N}(1+x_iv)\right)f(\lambda),\qquad f(\lambda)=\mathbb{F}_\lambda(x_1,\dots,x_N).$$

Defining $\mathfrak{H}_k$ by

$$(1-vs)^{N-1}\mathfrak{H}^{\mathrm{sHL}}=\sum_{k=0}^{N}v^k\mathfrak{H}_k, \tag{8.1}$$

we see that

$$\mathfrak{H}_k\mathbb{F}_\lambda=e_k(x_1,\dots,x_N)\mathbb{F}_\lambda. \tag{8.2}$$

The action of the $\mathfrak{H}_k$'s on functions $f(\lambda)$ can in principle be recovered using the vertex weights w^* in Figure 4 (without the denominator $1-vs$) which compose the sHL functions $(1-vs)^{N-1}\mathsf{F}^*_{\lambda'/\mu'}$. In the simplest cases $k=0$ or N one can verify that

$$\mathfrak{H}_0=\mathrm{Id},\qquad \mathfrak{H}_N=\mathcal{T}_{\mu_1}\cdots\mathcal{T}_{\mu_N},$$

where $\mathcal{T}$ is the shift operator

$$(\mathcal{T}_{\mu_i}f)(\mu)=\begin{cases} f(\mu+\mathrm{e}_i), & \text{if } \mu_i<\mu_{i-1} \text{ or } i=1;\\ 0, & \text{otherwise.}\end{cases}$$

Indeed, $\mathfrak{H}_0$ requires no vertical arrows to change from μ to λ, and $\mathfrak{H}_N$ corresponds to adding a full horizontal path starting at zero and ending at N, so that the arrow configuration corresponding to λ is obtained from the one for μ by adding one vertical arrow at location N.

When $1\le k\le N-1$, explicit formulas for $\mathfrak{H}_k$ look significantly more involved. We need only the one for $k=1$, and will not discuss the other operators $\mathfrak{H}_k$.[5] In the next statement, by agreement, we set $\mu_0=+\infty$, $\mu_{N+1}=-\infty$.

[5] Appropriate scaling limits of the higher operators $\mathfrak{H}_k$ could potentially lead to higher order differential operators commuting with the deformed quantum Toda Hamiltonian $\mathcal{H}_2$ introduced below in this section. We leave this investigation to a future work.

Proposition 8.1. *We have*

$$\mathfrak{H}_1 = h_{0,0}\,\mathrm{Id} + \sum_{0\le k<\ell\le N} h_{k,\ell}\mathcal{T}_{\mu_{k+1}}\cdots\mathcal{T}_{\mu_\ell},$$

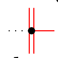
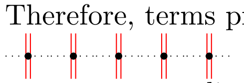
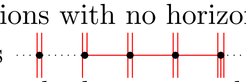
with

$$h_{0,0} = -s\sum_{j=1}^{N-1} q^{\mu_j-\mu_{j+1}},$$
$$h_{k,\ell} = (1-q^{\mu_k-\mu_{k+1}})(-s)^{\ell-k-1}q^{\mu_{k+1}-\mu_\ell}(1-s^2q^{\mu_\ell-\mu_{\ell+1}}).$$

Proof. Express the action of $\mathfrak{H}_1$ as

$$\mathfrak{H}_1 f(\mu) = \sum_{\mu\prec\lambda} H(\mu;\lambda)f(\lambda),$$

where the term $H(\mu;\lambda)$ corresponds to the weight of a row of vertices having a configuration λ at the top and μ at the bottom. Recall that we are using down-right directed paths as in Figure 4, and all the individual vertex weights are multiplied by $(1-vs)$.

Observe that each vertex  somewhere in the bulk comes with the weight $v(1-q^{\mu_i-\mu_{i+1}})$. Therefore, terms proportional to v can only come from configurations with no horizontal arrows  or with a single sequence of horizontal arrows . The first case, corresponding to $\lambda=\mu$, provides us with the term $h_{0,0}$. For the latter case, let k be the column of the leftmost vertex emanating an horizontal arrow, and let ℓ be the column where the single horizontal path stops. Such configuration corresponds to a partition $\lambda=\mu+\mathbf{e}_{k+1}+\cdots+\mathbf{e}_\ell$. Isolating the coefficient of v in the expansion of product of vertex weights, we recover $h_{k,\ell}$. □

8.2. **Scaling of the Pieri operators.** Introduce the differential operators acting in the variables $u_1,\dots,u_N$:

$$\mathcal{H}_1 \coloneqq \sum_{i=1}^{N}\partial_{u_i}; \tag{8.3}$$

$$\mathcal{H}_2 \coloneqq -\frac{1}{2}\sum_{i=1}^{N}\partial_{u_i}^2 + \sum_{1\le i<j\le N} S^{-2(j-i)}e^{u_j-u_i}(S-\partial_{u_i})(S+\partial_{u_j}). \tag{8.4}$$

In the second operator, the product is understood in the usual way as

$$(S-\partial_{u_i})(S+\partial_{u_j}) = S^2\,\mathrm{Id} + S(\partial_{u_j}-\partial_{u_i}) - \partial_{u_i}\partial_{u_j}.$$

For the next result we define the rescaling

$$q=e^{-\varepsilon},\qquad q^{\lambda_i}=\frac{e^{-u_i}}{S^{N+1-2i}},\qquad s=-e^{-\varepsilon S}. \tag{8.5}$$

Proposition 8.2. *Under the scaling* (8.5)*, we have*

$$\mathcal{H}_1 = \lim_{\varepsilon\to0}\frac{1}{\varepsilon}\left(\mathfrak{H}_1 - N\right),$$
$$\mathcal{H}_2 = -\lim_{\varepsilon\to0}\frac{1}{\varepsilon^2}\left(\mathfrak{H}_1 - N + \frac{1}{2}\mathfrak{H}_N^2 - 2\,\mathfrak{H}_N + \frac{3}{2}\right).$$

We remark that the combinations of the refined Pieri operators leading to $\mathcal{H}_1$ and $\mathcal{H}_2$ are the same as in the q-Whittaker case [GLO12b, Proposition 2.1], and correspond to the scaling of eigenvalues in the proof of Theorem 8.3 below. The scaling (8.5) of the variables, however, is different.

Proof of Proposition 8.2. First, expand the shift operator as $\varepsilon \to 0$. From (8.5) we see that the increment by 1 in μ_i corresponds to the increment by $\log(q^{-1}) = \varepsilon$ in the scaled variable u_i. Therefore,

$$\mathcal{T}_{\mu_k} = \mathrm{Id} + \varepsilon \partial_{u_k} + \frac{\varepsilon^2}{2} \partial_{u_k}^2 + \mathsf{o}(\varepsilon^2),$$

and hence

$$\mathcal{T}_{\mu_{k+1}} \cdots \mathcal{T}_{\mu_\ell} = \mathrm{Id} + \varepsilon \sum_{\alpha=k+1}^{\ell} \partial_{u_\alpha} + \frac{\varepsilon^2}{2} \sum_{k+1 \le \alpha, \beta \le \ell} \partial_{u_\alpha, u_\beta} + \mathsf{o}(\varepsilon^2).$$

This implies that

$$\frac{1}{2}\mathfrak{H}_N^2 - 2\mathfrak{H}_N + \frac{3}{2} = -\varepsilon \sum_{i=1}^{N} \partial_{u_i} + \mathsf{o}(\varepsilon^2). \tag{8.6}$$

Next we address the scaling of $\mathfrak{H}_1$. Set

$$a_{k,\ell} = \begin{cases} S^{-2(\ell-k)} e^{u_\ell - u_k} & \text{if } 1 \le k \le \ell \le N, \\ 0 & \text{else.} \end{cases}$$

Expanding the coefficients $h_{k,\ell}$, we have

$$h_{0,0} = \left(1 - \varepsilon S + \frac{\varepsilon^2}{2} S^2\right) \sum_{j=1}^{N-1} a_{j,j+1} + \mathsf{o}(\varepsilon^2),$$

and (recall that we assume $k < \ell$)

$$\begin{aligned} h_{k,\ell} = &(a_{k+1,\ell} - a_{k,\ell} - a_{k+1,\ell+1} + a_{k,\ell+1}) \\ &+ \varepsilon S \Big\{ -(\ell - k - 1)(a_{k+1,\ell} - a_{k,\ell}) + (\ell - k + 1)(a_{k+1,\ell+1} - a_{k,\ell+1}) \Big\} \\ &+ \frac{\varepsilon^2}{2} S^2 \Big\{ (\ell - k - 1)^2 (a_{k+1,\ell} - a_{k,\ell}) - (\ell - k + 1)^2 (a_{k+1,\ell+1} - a_{k,\ell+1}) \Big\} + \mathsf{o}(\varepsilon^2). \end{aligned}$$

Together with the action of the shifts $\mathcal{T}$, we see that $h_{k,\ell}\mathcal{T}_{k+1}\cdots\mathcal{T}_\ell$ expands as

$$\begin{aligned}
&(a_{k+1,\ell}-a_{k,\ell}-a_{k+1,\ell+1}+a_{k,\ell+1})\,\mathrm{Id}\\
&+\varepsilon\Big\{-S(\ell-k-1)(a_{k+1,\ell}-a_{k,\ell})\,\mathrm{Id}+S(\ell-k+1)(a_{k+1,\ell+1}-a_{k,\ell+1})\,\mathrm{Id}\\
&\qquad\qquad+(a_{k+1,\ell}-a_{k,\ell}-a_{k+1,\ell+1}+a_{k,\ell+1})\sum_{\alpha=k+1}^{\ell}\partial_{u_\alpha}\Big\}\\
&+\frac{\varepsilon^2}{2}\Big\{S^2(\ell-k-1)^2(a_{k+1,\ell}-a_{k,\ell})\,\mathrm{Id}-S^2(\ell-k+1)^2(a_{k+1,\ell+1}-a_{k,\ell+1})\,\mathrm{Id}\\
&\qquad\qquad+(a_{k+1,\ell}-a_{k,\ell}-a_{k+1,\ell+1}+a_{k,\ell+1})\sum_{k+1\le\alpha,\beta\le\ell}\partial^2_{u_\alpha,u_\beta}\\
&\qquad+2S\big(-(\ell-k-1)(a_{k+1,\ell}-a_{k,\ell})+(\ell-k+1)(a_{k+1,\ell+1}-a_{k,\ell+1})\big)\sum_{\alpha=k+1}^{\ell}\partial_{u_\alpha}\Big\}+\mathsf{o}(\varepsilon^2).
\end{aligned}$$

To evaluate the summation $\sum_{0\le k<\ell\le N}h_{k,\ell}\mathcal{T}_{\mu_{k+1}}\cdots\mathcal{T}_{\mu_\ell}$, we use the identities in Proposition D.1. We obtain

$$\begin{aligned}
&h_{0,0}+\sum_{0\le k<\ell\le N}h_{k,\ell}\mathcal{T}_{\mu_{k+1}}\cdots\mathcal{T}_{\mu_\ell}\\
&\quad=N+\varepsilon\sum_{i=1}^{N}\partial_{u_i}+\varepsilon^2\left\{\frac{1}{2}\sum_{i=1}^{N}\partial^2_{u_i}-\sum_{1\le i<j\le N}S^{2(i-j)}e^{u_j-u_i}\left(S-\partial_{u_i}\right)\left(S+\partial_{u_j}\right)\right\}+\mathsf{o}(\varepsilon^2).
\end{aligned}$$

Together with (8.6) this yields the proof. □

For the next result we employ the spin Whittaker functions in the *additive parameters* u_i, where the multiplicative parameters $\underline{L}_N=(L_{N,N},\dots,L_{N,1})$ are expressed through the u_i's as

$$L_{N,i}=S^{N+1-2i}e^{u_i}. \tag{8.7}$$

Denote the spin Whittaker function $\mathfrak{f}_{\underline{X}}(\underline{L}_N)$ in the additive parameters by $\mathfrak{f}^{add}_{\underline{X}}(u_1,\dots,u_N)$. Here $\underline{X}=(X_1,\dots,X_N)$ are such that $|X_i|<S$ for all i.

Theorem 8.3. *The spin Whittaker functions $\mathfrak{f}^{add}_{\underline{X}}(u_1,\dots,u_N)$ in the additive variables (8.7) are eigenfunctions of the differential operators $\mathcal{H}_1$ (8.3) and $\mathcal{H}_2$ (8.4). In particular, we have*

$$\begin{aligned}
\mathcal{H}_1\mathfrak{f}^{add}_{\underline{X}}(u_1,\dots,u_N)&=-\left(X_1+\cdots+X_N\right)\mathfrak{f}^{add}_{\underline{X}}(u_1,\dots,u_N),\\
\mathcal{H}_2\mathfrak{f}^{add}_{\underline{X}}(u_1,\dots,u_N)&=-\frac{1}{2}\left(X_1^2+\cdots+X_N^2\right)\mathfrak{f}^{add}_{\underline{X}}(u_1,\dots,u_N).
\end{aligned}$$

Proof. This result is a combination of the refined Pieri rules (8.2) viewed as eigenrelations for the spin q-Whittaker functions, and the convergence of the functions (Theorem 6.14) and the operators (Proposition 8.2). More precisely, under the scaling $x_i=e^{-\varepsilon X_i}$ for the eigenvalues $e_k(x_1,\dots,x_N)$ we have

$$\frac{1}{\varepsilon}\big(e_1(x_1,\dots,x_N)-N\big)\xrightarrow[\varepsilon\to0]{}-X_1-\cdots-X_N,$$

and

$$\frac{1}{\varepsilon^2}\left(e_1(x_1,\ldots,x_N)-N+\frac{1}{2}e_N(x_1,\ldots,x_N)^2-2e_N(x_1,\ldots,x_N)+\frac{3}{2}\right)\xrightarrow[\varepsilon\to 0]{}\frac{1}{2}(X_1^2+\cdots X_N^2).$$

This leads to the desired results. □

8.3. **Reduction to the quantum Toda Hamiltonian.** It is natural to call the second order differential operator $\mathcal{H}_2$ (8.4) the *deformed quantum Toda Hamiltonian.* Namely, it is diagonal in the spin Whittaker functions which (formally) reduce, as $S\to+\infty$, to the classical $\mathfrak{gl}_n$ Whittaker functions (Section 6.8). Further justification to this name comes from the fact that the operator $\mathcal{H}_2$ itself degenerates as $S\to+\infty$ to the usual quantum Toda Hamiltonian

$$\mathcal{H}_2^{\mathrm{Toda}}:=-\frac{1}{2}\sum_{i=1}^{N}\partial_{u_i}^2+\sum_{i=1}^{N-1}e^{u_{i+1}-u_i}. \tag{8.8}$$

Proposition 8.4. *As $S\to+\infty$, the operator $\mathcal{H}_2$ (8.4) converges to the quantum Toda Hamiltonian $\mathcal{H}_2^{\mathrm{Toda}}$ (8.8).*

Proof. The factors $S^{-2(j-i)}$, $1\le i\le j\le N$, in the sum in (8.4) decay at least as fast as S^{-2} as $S\to+\infty$. Therefore, the only surviving contribution in the limit $S\to+\infty$ comes from the terms with $j=i+1$, for which we have

$$S^{-2}e^{u_{i+1}-u_i}(S-\partial_{u_i})(S+\partial_{u_{i+1}})\to e^{u_{i+1}-u_i},\qquad S\to+\infty.$$

This completes the proof. □

9. Desired properties and conjectures

This paper developed the spin q-Whittaker polynomials and spin Whittaker functions, and established many of their properties which are one-parameter generalizations of the corresponding facts about the q-Whittaker polynomials and $\mathfrak{gl}_n$ Whittaker functions. In this final section we briefly discuss further desired properties and conjectures corresponding to our deformed situation.

9.1. **Orthogonality and spectral theory for spin q-Whittaker polynomials.** The q-Whittaker polynomials satisfy orthogonality relations coming from (the $t=0$ degeneration of) the Macdonald torus scalar product [Mac95, Ch. VI.9]. This relation states that the $s=0$ versions of $\mathbb{F}_\lambda(z_1,\ldots,z_N)$ and $\mathbb{F}_\mu(1/z_1,\ldots,1/z_N)$ are orthogonal to each other when $\mu\ne\lambda$ with respect to a certain weight on the N-dimensional torus $\mathbb{T}^N=\{|z_i|=1,\, i=1,\ldots,N\}$.

Remark 9.1. Under the generalization with a spin parameter, the spin Hall–Littlewood polynomials also satisfy a version of the torus orthogonality (called spatial orthogonality in [BCPS15, Corollary 3.10], see also [Bor17], [BMP19, Proposition 8.6]), as well as another biorthogonality involving the summation of over λ instead of integration over z. Here we discuss only the former conjectural orthogonality of the spin q-Whittaker polynomials.

Define

$$m_{q,s}^N(z_1,\ldots,z_N):=\frac{1}{N!}\prod_{1\le i\ne j\le N}\frac{(s^2,z_i/z_j;q)_\infty}{(-sz_i,-s/z_i;q)_\infty}\prod_{i=1}^{N}\frac{1}{2\pi\mathrm{i}z_i},\qquad (z_1,\ldots,z_N)\in\mathbb{T}^N.$$

When $s=0$, $m_{q,s}^N$ reduces to the orthogonality measure of the q-Whittaker polynomials on $\mathbb{T}^N$, which is a $t=0$ degeneration of the Macdonald's torus scalar product $\Delta(z_1,\ldots,z_N;q,t)$, cf. [Mac95, VI.(9.2)].

Lemma 9.2. *Both eigenoperators* $\mathfrak{D}_1$, $\overline{\mathfrak{D}}_1$ (3.9), (3.10) *for the spin q-Whittaker polynomials are self-adjoint with respect to the scalar product*

$$\langle f, g\rangle_{q,s} := \int_{\mathbb{T}^N} f(z_1, \ldots, z_N)\, \overline{g(z_1, \ldots, z_N)}\, m^N_{q,s}(z_1, \ldots, z_N)\, dz_1 \ldots dz_N,$$

where f, g *are Laurent polynomials with coefficients in* $\mathbb{R}(q, s)$.

Proof. A direct verification. □

Conjecture 9.3. *We have for all signatures* λ, μ*:*

$$\int_{\mathbb{T}^N} \mathbb{F}_\lambda(z_1, \ldots, z_N)\, \mathbb{F}_\mu(1/z_1, \ldots, 1/z_N)\, m^N_{q,s}(z_1, \ldots, z_N)\, dz_1 \cdots dz_N = c_\lambda \mathbf{1}_{\lambda=\mu}, \tag{9.1}$$

where

$$c_\lambda = \prod_{i=1}^{N-1} \frac{(s^2; q)_\infty}{(q; q)_\infty} \frac{(q; q)_{\lambda_i-\lambda_{i+1}}}{(s^2; q)_{\lambda_i-\lambda_{i+1}}}. \tag{9.2}$$

Note that for $N \le 2$ the statement (up to the concrete formula for the norm c_λ) follows from Lemma 9.2 and the eigenrelations of Theorems 3.9 and 3.10. However, for $N \ge 3$ the two operators $\mathfrak{D}_1$, $\overline{\mathfrak{D}}_1$ are not sufficient to conclude orthogonality.

Remark 9.4. When $s = 0$, the constant c_λ (9.2) coincides with the $t = 0$ degeneration of the torus scalar product norm of a Macdonald polynomial [Mac95, Ch. VI.9, Example 1].

Let us present one further argument in favor of Conjecture 9.3. It was proven in [IMS19, Proposition 4.10] that the probability mass function of a tagged particle in the homogeneous q-Hahn Tasep with parameters $\nu = s^2$ and $\mu = -ys$ is

$$\begin{aligned}\mathbb{P}(\mathcal{H}(N, t) = \ell) = \left(\frac{(q; q)_\infty}{(s^2; q)_\infty}\right)^{N-1} \int_{\mathbb{T}^N} m^N_{q,s}(z_1, \ldots, z_N) \prod_{j=1}^N \left(\frac{\Pi(z_j; y)}{\Pi(-s; y)}\right)^t \\ \times \left(\frac{(-s)^N}{z_1 \cdots z_N}\right)^\ell \frac{(\frac{(-s)^N}{z_1\cdots z_N}; q)_\infty (s^2; q)^{N-1}_\infty}{(-sz_1, \ldots, -sz_N; q)_\infty} dz_1 \cdots dz_N,\end{aligned} \tag{9.3}$$

where $\Pi(z; y) = \dfrac{(-sz; q)_\infty}{(-sy; q)_\infty}$. The same probability can be expressed as

$$\sum_{\substack{\lambda\in\mathrm{Sign}_N\\ \lambda_N=\ell}} \mathbb{F}_\lambda(-s, \ldots, -s)\, \frac{\mathbb{F}^*_\lambda(y, \ldots, y)}{\Pi(-s; y)},$$

Assuming Conjecture 9.3, this sum becomes

$$\int_{\mathbb{T}^N} m^N_{q,s}(z_1, \ldots, z_N) \prod_{j=1}^N \left(\frac{\Pi(z_j; y)}{\Pi(-s; y)}\right)^t \left(\frac{(-s)^N}{z_1 \cdots z_N}\right)^\ell \sum_{\substack{\lambda\in\mathrm{Sign}_N\\ \lambda_N=0}} \frac{(-s)^{|\lambda|}}{c_\lambda} \mathbb{F}_\lambda(1/z_1, \ldots, 1/z_N)\, dz_1 \cdots dz_N.$$

Here we used the Cauchy Identity and the torus scalar product to express the dual function $\mathbb{F}^*_\lambda$, and the shifting rule of Proposition 2.9 to take out the monomial of degree ℓ. Evaluating the sum inside the integral as in (2.13), we recover exactly (9.3).

9.2. **Accessing spin q-Whittaker polynomials via free field.** In this paper we did not focus on Fredholm determinantal structures for marginals of spin q-Whittaker processes. These aspects have been covered quite extensively in literature for specific models in the last few years [Cor14], [CP16], [IS19], [IMS19], [BMP19]. Techniques to access such Fredholm determinantal formulas usually rely on manipulations with integral representations of q-moments (as in [BC14], [BCS14]).

In the realm of Macdonald processes there exists an alternative approach to expose the determinantal nature of specific observables. This is done via a free field realization of Macdonald functions and Macdonald operators [FHH$^+$09], [Kos19], see also [BCGS16] and, e.g., [FW09] for the Hall–Littlewood case. In the yet simpler case of the Schur processes this reduces to the infinite wedge representation of Schur functions [Oko01], [OR03]. It would be of great interest to understand to what extent our spin q-Whittaker functions, operators, and processes admit a description in terms of Fock type representations of a hypothetical (q, s)-deformed Heisenberg algebra. It is worth mentioning that an example where symmetric functions coming from solvable vertex models have been incorporated in the language of Fock space representation can be found in [BBBG18].

9.3. **Sampling and RSK like constructions.** The sqW/sHL and sqW/sqW random fields of signatures (described in Section 4) can be sampled using the bijectivization of the corresponding Yang–Baxter equations. While these sampling algorithms are well-adapted to particle system marginals, there could be other randomized procedures to sample the whole signatures (and resulting in potentially different random fields).

In particular, there could exist distinguished "least random" (i.e., using the least possible number of random variables) sampling procedures resembling the classical Robinson-Schensted-Knuth (RSK) insertion algorithms. At $s = 0$, such (rather complicated) RSK-like algorithms were developed in [MP17] for sampling q-Whittaker processes. Further setting $q = 0$ recovers the classical RSK algorithms related to Schur processes.[6] It would be interesting to extend RSK-like sampling algorithms to the spin q-Whittaker level.

In the scaling limit as $q \to 1$, the RSK-like sampling algorithms of [MP17] degenerate into the well-known geometric RSK algorithms introduced and studied in [Kir01], [NY04]. The geometric RSK's are naturally associated with Brownian and log-gamma polymer models and $\mathfrak{gl}_n$ Whittaker functions [O'C12], [COSZ14]. It would be very interesting to lift geometric RSK's to the spin Whittaker processes / beta polymer level developed in the present paper. This beta polymer version of the RSK could arise in the corresponding scaling limit of the spin q-Whittaker RSK.

9.4. **Higher polymer interpretations and random walks.** The strict-weak log-gamma polymer model is matched in distribution to the last row marginals of the Whittaker process (cf. Remark 7.8). Moreover, this connection extends (via the geometric RSK) to the so-called higher polymer partition functions, i.e., partition functions of k-tuples of noncrossing paths (in the same log-gamma environment), $k = 1, 2, \ldots$, where $k = 1$ corresponds to the original strict-weak log-gamma polymer [NY04], [COSZ14]. The higher log-gamma partition functions are matched with joint distributions of several components of the Whittaker process. It is very interesting to find similar higher polymer like interpretations of joint distributions of multiple components in the spin Whittaker process introduced in Section 7.

[6] We refer to [BMP19, Section 2.6] for a detailed historical discussion of sampling random Young diagrams / signatures whose probability weights are expressed through various families of symmetric functions.

The strict-weak beta polymer partition function admits an alternative description as a certain probability for the random walk in beta random environment [BC16a]. Could multiple components in the spin Whittaker process be matched to certain probabilities of interacting random walks in beta random environment?

9.5. **Further properties of spin Whittaker functions.** Whittaker functions have a number of important properties whose generalization to the spin Whittaker level seems potentially very interesting. This includes connections to representation theory [Kos80], Mellin-Barnes integral representation [GKLO06], and orthogonality relations [STS94]. We only make a conjecture about the latter which is in effect a scaling limit of Conjecture 9.3.

Conjecture 9.5. *For all $\underline{L}_N, \underline{L}'_N \in \mathcal{W}_N$ we have*

$$\int_{(\mathrm{i}\mathbb{R})^N} \mathfrak{f}_{\underline{Z}}(\underline{L}_N)\, \mathfrak{f}_{-\underline{Z}}(\underline{L}'_N)\, \mathfrak{M}_S^N(\underline{Z})\, dZ_1 \dots dZ_N = \prod_{i=1}^{N-1} \left(1 - \frac{L_{N,i+1}}{L_{N,i}}\right)^{1-2S} \delta_{\underline{L}_N - \underline{L}'_N}, \tag{9.4}$$

where $\mathfrak{M}_S^N$ is the S-deformation of the Sklyanin measure

$$\mathfrak{M}_S^N(\underline{Z}) = \frac{1}{N!(2\pi \mathrm{i})^N} \prod_{1 \le i \ne j \le N} \frac{\Gamma(S+Z_i)\Gamma(S-Z_i)}{\Gamma(2S)\Gamma(Z_i - Z_j)},$$

and $\delta_{\underline{L}_N - \underline{L}'_N}$ is a delta function.

In support of this conjecture we note that the eigenoperators $\mathcal{D}_1$ and $\overline{\mathcal{D}}_1$ for the spin Whittaker functions (Definition 6.23) are self-adjoint with respect to the scalar product defined by the S-deformed Sklyanin measure $\mathfrak{M}_S^N$ (this can also be checked directly). This implies the desired statement for $N \le 2$, up to the concrete expression for the norm in the right-hand side of (9.4).

The theory quantum Toda Hamiltonians and Whittaker functions extends from $\mathfrak{gl}_n$ to other classical Lie groups [Kos80], [GLO12a]. It would be interesting to extend our deformation (8.4) of the $\mathfrak{gl}_n$ quantum Toda Hamiltonian to other symmetry (Killing-Cartan) types.

Appendix A. Special functions and probability distributions

We use the q-Pochhammer symbol notation (1.5).

A.1. **q-beta binomial distribution.** Recall the definition of the q-deformed beta-binomial distribution $\varphi_{q,\mu,\nu}$ from [Pov13], [Cor14].

Definition A.1. For $m \in \mathbb{Z}_{\ge 0}$, consider the following distribution on $\{0, 1, \dots, m\}$:

$$\varphi_{q,\mu,\nu}(j \mid m) = \mu^j \, \frac{(\nu/\mu;q)_j (\mu;q)_{m-j}}{(\nu;q)_m} \frac{(q;q)_m}{(q;q)_j (q;q)_{m-j}}, \qquad 0 \le j \le m. \tag{A.1}$$

When $m = +\infty$, extend the definition as

$$\varphi_{q,\mu,\nu}(j \mid \infty) = \mu^j \frac{(\nu/\mu;q)_j}{(q;q)_j} \frac{(\mu;q)_\infty}{(\nu;q)_\infty}, \qquad j \in \mathbb{Z}_{\ge 0}. \tag{A.2}$$

The distribution depends on q and two other parameters μ, ν.

When $0 \le \mu \le 1$ and $\nu \le \mu$, the weights $\varphi_{q,\mu,\nu}(j \mid m)$ are nonnegative.[7] They also sum to one:

$$\sum_{j=0}^{m} \varphi_{q,\mu,\nu}(j \mid m) = 1, \qquad m \in \{0, 1, \dots\} \cup \{+\infty\}.$$

A.2. **q-hypergeometric function and related quantities.** The unilateral basic hypergeometric series ${}_{k+1}\phi_k$ is defined via

$$ {}_{k+1}\phi_k \left(\begin{matrix} a_1 & \dots & a_{k+1} \\ b_1 & \dots & b_k \end{matrix} ; q, z \right) = \sum_{n=0}^{\infty} \frac{(a_1, \dots, a_{k+1}; q)_n}{(b_1, \dots, b_k, q; q)_n} z^n. \tag{A.3}$$

If one of a_j is q^{-y} for a positive integer y, then this series is terminating. Otherwise we assume $|q|, |z| < 1$ for the sum to be convergent. In the terminating case, we also define the regularized version by

$$\begin{aligned} {}_{k+1}\overline{\phi}_k \left(\begin{matrix} q^{-n} & a_1 & \dots & a_k \\ & b_1 & \dots & b_k \end{matrix} ; q, z \right) &:= (b_1, \dots, b_k; q)_n \cdot {}_{k+1}\phi_k \left(\begin{matrix} q^{-n} & a_1 & \dots & a_{k+1} \\ b_1 & b_2 & \dots & b_k \end{matrix} ; q, z \right) \\ &= \sum_{j=0}^{n} z^j \, \frac{(q^{-n}; q)_j}{(q; q)_j} \, (a_1, \dots, a_k; q)_j \, (q^j b_1, \dots, q^j b_k; q_{n-j}). \end{aligned} \tag{A.4}$$

The q-gamma and the q-beta functions are

$$\Gamma_q(X) = \frac{(q;q)_\infty}{(q^X;q)_\infty}(1-q)^{1-X}, \qquad \mathrm{B}_q(X,Y) = \frac{\Gamma_q(X)\Gamma_q(Y)}{\Gamma_q(X+Y)}, \qquad \text{for } X, Y > 0. \tag{A.5}$$

The q-hypergeometric distribution is

$$\psi_{q,a,b,c}(n) = \left(\frac{c}{ac}\right)^n \frac{(a,b;q)_n}{(c,q;q)_n} \frac{(c, c/(ab);q)_\infty}{(c/a, c/b;q)_\infty}. \tag{A.6}$$

The fact that the weights (A.6) sum to one over $n \in \mathbb{Z}_{\ge 0}$ follows from the Heine summation formula [GR04, (II.8)]:

$$ {}_2\phi_1 \left(\begin{matrix} a & b \\ & c \end{matrix} ; q, c/(ab) \right) = \frac{(c/a, c/b; q)_\infty}{(c, c/(ab); q)_\infty}.$$

A.3. **Spin Whittaker level quantities.** It is well-known that $\Gamma_q(X)$ converges to $\Gamma(X)$ as $q \to 1$ uniformly for $X > 0$, where Γ is the usual gamma function $\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt$, $z > 0$ (e.g., see [And86]). Hence $\mathrm{B}_q(X,Y) \to \mathrm{B}(X,Y)$ uniformly for $X, Y > 0$, where B is the beta function

$$\mathrm{B}(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \int_0^1 t^{x-1}(1-t)^{y-1} dt, \qquad x, y > 0. \tag{A.7}$$

The *inverse gamma distribution* $\Gamma^{-1}(\alpha)$ on $(0, +\infty)$ with a parameter $\alpha > 0$ is

$$\Gamma^{-1}(\alpha)[x] = \frac{x^{-1-\alpha} e^{-1/x}}{\Gamma(\alpha)}. \tag{A.8}$$

The *beta distribution* on $(0,1)$ with (real) parameters $m, n > 0$ has density

$$\mathcal{B}(m,n)[x] = \frac{x^{m-1}(1-x)^{n-1}}{\mathrm{B}(n,m)} \qquad \text{for } x \in (0,1).$$

[7] These conditions do not exhaust the full range of (q, μ, ν) for which the weights are nonnegative. See, e.g., [BP18, Section 6.6.1] for additional families of parameters leading to nonnegative weights.

We also recall that a random variable with *negative binomial* distribution has probability mass function

$$\mathcal{NB}(r,p)[k] = p^k(1-p)^r \binom{k+r-1}{k}, \qquad \text{for } k \in \mathbb{Z}_{\geq 0}$$

and $r > 0$, $0 \leq p \leq 1$. Sampling x in the interval $(0,1)$ with $\mathcal{B}(m, n+k)$ law, where k is a $\mathcal{NB}(r,p)$ independent random variable generates the *negative beta binomial* distribution on $(0,1)$. It has the probability density

$$\mathcal{NBB}(r,p,m,n)[x] = \frac{(1-p)^r x^{m-1}(1-x)^{n-1}}{\mathrm{B}(n,m)} \, {}_2F_1\left(\begin{matrix} r, n+m \\ n \end{matrix} \middle| \, p(1-x) \right), \tag{A.9}$$

where we used the Gauss hypergeometric function

$${}_2F_1\left(\begin{matrix} a\,,\, b \\ c \end{matrix} \middle| \, z \right) = \sum_{k\geq 0} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!}, \tag{A.10}$$

and $(r)_k = r(r+1)\cdots(r+k-1)$ is the Pochhammer symbol. Note that the inverse gamma, beta, and the negative beta binomial are continuous distributions, while the negative binomial is a discrete distribution.

Appendix B. Yang–Baxter equations

In this section we list the Yang–Baxter equations used throughout the paper. We employ the special function notation from Appendix A.

B.1. sqW/sqW Yang–Baxter equation.

Let us introduce the cross vertex weight

$$R_{x,y}(i_1, j_1; i_2, j_2) := \mathbf{1}_{i_1+j_1=i_2+j_2} \, \mathbf{1}_{i_1 \geq j_2} \, (y/x)^{j_2} \, \frac{(-s/y;q)_{j_2}(y/x;q)_{i_1-j_2}(q;q)_{i_1}}{(q;q)_{j_2}(q;q)_{i_1-j_2}(-s/x;q)_{i_1}}. \tag{B.1}$$

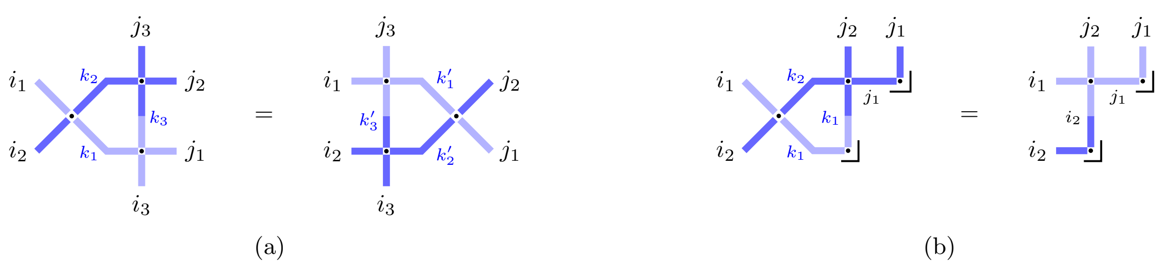


Figure 13. Yang–Baxter equations (B.2), (B.3) correspond to local changes in the lattice illustrated by (a) and (b), respectively.

This cross vertex weight is involved in the following Yang–Baxter equations:

Proposition B.1. *For any $i_1, i_2, i_3, j_1, j_2, j_3 \in \mathbb{Z}_{\geq 0}$, we have*

$$\begin{aligned} &\sum_{k_1,k_2,k_3\geq 0} R_{x,y}(i_2,i_1;k_2,k_1) W^{+}_{y,s}(i_3,k_1;k_3,j_1) W^{+}_{x,s}(k_3,k_2;j_3,j_2) \\ &\qquad = \sum_{k_1',k_2',k_3'\geq 0} W^{+}_{x,s}(i_3,i_2;k_3',k_2') W^{+}_{y,s}(k_3',i_1;j_3,k_1') R_{x,y}(k_2',k_1';j_2,j_1), \end{aligned} \tag{B.2}$$

where W^{+} are the bulk weights defined by (2.3). *See Figure 13*(a) *for a graphical interpretation.*

Proof. This is obtained in [BW17, Corollary 4.3] via fusion from the elementary Yang–Baxter equation for the higher spin $\mathfrak{sl}_2$ vertex model. Note that the claim of [BW17, Corollary 4.3] contains a typo: the spectral parameters x, y in the definition of the cross vertex weight should be swapped. This is corrected here by defining $R_{x,y}$ in (B.1) with parameters already swapped. $\square$

Proposition B.2. *For any $i_1, i_2, j_1, j_2 \in \mathbb{Z}_{\geq 0}$, we have*

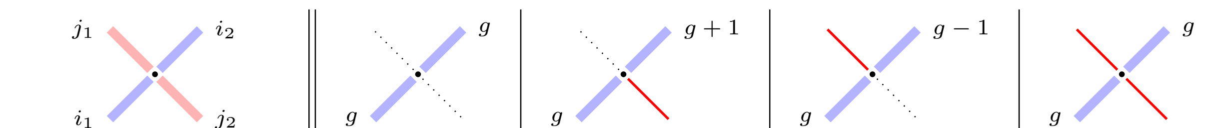
$$\sum_{k_1,k_2\geq 0} R_{x,y}(i_2,i_1;k_2,k_1) W^{\lrcorner}_{y,s}(k_1) W^{+}_{x,s}(k_1,k_2;j_2,j_1) W^{\lrcorner}_{x,s}(j_1) \\ = W^{\lrcorner}_{x,s}(i_2) W^{+}_{y,s}(i_2,i_1;j_2,j_1) W^{\lrcorner}_{y,s}(j_1), \tag{B.3}$$

where $W^{\lrcorner}$ are the right corner weight defined by (2.4)*. See Figure 13*(b) *for an illustration.*

Proof. Expanding both right and left-hand side of (B.3) and simplifying common factors we end up with the identity

$$\sum_{k=j_1}^{i_2} (y/x)^{k-j_1} \frac{(y/x;q)_{i_2-k}}{(q;q)_{i_2-k}} \frac{(-sx;q)_{k-j_1}}{(q;q)_{k-j_1}} = \frac{(-sy;q)_{i_2-j_1}}{(q;q)_{i_2-j_1}},$$

which follows from the q-Chu–Vandermonde identity (e.g., [GR04, (II.6)]). $\square$



$\mathcal{R}_{x,v,s}(i_1,j_1;i_2,j_2)$	$\frac{1-q^g sv}{1-sv}$	$\frac{xv+q^g sv}{1-sv}$	$\frac{1-q^g}{1-sv}$	$\frac{q^g+vx}{1-sv}$

FIGURE 14. The cross vertex weights involved in the Yang–Baxter equations for the sHL and sqW vertex weights. Note that these weights vanish unless $i_1 + j_2 = j_1 + i_2$.

B.2. **Yang–Baxter equations with dual weights.** Our additional Yang–Baxter equations involve the dual sHL weights $w^*_{v,s}$ which are given in Figure 4 in the text and the dual sqW weights (2.25)–(2.26). We use the cross vertex weights $\mathcal{R}_{x,v,s}$ given in Figure 14 and the following cross vertex weights:

$$\mathbb{R}_{x,y,s}(i_1,j_1;i_2,j_2) := \mathbf{1}_{i_2+j_1=i_1+j_2} \frac{q^{i_2 i_1+\frac{1}{2}j_2(j_2-1)}(sx)^{j_2}(q;q)_{j_1}}{(s^2;q)_{j_1+i_2}(q;q)_{j_2}(q;q)_{i_2}(-q/(sx);q)_{i_1-j_1}} \\ \times {}_4\overline{\phi}_3 \left(\begin{matrix} q^{-i_2}; q^{-i_1}, -sy, -q/(sx) \\ -s/x, q^{1+j_2-i_2}, -yq^{1-i_1-j_2}/s \end{matrix} \,\Big|\, q, q \right). \tag{B.4}$$

Here ${}_4\overline{\phi}_3$ is the regularized q-hypergeometric function (A.4). We remark that one of the first ${}_4\overline{\phi}_3$ type formulas for vertex weights of the fused six vertex model appeared in [Man14]. See also [CP16], [BP18] for a probabilistic explanation of the fusion procedure which goes back to [KRS81].

Next we list Yang–Baxter equations involving a usual and a dual family of vertex weights. There are two instances of these Yang–Baxter equations, one with sqW/sHL weights, and another with sqW/sqW weights. Moreover, each of these has two different forms, in the bulk and at the boundary. In total there are four statements. The bulk statements are available from [BMP19] (and also can be found in [BW18]), and the statements on the boundary need to be proven.

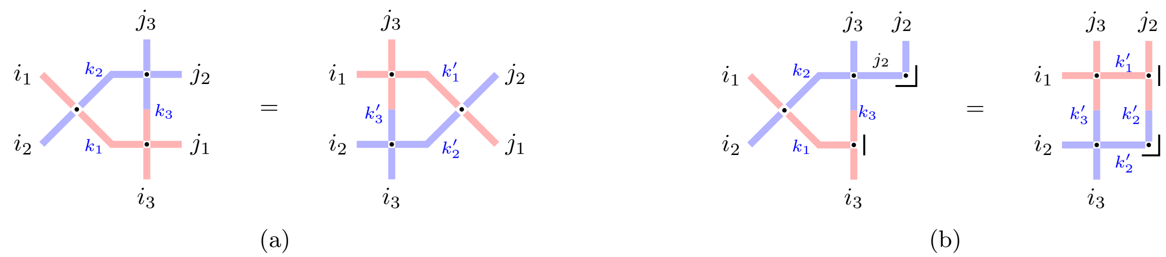


FIGURE 15. Graphical representation of the Yang–Baxter equation for dual weights.

Proposition B.3. *For any* $i_1, j_1 \in \{0,1\}$ *and* $i_2, i_3, j_2, j_3 \in \mathbb{Z}_{\geq 0}$ *we have*

$$\sum_{k_1,k_2,k_3\geq 0} \mathcal{R}_{x,v,s}(i_2,i_1;k_2,k_1)\, w^{*,+}_{v,s}(i_3,k_1;k_3,j_1)\, W^{+}_{x,s}(k_3,k_2;j_3,j_2)$$
$$= \sum_{k_1',k_2',k_3'\geq 0} W^{+}_{x,s}(i_3,i_2;k_3',k_2')\, w^{*,+}_{v,s}(k_3',i_1;j_3,k_1')\, \mathcal{R}_{x,v,s}(k_2',k_1';j_2,j_1). \tag{B.5}$$

See Figure 15 (a) *for a graphical interpretation.*

Proof. This is [BMP19, (A.11)]. □

Proposition B.4. *For any* $i_1 \in \{0,1\}$ *and* $i_2, i_3, j_2, j_3 \in \mathbb{Z}_{\geq 0}$, *we have*

$$\sum_{k_1,k_2,k_3\geq 0} \mathcal{R}_{x,v,s}(i_2,i_1;k_2,k_1)\, W^{*,\urcorner}(i_3,k_1;k_3)\, W^{+}_{x,s}(k_3,k_2;j_3,j_2)\, W^{\lrcorner}_{x,s}(j_2)$$
$$= \sum_{k_1',k_2',k_3'\geq 0} W^{+}_{x,s}(i_3,i_2;k_3',k_2')\, W^{\lrcorner}_{x,s}(k_2')\, w^{*}_{v,s}(k_3',i_1;j_3,k_1')\, W^{*,\urcorner}(k_2',k_1';j_2). \tag{B.6}$$

See Figure 15 (b) *for a graphical interpretation.*

Proof. Consider separately the cases $i_1 = 0$ and $i_1 = 1$. Start with $i_1 = 0$. We see that (B.6) is nontrivial only when $i_2 + i_3 = j_2 + j_3$ (say $j_2 = i_2 + i_3 - j_3$) and $j_3 \geq i_2$. Under these conditions we have

$$\begin{aligned}
&\mathcal{R}_{x,v,s}(i_2,0;i_2,0)W^{+}_{x,s}(i_3,i_2;j_3,i_2+i_3-j_3)W^{\lrcorner}_{x,s}(i_2+i_3-j_3)\\
&\qquad + \mathcal{R}_{x,v,s}(i_2,0;i_2+1,1)W^{+}_{x,s}(i_3-1,i_2+1;j_3,i_2+i_3-j_3)W^{\lrcorner}_{x,s}(i_2+i_3-j_3)\\
&= w^{*}_{v,s}(j_3,0;j_3,0)W^{+}_{x,s}(i_3,i_2;j_3,i_2+i_3-j_3)W^{\lrcorner}_{x,s}(i_2+i_3-j_3)\\
&\qquad + w^{*}_{v,s}(j_3-1,0;j_3,1)W^{+}_{x,s}(i_3,i_2;j_3-1,i_2+i_3-j_3+1)W^{\lrcorner}_{x,s}(i_2+i_3-j_3+1).
\end{aligned}$$

After the required simplifications, the previous relation reduces to

$$(1-q^{i_2}sv)(1+sxq^{j_3-i_2-1}) + (xv+svq^{i_2})(1-q^{j_3-i_2})$$
$$= (1-svq^{j_3})(1+sxq^{j_3-i_2-1}) + xv(1-q^{j_3-i_2})(1-s^2q^{j_3-1}),$$

that can be checked directly.

When $i_1 = 1$, as in the $i_1 = 0$ case, (B.6) is an equality between sums of at most two terms that after simplification reduces to

$$(1 - q^{i_2})(1 + sxq^{j_3-i_2}) + (q^{i_2} + xv)(1 - q^{j_3-i_2+1}) \\ = (1 - q^{j_3+1})(1 + sxq^{j_3-i_2}) + x(1 - q^{j_3-i_2+1})(v - sq^{j_3}),$$

which is again checked directly. ☐

Proposition B.5. *For any $i_1, j_1, i_2, i_3, j_2, j_3 \in \mathbb{Z}_{\geq 0}$ we have*

$$\sum_{k_1,k_2,k_3\geq 0} \mathbb{R}_{x,y,s}(i_2, i_1; k_2, k_1)\, W^{*,+}_{y,s}(i_3, k_1; k_3, j_1)\, W^{+}_{x,s}(k_3, k_2; j_3, j_2) \\ = \sum_{k_1',k_2',k_3'\geq 0} W^{+}_{x,s}(i_3, i_2; k_3', k_2')\, W^{*,+}_{y,s}(k_3', i_1; j_3, k_1')\, \mathbb{R}_{x,y,s}(k_2', k_1'; j_2, j_1). \tag{B.7}$$

See Figure 15 (a) *for a graphical interpretation.*

Proof. This is [BMP19, (A.13)]. ☐

Proposition B.6. *For any $i_1, i_2, i_3, j_1, j_2, j_3 \in \mathbb{Z}_{\geq 0}$, we have*

$$\sum_{k_1,k_2,k_3\geq 0} \mathbb{R}_{x,y,s}(i_2, i_1; k_2, k_1) W^{*,\dashv}(i_3, k_1; k_3) W^{+}_{x,s}(k_3, k_2; j_3, j_2) W^{\lrcorner}_{x,s}(j_2) \\ = \frac{(-sy;q)_\infty}{(s^2;q)_\infty} \sum_{k_1',k_2',k_3'\geq 0} W^{+}_{x,s}(i_3, i_2; k_3', k_2') W^{\lrcorner}_{x,s}(k_2') W^{*,+}_{y,s}(k_3', i_1; j_3, k_1') W^{*,\dashv}(k_2', k_1'; j_2). \tag{B.8}$$

Proof. This follows from the analogous relation (B.6). In fact both the R-matrix $\mathbb{R}$ and the vertex weight $W^{*,+}$ can be constructed respectively from $\mathcal{R}$ and $w^{*,+}$ via fusion with respect to the spectral parameter v (see [BW17], [BMP19] for details). The coefficient $\frac{(-sy;q)_\infty}{(s^2;q)_\infty}$ arises from fusion of $w^{*,+}$ and does not simplify since in the left hand side of (B.6) the bulk weight $w^{*,+}$ is missing. One can check that the fusion procedure preserves identity (B.6) and hence (B.8) holds. ☐

We also need a useful summation identity that was stated in a slightly a more general form in Proposition A.5 of [BMP19]:

Proposition B.7. *For $|xy| < 1$ and under the usual conditions $0 < q < 1$ and $-1 < s < 0$, we have*

$$\sum_{i,j\geq 0} \mathbb{R}_{x,y,s}(0, 0; i, j) = \sum_{j=0}^{\infty} (xy)^j \frac{(-s/x;q)_j(-s/y;q)_j}{(s^2;q)_j(q;q)_j} = \frac{(-sx;q)_\infty(-sy;q)_\infty}{(s^2;q)_\infty(xy;q)_\infty}. \tag{B.9}$$

The R-matrix $\mathbb{R}_{x,y,s}$ is positive if we assume its parameters to be in a specific range:

Proposition B.8 ([BMP19, Proposition A.8], see also [CMP19, Proposition 3.1])**.** *Let us take the parameters $q \in (0, 1)$, $s \in (-\sqrt{q}, 0)$ and $x, y \in [-s, -s^{-1}]$. Then $\mathbb{R}_{x,y,s}(i_1, j_1; i_2, j_2) \geq 0$ for all $i_1, i_2, j_1, j_2 \geq 0$.*

Appendix C. Proof of Proposition 6.15

Lemma C.1 ([BC16a], Lemma 2.2)**.** *Let $A, B > 0$. Then*

$$\lim_{q\to 1} \frac{(\ell q^A; q)_\infty}{(\ell q^B; q)_\infty} = (1-\ell)^{B-A}, \tag{C.1}$$

uniformly in ℓ belonging to any compact subset of $(0,1)$.

Note that the uniformity in ℓ in (C.1) is not claimed in [BC16a] but easily follows from the uniformity of all Taylor expansions involved in the proof in the cited paper (which we do not reproduce).

Lemma C.2. *Let $A, B > 0$. Then, for all $n \in \mathbb{Z}_{\geq 1}$ and all $q \in (\frac{1}{2}, 1)$, we have*

$$\frac{(q^{A+n}; q)_\infty}{(q^{B+n}; q)_\infty} \leq c\,(1-q^n)^{B-A}, \tag{C.2}$$

where c is a constant independent of q, n.

Proof. Set $q = e^{-\varepsilon}$. The result of the Lemma is restated, taking the logarithm of both sides of (C.2), as

$$\sum_{k\geq 0} \log \frac{\left(1-e^{-\varepsilon(A+n+k)}\right)}{\left(1-e^{-\varepsilon(B+n+k)}\right)} - (B-A)\log(1-e^{-\varepsilon n}) \leq c', \tag{C.3}$$

for all $\varepsilon \in (0, -\log 2)$ and a constant c' independent of ε, n. Using Lagrange mean value theorem, we can rewrite the generic term of the infinite sum as

$$\log \frac{\left(1-e^{-\varepsilon(A+n+k)}\right)}{\left(1-e^{-\varepsilon(B+n+k)}\right)} = (A-B)\,\frac{\varepsilon}{e^{\varepsilon(\widetilde{t}_k+n+k)}-1},$$

where numbers $\widetilde{t}_k$ belong to the interval $(\min(A,B), \max(A,B))$. We show that for any positive bounded sequence $\{t_k\}_k \subset (0, M)$, with M fixed, the quantity

$$\sum_{k\geq 0} \frac{\varepsilon}{e^{\varepsilon(t_k+n+k)}-1} + \log(1-e^{-\varepsilon n}) \tag{C.4}$$

is absolutely bounded uniformly in ε and n and this would prove (C.3) and hence (C.2). To evaluate the infinite sum over k we fix a positive constant δ and distinguish two cases.

Case 1, $k \geq \delta/\varepsilon$. We use the estimate

$$\frac{\varepsilon}{e^{\varepsilon(t_k+n+k)}-1} \leq e^{-\varepsilon k}\frac{\varepsilon}{1-e^{-\delta}},$$

that implies, summing over k,

$$\sum_{k\geq \delta/\varepsilon} \frac{\varepsilon}{e^{\varepsilon(t_k+n+k)}-1} \leq \frac{1}{e^{\delta}-1}\frac{\varepsilon}{1-e^{-\varepsilon}} \leq \frac{2\log 2}{e^{\delta}-1}. \tag{C.5}$$

Case 2, $k < \delta/\varepsilon$. In this case we use again Lagrange mean value theorem to express the denominator of the generic term of the summation of (C.4) as

$$\frac{\varepsilon}{e^{\varepsilon(t_k+n+k)}-1} = \frac{e^{-\varepsilon\xi_{k,n}}}{t_k+n+k}, \qquad \text{for some } \xi_{k,n} \in (0, t_k+n+k).$$

This implies the bounds

$$\frac{e^{-\varepsilon(M+n+k)}}{M+n+k} \leq \frac{\varepsilon}{e^{\varepsilon(t_k+n+k)}-1} \leq \frac{1}{n+k}. \tag{C.6}$$

We focus first on the lower bound given by the first inequality in (C.6). Summing over k we find

$$\sum_{k=0}^{\delta/\varepsilon} \frac{e^{-\varepsilon(M+n+k)}}{M+n+k} \geq \int_0^{\delta/\varepsilon} \frac{e^{-\varepsilon(M+n+k)}}{M+n+k} dk = \int_{\varepsilon(M+n)}^{\delta+\varepsilon(M+n)} \frac{e^{-k'}}{k'} dk' \geq \int_{\varepsilon(M+n)}^{\delta+\varepsilon(M+n)} \left(\frac{1}{k'}-1\right) dk',$$

which gives

$$\sum_{k=0}^{\delta/\varepsilon} \frac{\varepsilon}{e^{\varepsilon(t_k+n+k)}-1} \geq \log\left(1+\frac{\delta}{\varepsilon(M+n)}\right) - \delta. \tag{C.7}$$

We turn now our attention to the second inequality in (C.6) and, since

$$\sum_{k=0}^{\delta/\varepsilon} \frac{1}{n+k} \leq \int_n^{\delta/\varepsilon+n} \frac{dk}{k-1/2},$$

we obtain

$$\sum_{k=0}^{\delta/\varepsilon} \frac{\varepsilon}{e^{\varepsilon(t_k+n+k)}-1} \leq \log\left(1+\frac{\delta}{\varepsilon(n-1/2)}\right). \tag{C.8}$$

Combining results obtained from the analysis of cases $k \geq \delta/\varepsilon$ and $k < \delta/\varepsilon$ in (C.5), (C.7) (C.8) we can finally write

$$\log\left((1+\frac{\delta}{\varepsilon(M+n)})(1-e^{-\varepsilon n})\right) + \mathsf{O}(\delta) \leq \text{(C.4)} \leq \log\left((1+\frac{\delta}{\varepsilon(n-1/2)})(1-e^{-\varepsilon n})\right) + \mathsf{O}(\delta).$$

This concludes our proof since the arguments of the logarithms in the left and right-hand side are bounded functions for $\varepsilon \in (0, \log 2)$ and $n \geq 1$. □

For the next lemma we define the quantity

$$\mathcal{A}_{S,X}^{(q)}(\ell_3,\ell_2,\ell_1) = \frac{1}{\Delta_q(\ell_3,\ell_2,\ell_1)} \frac{(q^{S-X};q)_{n_1-n_2}}{(q;q)_{n_1-n_2}} \frac{(q^{S+X};q)_{n_2-n_3}}{(q;q)_{n_2-n_3}} \frac{(q;q)_{n_1-n_3}}{(q^{2S};q)_{n_1-n_3}},$$

where we assumed $1 \leq \ell_3 \leq \ell_2 \leq \ell_1$ and $n_i = \lfloor \log_q(1/\ell_i) \rfloor$. Here Δ_q is defined in (6.21). We think of $\mathcal{A}_{S,X}^{(q)}$ as a q-deformation of $\mathcal{A}_{S,X}$ (6.8).

Lemma C.3. *For any continuous function $f(\ell_2)$ we have*

$$\lim_{q\to 1} \int_{\ell_3}^{\ell_1} f(\ell_2)\mathcal{A}_{S,X}^{(q)}(\ell_3,\ell_2,\ell_1)\frac{d\ell_2}{\ell_2} = \int_{\ell_3}^{\ell_1} f(\ell_2)\mathcal{A}_{S,X}(\ell_3,\ell_2,\ell_1)\frac{d\ell_2}{\ell_2}, \tag{C.9}$$

uniformly for any $\ell_3 \leq \ell_1$ bounded away from ∞.

Proof. Fix small positive δ. We will prove our claim distinguishing two cases, based on the distance between ℓ_3 and ℓ_1.

Case 1, $\ell_1 - \ell_3 > \delta$. The integral in the left-hand side of (C.9) can be decomposed as

$$\int_{\ell_3}^{\ell_1} = \int_{\ell_3+\delta/2}^{\ell_1-\delta/2} + \int_{\ell_3}^{\ell_3+\delta/2} + \int_{\ell_1-\delta/2}^{\ell_1}.$$

When $\ell_3 + \delta/2 \le \ell_2 \le \ell_1 - \delta/2$, by virtue of Lemma C.1, we have

$$\int_{\ell_3+\delta/2}^{\ell_1-\delta/2} f(\ell_2)\mathcal{A}^{(q)}_{S,X}(\ell_3,\ell_2,\ell_1)\frac{d\ell_2}{\ell_2} \xrightarrow[q\to 1]{} \int_{\ell_3+\delta/2}^{\ell_1-\delta/2} f(\ell_2)\mathcal{A}_{S,X}(\ell_3,\ell_2,\ell_1)\frac{d\ell_2}{\ell_2},$$

uniformly.

On the other hand, when $\ell_3 \le \ell_2 < \ell_3 + \delta/2$ we use estimates

$$\mathcal{A}^{(q)}_{S,X}(\ell_3,\ell_2,\ell_1) \le \begin{cases} C\mathcal{A}_{S,X}(\ell_3,\ell_2,\ell_1) & \text{if } n_3 < n_2, \\ \dfrac{C}{\Delta_q(\ell_3,\ell_2,\ell_1)}\left(\dfrac{1-q}{1-\ell_3/\ell_1}\right)^{S+X} & \text{if } n_3 = n_2. \end{cases},$$

valid for some constant C independent of q of ℓ_2 and that can be deduced using Lemma C.2 and identity (6.24). This implies that

$$\int_{\ell_3}^{\ell_3+\delta/2} f(\ell_2)\mathcal{A}^{(q)}_{S,X}(\ell_3,\ell_2,\ell_1)\frac{d\ell_2}{\ell_2} = \mathsf{O}(\delta) + \mathsf{O}\left(\frac{1-q}{\delta}\right)^{S+X}.$$

In an analogous fashion one can also show that

$$\int_{\ell_1-\delta/2}^{\ell_1} f(\ell_2)\mathcal{A}^{(q)}_{S,X}(\ell_3,\ell_2,\ell_1)\frac{d\ell_2}{\ell_2} = \mathsf{O}(\delta) + \mathsf{O}\left(\frac{1-q}{\delta}\right)^{S-X}.$$

This concludes the proof of (C.9) when $\ell_1 - \ell_3 > \delta$.

Case 2, $\ell_1 - \ell_3 \le \delta$. Assuming δ is very small, for any $\ell_2 \in [\ell_3, \ell_1]$, we can write, by continuity, $f(\ell_2) = f(\ell_1) + \mathsf{o}(1)$, where $\mathsf{o}(1)$ tends to 0 as $\delta \to 0$. Thus, we have

$$\begin{aligned}\int_{\ell_3}^{\ell_1} f(\ell_2)\mathcal{A}^{(q)}_{S,X}(\ell_3,\ell_2,\ell_1)\frac{d\ell_2}{\ell_2} &= \sum_{n_2=n_3}^{n_1} \left(f(\ell_1)+\mathsf{o}(1)\right) q^{-(S+X)(n_1-n_2)}\varphi_{q,q^{S+X},q^{2S}}(n_1-n_2|n_1-n_3) \\ &= f(\ell_1)+\mathsf{o}(1),\end{aligned}$$

and by Lemma 6.12 this concludes the analysis of the case $\ell_1 - \ell_3 \le \delta$.

Since all the estimates we provided are controlled as functions of δ, the convergence in (C.9) is uniform provided that ℓ_1 stays bounded. □

Proof of Proposition 6.15. The integral in the left-hand side of (6.25) is equal to

$$\begin{aligned}&\int_{L_{N,N}}^{L_{N,N-1}} \frac{dL_{N-1,N-1}}{L_{N-1,N-1}}\mathcal{A}^{(q)}_{S,X}(L_{N,N},L_{N-1,N-1},L_{N,N-1})\cdots \\ &\cdots\int_{L_{N,2}}^{L_{N,1}} \frac{dL_{N-1,1}}{L_{N-1,1}}\mathcal{A}^{(q)}_{S,X}(L_{N,2},L_{N-1,1},L_{N,1})\left(\frac{L_{N-1,N-1}\cdots L_{N-1,1}}{L_{N,N}\cdots L_{N,1}}\right)^X f(\underline{L}_{N-1})\end{aligned}$$

and we can take the $q \to 1$ limit in each of the $N-1$ integrals using Lemma C.3. This establishes the convergence to the right-hand side of (6.25) as $q \to 1$, uniformly on any compact subset of $\mathcal{W}_N$. □

Appendix D. Triangular Sums

Here we write down a number of identities of summations of certain symbols $a_{k,\ell}, b_\alpha$ used in the proof of Proposition 8.2. Fix a positive integer N, and assume that the symbols b_α, $\alpha = 1, \dots, N$ commute with each other. Let $a_{k,\ell}$ be

$$a_{k,\ell} = \begin{cases} 0 & \text{if } 0 = k, \text{ or } \ell = N+1; \\ 1 & \text{if } 0 \le k = \ell \le N; \\ \in \mathbb{R} & \text{else.} \end{cases}$$

Proposition D.1. *For any $N \ge 1$, the following identities hold*

$$\sum_{0 \le k < \ell \le N} a_{k+1,\ell} - a_{k,\ell} - a_{k+1,\ell+1} + a_{k,\ell+1} = N - \sum_{j=1}^{N-1} a_{j,j+1};$$

$$\sum_{0 \le k < \ell \le N} (\ell - k + 1)(a_{k+1,\ell+1} - a_{k,\ell+1}) - (\ell - k - 1)(a_{k+1,\ell} - a_{k,\ell}) = \sum_{j=1}^{N-1} a_{j,j+1};$$

$$\sum_{0 \le k < \ell \le N} (a_{k+1,\ell} - a_{k,\ell} - a_{k+1,\ell+1} + a_{k,\ell+1}) \sum_{\alpha=k+1}^{\ell} b_\alpha = \sum_{\alpha=1}^{N} b_\alpha;$$

$$\sum_{0 \le k < \ell \le N} (\ell - k - 1)^2 (a_{k+1,\ell} - a_{k,\ell}) - (\ell - k + 1)^2 (a_{k+1,\ell+1} - a_{k,\ell+1})$$
$$= - \sum_{j=1}^{N-1} a_{j,j+1} - 2 \sum_{1 \le k < \ell \le N} a_{k,\ell};$$

$$\sum_{0 \le k < \ell \le N} (a_{k+1,\ell} - a_{k,\ell} - a_{k+1,\ell+1} + a_{k,\ell+1}) \sum_{k+1 \le \alpha, \beta \le l} b_\alpha b_\beta = \sum_{\alpha=1}^{N} b_\alpha^2 + 2 \sum_{1 \le k < \ell \le N} a_{k,\ell} b_k b_\ell;$$

$$\sum_{0 \le k < \ell \le N} \Big((\ell - k + 1)(a_{k+1,\ell+1} - a_{k,\ell+1}) - (\ell - k - 1)(a_{k+1,\ell} - a_{k,\ell}) \Big) \sum_{\alpha=k+1}^{\ell} b_\alpha$$
$$= \sum_{1 \le k < \ell \le N}^{N} a_{k,\ell} (b_k - b_\ell).$$

Proof. All these identities are elementary and can be proven by induction in a straightforward way. □

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