PERTURBED BETA CORNERS PROCESS

LEONID PETROV AND JIAMING XU

Abstract. We introduce and study the perturbed β-corners process, a deformation of the classical β-corners process. The latter is a probability measure on interlacing arrays of real numbers that, for the classical values $\beta = 1, 2, 4$, describes the joint distribution of eigenvalues of principal submatrices of a Gaussian random matrix with the corresponding GOE/GUE/GSE symmetry.

For classical β, the perturbed process arises from adding a deterministic diagonal matrix $A = \mathrm{diag}(a_1, \ldots, a_N)$ to a GOE/GUE/GSE matrix. The eigenvalues of the resulting random matrix depend symmetrically on $a_1, \ldots, a_N$, but this symmetry does not extend to the whole corners process.

We extend the construction to all $\beta > 0$ via multivariate Bessel functions, and analyze the crystallization ($\beta \to \infty$) of the resulting interlacing array, proving a law of large numbers and a central limit theorem in two regimes. For fixed perturbation, the eigenvalue repulsion dominates, and the array crystallizes on the same lattice of polynomial-derivative roots, with the same discrete Gaussian free field fluctuations, as in the unperturbed case treated by Gorin and Marcus [GM20]. New phenomena appear in the second regime, where the a_i's grow linearly in β. Then the external source competes with the repulsion at leading order. Here the array freezes on a deformed lattice characterized by a coupled system of optimality equations. The fluctuations are governed by the same discrete Gaussian free field, now attached to the deformed lattice.

The deformed lattice equations decouple in the special case of a single spike in the last coordinate. Then the deformed lattice is obtained explicitly by applying one shifted derivative $D_c f = f' + cf$ followed by iterated ordinary derivatives.

Contents

1. Introduction

1.1. **Overview.** We introduce and study the *perturbed β-corners process*, a deformation of the Gaussian β-corners process depending on parameters $\mathbf{a} = (a_1, \ldots, a_N)$. For the classical values $\beta = 1, 2, 4$, the construction reduces to computing the joint eigenvalue distribution of nested principal submatrices of $A + G$, where $A = \mathrm{diag}(a_1, \ldots, a_N)$ is deterministic and G is a matrix

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from the Gaussian Orthogonal/Unitary/Symplectic Ensemble (GOE/GUE/GSE), respectively. We extend the definition to all $\beta > 0$ by analytic continuation, in which the group-theoretic Harish-Chandra–Itzykson–Zuber (HCIZ) integral is replaced by multivariate Bessel functions, the symmetric eigenfunctions of the Dunkl operators. We then study the zero-temperature limit $\beta \to \infty$ at fixed particle number N.

The Gaussian β-ensemble (GβE) is the joint eigenvalue density proportional to

$$\prod_{1 \le i < j \le N} |\lambda_i - \lambda_j|^\beta \, e^{-(1/2)\sum \lambda_i^2},$$

which for $\beta = 1, 2, 4$ corresponds to the eigenvalues of GOE/GUE/GSE matrices.

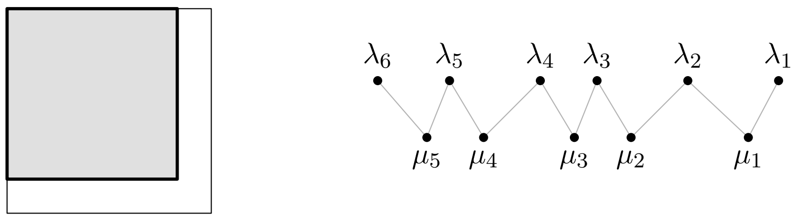

FIGURE 1. Cutting the upper left $(N-1) \times (N-1)$ corner. The eigenvalues $\boldsymbol{\mu}$ of the corner interlace with the eigenvalues $\boldsymbol{\lambda}$ of the matrix.

The *corners process* describes the joint distribution of eigenvalues of nested principal submatrices (Figure 1). For $\beta = 2$ (GUE), this structure was identified by Baryshnikov [Bar01] and Johansson–Nordenstam [JN06]; for general $\beta > 0$, the Gaussian case was constructed by Gorin and Shkolnikov [GS15] via multilevel Dyson Brownian motions, and the Jacobi case by Borodin and Gorin [BG15], who also proved that the global fluctuations of the β-corners process as $N \to \infty$ and β fixed are governed by the Gaussian free field. An alternative approach to β-corners processes, based on beta integrals over Rayleigh triangles, was developed by Neretin [Ner03]. We note that there is a tridiagonal random matrix model for GβE for all $\beta > 0$ introduced by Dumitriu and Edelman [DE02], which, however, cannot be used to construct the corners process, even in the unperturbed case [GK24].

In the log-gas picture, β plays the role of the inverse temperature, so that $\beta \to 0$ is the high-temperature regime and $\beta \to \infty$ the zero-temperature one. The former was studied by Benaych-Georges, Cuenca, and Gorin [BGCG22] and by Cuenca and Dołega [CD25]. The zero-temperature regime (leading to crystallization of the eigenvalues) of the β-corners process was established by Gorin and Marcus [GM20]: as $\beta \to \infty$ the eigenvalue array freezes on a deterministic lattice given by iterated polynomial derivatives, with discrete Gaussian free field fluctuations on top of it. This has been further developed by Gorin and Kleptsyn [GK24] (the G∞E and Airy$_\infty$ ensembles), and extended to Laguerre corners by Lerner-Brecher [LB25]. In the language of finite free probability of Marcus, Spielman, and Srivastava [MSS22] and Marcus [Mar21], passing to eigenvalues of a corner one size less at $\beta = \infty$ is the finite free projection, that is, differentiation of the characteristic polynomial.

Adding a deterministic matrix to a random one — an *external source* or *spike* — is also classical. Baik, Ben Arous, and Péché [BBP05] found, for sample covariance matrices, the phase transition at which a sufficiently strong spike makes the largest eigenvalue detach from the bulk; the additive counterpart, for Hermitian matrices shifted by a finite-rank deterministic matrix, is due to Péché [Péc06]. This has led to numerous follow-up works, which we do not survey here.

The idea of adding a deterministic matrix was extended to random matrix corners. For $\beta = 2$, Ferrari and Frings [FF14] considered the matrix diffusion $H(t) + tA$, where H is the Hermitian Brownian motion started at the null matrix, so that at $t = 1$ one recovers $A + G$. Under $H(t)$, the eigenvalues perform the Dyson Brownian motion [Dys62]. Ferrari and Frings computed the correlation kernel of the corners process of $H(t) + tA$ at a fixed time, and identified the resulting corners distribution with the fixed-time distribution of a Warren process [War07] in which all Brownian motions at level k carry the same drift a_k and reflect off those at level $k - 1$. The same correlation kernel was obtained independently by Adler, van Moerbeke, and Wang [AvMW13]. Petrov and Tikhonov [PT20] studied the interplay between the symmetry of the level-k eigenvalue distribution in $a_1, \dots, a_k$ and the failure of this symmetry for the whole corners distribution, via certain Markov jump operators acting on individual particles. For general β, Bloemendal and Virág [BV13] and Lamarre and Shkolnikov [GLS19] studied the edge of the single-level Gaussian β-ensemble with one spike (through the tridiagonal model of [DE02]). Two recent works rely on the same machinery as the present paper, namely Dunkl operators acting on multivariate Bessel generating functions: Keating and Xu [KX24] consider the β-addition of a Gaussian and finitely many Laguerre β-ensembles (the same operation as ours, except that the added spectrum is random instead of the fixed diagonal A) and prove that its edge limit is the Airy$_\beta$ process. Gorin, Xu, and Zhang [GXZ24] characterize the Airy$_\beta$ line ensemble and identify it as the edge limit of the unperturbed GβE corners process. In both cases the limit is $N \to \infty$ at fixed β. We instead fix N, let $\beta \to \infty$, and follow the perturbation through the whole interlacing array.

1.2. **Main results.** Let us outline our main contributions.

First, for the classical values $\beta \in \{1, 2, 4\}$, we derive the level-by-level Markov transition kernels (*links*) of the corners process of $C = A+G$ (where A is fixed and G is random GOE/GUE/GSE) by a change-of-variables calculation (Proposition 2.1). We then extend the density of the perturbed corners process to all $\beta > 0$ via multivariate Bessel functions (see Opdam [Opd93] and Heckman and Opdam [HO97]), which replace the classical HCIZ integrals available only for $\beta = 1, 2, 4$. The resulting definition of the perturbed β-corners process (Definition 3.4) recovers the classical β formulas and preserves the interlacing structure.

Second, we analyze the fixed N, $\beta \to \infty$ limit of the perturbed β-corners process with a fixed top row in two regimes:

- *Fixed perturbation* (all a_i fixed), when the eigenvalue repulsion dominates. The corners eigenvalues crystallize on the same lattice as in the unperturbed model of Gorin and Marcus [GM20], and the perturbation becomes invisible (Theorem 4.6). The rescaled fluctuations converge to the discrete Gaussian free field described in [GM20].
- *Scaled perturbation* ($a_i = \frac{\beta}{2}\mathsf{a}_i$), when the external source competes with the repulsion. This regime produces a *deformed* crystal lattice whose positions are the unique solution of a coupled system of optimality equations (Definition 5.1 and Theorem 5.8). The eigenvalue fluctuations converge to a deformed discrete Gaussian free field (Theorem 5.12). When the perturbation is constant ($\mathsf{a}_1 = \cdots = \mathsf{a}_N$), we recover the same limits as in the unperturbed case.

For the asymptotic analysis as $\beta \to \infty$, a key simplification is that the multivariate Bessel function of the top row appears in the perturbed GβE density and, inverted, in the perturbed links, so it cancels from their product. The joint density is then governed by an explicit log-gas energy (which we refer to as the effective Hamiltonian).

In both cases (fixed and scaled perturbation), the array of the perturbed β-corners process crystallizes on the minimizer of the effective Hamiltonian. In the unperturbed case, treated in [GM20], the minimizer is explicit: it is the lattice of roots of iterated polynomial derivatives. The deformed lattice admits no such formula in general (but does so for a single spike, see Section 5.3). We proceed by showing the strict convexity of the effective Hamiltonian (and, moreover, that it blows up at the boundary of the Gelfand–Tsetlin polytope); see Proposition 5.5 and Lemma 5.3. The unique global minimizer is then the deformed lattice. Once the convexity is established, the law of large numbers and the central limit theorem follow by a standard Laplace-type argument [Won01, Chapter 9]. See Remark 5.7 for a detailed comparison of our technique with the approach of [GM20].

In Section 6, we take the same zero-temperature limit in the other direction, of the conditional distribution of the top row $\boldsymbol{\lambda}$ given the lower row $\boldsymbol{\mu}$ (cf. Figure 1), in both regimes of fixed and scaled perturbation. These limits of upward transitions are powered by Rodrigues-type weighted derivatives $w^{-1}(wf)'$, where w depends on the regime and can be Gaussian, exponential, or constant.

1.3. **Further directions.** We work at fixed N throughout; the $N \to \infty$ limit of the perturbed β-corners process is out of scope of this paper. In particular, we do not consider a possible BBP-type transition for a perturbed version of the Airy$_\beta$ line ensemble. For a single level and a single spike this transition is known [BV13], [GLS19], and for finitely many spikes it is known at $\beta = 1, 2, 4$ [BV16]. Extending this to several spikes at general β, and to the whole interlacing array rather than its top row alone, remains open. We also do not know whether the distributional symmetry of the perturbed GUE corners process discovered in [PT20] extends to general β; their construction rests on the uniform Gibbs property of the GUE, which nontrivially deforms for general β. Two further extensions are a perturbed *Jacobi* process, whose eigenvalues are confined to a compact interval and which carries additional boundary parameters, and the high-temperature regime $\beta \to 0$, where the repulsion weakens instead of freezing the array. Matrix addition in that regime is treated by Benaych-Georges, Cuenca, and Gorin [BGCG22].

1.4. **Notation.** We collect the main notation used throughout the paper. The parameter $\beta > 0$ is the inverse temperature (with $\beta = 1, 2, 4$ corresponding to the GOE, GUE, and GSE, respectively), and $N \geq 1$ is the matrix size, which remains fixed throughout the asymptotic analysis.

Bold letters denote ordered tuples: $\boldsymbol{\lambda} = (\lambda_1 \geq \cdots \geq \lambda_N)$, $\boldsymbol{\mu} = (\mu_1 \geq \cdots \geq \mu_{N-1})$, $\mathbf{a} = (a_1, \ldots, a_N)$ (perturbation parameters), $\mathbf{z} = (z_1 > \cdots > z_N)$ (fixed top row). The interlacing array has levels $\mathbf{y}^k = (y_1^k \geq \cdots \geq y_k^k)$ for $k = 1, \ldots, N$, and we write $\mathbf{Y} = (\mathbf{y}^1, \ldots, \mathbf{y}^{N-1})$ for the collection of lower levels.

Interlacing $\boldsymbol{\mu} \prec \boldsymbol{\lambda}$ means $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \cdots \geq \mu_{N-1} \geq \lambda_N$. The arrays $\mathbf{Y}$ satisfying $\mathbf{y}^1 \prec \cdots \prec \mathbf{y}^{N-1} \prec \mathbf{z}$ with the top row $\mathbf{z}$ held fixed form the *Gelfand–Tsetlin polytope*. We write $\mathbf{1}_A$ for the indicator of an event or condition A. For a tuple $\mathbf{w}$ of distinct reals we write $V(\mathbf{w}) = \prod_{i<j} |w_i - w_j|$ for its Vandermonde. The notation $:=$ stands for "is defined as."

Key objects defined in the text: $B_{\mathbf{a}}(\mathbf{z}; \beta)$ is the multivariate Bessel function (Section 3.1), $f_{\mathbf{a}}$ is the perturbed GβE density (3.6), $\Lambda_{\mathbf{a}}$ is the perturbed link (3.11), and $F_{\mathbf{a}} = f_{\mathbf{a}} \cdot \Lambda_{\mathbf{a}}$ is the joint density of the perturbed corners process (Definition 3.4). In Sections 4 and 5 we reserve sans-serif letters for deterministic objects, italic ones remaining random: y_i^k denotes the unperturbed deterministic lattice (Definition 4.1), and $\bar{\mathsf{y}}_i^k$ the deformed lattice (Definition 5.1) arising under the scaled perturbation $a_i = \frac{\beta}{2}\mathsf{a}_i$ with $\mathsf{a} = (\mathsf{a}_1, \ldots, \mathsf{a}_N)$. The capital letters denote the whole arrays: the ∞-corners lattice $\mathsf{Y} = \{\mathsf{y}_i^k\}$ and the deformed lattice $\bar{\mathsf{Y}} = \{\bar{\mathsf{y}}_i^k\}$. The shifted derivative operator is $D_c f = f' + cf$ (5.24).

1.5. **Outline.** The remainder of the paper is organized as follows. Section 2 derives the perturbed corners process for $\beta = 1, 2, 4$ from the matrix model, and Section 3 extends it to all $\beta > 0$ via multivariate Bessel functions. The zero-temperature analysis fills the last three sections. A fixed perturbation leaves the unperturbed lattice in place (Section 4); a perturbation growing linearly in β deforms it (Section 5). Section 6 considers zero-temperature the behavior of the upper row given the lower row in the perturbed β-corners process, and identifies the limit in both regimes as a weighted derivative.

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AI statement. All the ideas in this paper are due to the authors. Some proof details in Sections 5 and 6, as well as parts of the text, were produced with AI assistance (ChatGPT Pro GPT-5.6 by OpenAI and Claude Code Opus 4.8 and Fable 5 by Anthropic). We checked every statement and every argument, and the responsibility for the correctness of the paper is ours alone.

2. Derivation for $\beta = 1, 2, 4$ from Gaussian matrix models

2.1. **Setup.** Consider the matrix addition

$$C = A + G, \tag{2.1}$$

where $A = \operatorname{diag}(a_1, \ldots, a_N)$ with fixed deterministic eigenvalues $a_1, \ldots, a_N \in \mathbb{R}$, and $G = \frac{X+X^*}{2}$ is the $N \times N$ GOE/GUE/GSE matrix. Here X^* denotes the transpose for the GOE ($\beta = 1$, real entries), the conjugate transpose for the GUE ($\beta = 2$, complex entries), and the quaternionic conjugate transpose for the GSE ($\beta = 4$, quaternionic entries). The matrix X is $N \times N$, with independent identically distributed real/complex/quaternionic Gaussian entries, normalized so that $\operatorname{Re}(X_{ij}) \sim \mathcal{N}(0, 1)$ for all $i, j = 1, \ldots, N$.

Denote the eigenvalues of $C = C_N$ by $\boldsymbol{\lambda} = (\lambda_1 \geq \cdots \geq \lambda_N)$, and the eigenvalues of C_{N-1}, the $(N-1) \times (N-1)$ upper left corner of C, by $\boldsymbol{\mu} = (\mu_1 \geq \cdots \geq \mu_{N-1})$. The goal of this section is to derive the conditional density $h(\boldsymbol{\lambda} \mid \boldsymbol{\mu})$ of $\boldsymbol{\lambda}$ given $\boldsymbol{\mu}$ (Proposition 2.1 below).

2.2. **Diagonalization.** Let us diagonalize C_{N-1} by some $U \in U_\beta(N-1)$ (here $U_\beta(n)$ stands for the Orthogonal/Unitary/Unitary Symplectic group of size n, respectively, for $\beta = 1, 2, 4$), and conjugate C_N by $\begin{bmatrix} U & 0 \\ 0 & 1 \end{bmatrix} \in U_\beta(N)$. We get

$$\tilde{C} := \begin{bmatrix} U & 0 \\ 0 & 1 \end{bmatrix} C \begin{bmatrix} U^* & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \mu_1 & 0 & & 0 & P_1 \\ 0 & \ddots & & 0 & P_2 \\ & & \ddots & & \\ & & & \mu_{N-1} & P_{N-1} \\ P_1^* & \ldots & \ldots & P_{N-1}^* & \xi_N \end{bmatrix}.$$

Here the last diagonal entry is $\xi_N = c_{NN} \sim \mathcal{N}(a_N, 1)$, and

$$P_i = \sum_{j=1}^{N-1} u_{ij} c_{jN}, \qquad i = 1, \ldots, N-1. \tag{2.2}$$

In vector form, (2.2) reads $(P_1, \ldots, P_{N-1})^t = U\,(c_{1N}, \ldots, c_{N-1,N})^t$: the P_i are the entries of the last column of C above the diagonal, written in the eigenbasis of the corner C_{N-1}. Their joint distribution can be understood as follows. First, A is diagonal, so the off-diagonal entries of C coincide with those of G, and their joint law does not involve the perturbation parameters. Second, in the GOE/GUE/GSE the entries on and above the diagonal are independent, so these $N-1$ entries are independent of the corner C_{N-1}; since U is a function of C_{N-1} alone, they are independent of U as well. Third, the joint law of $P_1, \dots, P_{N-1}$ is invariant under $U_\beta(N-1)$, so multiplying by U does not change it. Consequently, the P_i's are independent, with the real/complex/quaternionic Gaussian distribution normalized so that $\mathrm{Re}(P_i) \sim \mathcal{N}(0, \frac{1}{2})$, exactly as in the unperturbed case.

2.3. **Characteristic polynomial.** The eigenvalue equation $\det(x - \tilde{C}) = 0$,[1] divided by the product $\pm(x-\mu_1)\cdots(x-\mu_{N-1})$, takes the form

$$\sum_{i=1}^{N-1} \frac{\xi_i}{x-\mu_i} - (x - \xi_N) = 0. \tag{2.3}$$

The coefficients ξ_i, $i = 1, \dots, N-1$, are expressed through P_i (2.2) as

$$\xi_i = P_i P_i^* = \|P_i\|^2 \sim \sum_{j=1}^{\beta} \mathcal{N}(0, \tfrac{1}{2})^2 \sim \tfrac{1}{2}\chi_\beta^2,$$

where χ_β^2 is the chi-squared distribution with β degrees of freedom.

Since λ_i's are all the N roots of (2.3), we have the interlacing $\boldsymbol{\mu} \prec \boldsymbol{\lambda}$ thanks to the behavior of the left-hand side around its poles μ_i. Moreover, (2.3) can be written in the following equivalent forms:

$$\sum_{i=1}^{N-1} \frac{\xi_i}{x-\mu_i} = -\frac{(x-\lambda_1)\cdots(x-\lambda_N)}{(x-\mu_1)\cdots(x-\mu_{N-1})} + x - \xi_N, \qquad \text{or} \tag{2.4}$$

$$\sum_{i=1}^{N-1} \xi_i \prod_{j\neq i}(x-\mu_j) = \prod_{i=1}^{N-1}(x-\mu_i)(x-\xi_N) - (x-\lambda_1)\cdots(x-\lambda_N). \tag{2.5}$$

Conditioning on $\boldsymbol{\mu}$, identities (2.4) or (2.5) define a bijection

$$(\xi_1, \dots, \xi_{N-1}, \xi_N) \longleftrightarrow (\lambda_1, \dots, \lambda_N).$$

To find the conditional density of $\boldsymbol{\lambda}$ given $\boldsymbol{\mu}$, it remains to calculate the Jacobian $\left|\frac{\partial \xi_i}{\partial \lambda_j}\right|_{i,j=1,\dots,N}$.

2.4. **Jacobian.** Setting $x = \mu_i$ for $i = 1, \dots, N-1$ in (2.5), we have

$$\xi_i = -\frac{(\mu_i-\lambda_1)\cdots(\mu_i-\lambda_N)}{\prod_{j\neq i}(\mu_i-\mu_j)} \implies \frac{\partial \xi_i}{\partial \lambda_j} = \frac{\prod_{d\neq j}(\mu_i-\lambda_d)}{\prod_{d\neq i}(\mu_i-\mu_d)}, \tag{2.6}$$

where $j = 1, \dots, N$.

For $i = N$, by comparing the constant terms of the two polynomials in x in (2.5), we get

$$(-1)^{N+1}\prod_{i=1}^{N}\lambda_i + (-1)^N \xi_N \prod_{i=1}^{N-1}\mu_i = \sum_{i=1}^{N-1}\xi_i(-1)^{N-2}\prod_{j\neq i}\mu_j$$

[1]For $\beta = 4$ the matrix $\tilde{C}$ is quaternionic, and det denotes the reduced (Moore) determinant, that is, the monic degree-N polynomial $\prod_{i=1}^{N}(x-\lambda_i)$. It obeys the same block Schur-complement factorization as in the real and complex cases, so the derivation below is uniform in $\beta = 1, 2, 4$.

$$
\begin{aligned}
&\implies \xi_N = \frac{\sum_{i=1}^{N-1} \xi_i \prod_{k\neq i} \mu_k + \prod_{i=1}^{N} \lambda_i}{\prod_{i=1}^{N-1} \mu_i} \\
&\implies \frac{\partial \xi_N}{\partial \lambda_j} = \frac{\sum_{i=1}^{N-1} \frac{\partial \xi_i}{\partial \lambda_j} \prod_{k\neq i} \mu_k + \prod_{d\neq j} \lambda_d}{\prod_{i=1}^{N-1} \mu_i}.
\end{aligned} \tag{2.7}
$$

Here also $j = 1, \dots, N$.

Note that the terms in (2.7) involving $\frac{\partial \xi_i}{\partial \lambda_j}$ are linear combinations of the previous rows of the Jacobian matrix, and can be ignored when calculating the determinant. Thus, we have

$$
\det\left(\frac{\partial \xi_i}{\partial \lambda_j}\right)_{i,j=1,\dots,N}
$$

$$
= \frac{1}{\prod_{k=1}^{N-1}\prod_{d\neq k}(\mu_k-\mu_d)} \times \prod_{k=1}^{N-1}\prod_{d=1}^{N}(\mu_k-\lambda_d) \times \frac{\prod_{d=1}^{N}\lambda_d}{\prod_{k=1}^{N-1}\mu_k} \det\begin{bmatrix} \frac{1}{\mu_1-\lambda_1} & \cdots & \frac{1}{\mu_1-\lambda_N} \\ \cdots & & \cdots \\ \cdots & & \cdots \\ \frac{1}{\mu_{N-1}-\lambda_1} & \cdots & \frac{1}{\mu_{N-1}-\lambda_N} \\ \frac{1}{\lambda_1} & \cdots & \frac{1}{\lambda_N} \end{bmatrix}.
$$

The last determinant above is the Cauchy determinant, so it is equal to

$$
(-1)^{N-1} \frac{\prod_{1\le i<j\le N}(\lambda_i-\lambda_j)(\mu_j-\mu_i)}{\prod_{i=1}^{N}\prod_{j=1}^{N}(\lambda_i-\mu_j)},
$$

where we set $\mu_N = 0$ for convenience. Hence, up to $\boldsymbol{\mu}$-dependent factors, the absolute value of the Jacobian is the Vandermonde:

$$
\left|\det\left(\frac{\partial \xi_i}{\partial \lambda_j}\right)_{i,j=1,\dots,N}\right| \propto \prod_{1\le i<j\le N}(\lambda_i-\lambda_j).
$$

2.5. **Conditional density.** We can now put everything together for classical β:

Proposition 2.1. *For $\beta = 1, 2$, or 4, let $\boldsymbol{\lambda} = (\lambda_1 \ge \cdots \ge \lambda_N)$ be the eigenvalues of the perturbed $N\times N$ GOE/GUE/GSE matrix $C = A+G$ (2.1), and let $\boldsymbol{\mu} = (\mu_1 \ge \cdots \ge \mu_{N-1})$ be the eigenvalues of its $(N-1)\times(N-1)$ principal corner. Then the conditional density of $\boldsymbol{\lambda}$ given $\boldsymbol{\mu}$ is*

$$
h(\boldsymbol{\lambda} \mid \boldsymbol{\mu}) \propto \exp\left(-\frac{1}{2}\sum_{i=1}^{N}\lambda_i^2\right)\exp\left(a_N\sum_{i=1}^{N}\lambda_i\right)\prod_{1\le i<j\le N}(\lambda_i-\lambda_j)\prod_{i=1}^{N-1}\prod_{j=1}^{N}|\mu_i-\lambda_j|^{\beta/2-1}\mathbf{1}_{\boldsymbol{\lambda}\succ\boldsymbol{\mu}}, \tag{2.8}
$$

where the proportionality constant depends only on $\boldsymbol{\mu}$ and the parameters a_i.

Proof. The variables $(\xi_1, \dots, \xi_{N-1}, \xi_N)$ are conditionally independent given $\boldsymbol{\mu}$, with joint density

$$
f(\boldsymbol{\xi}) \propto \left[\prod_{i=1}^{N-1}\xi_i^{\beta/2-1}e^{-\xi_i}\right]e^{-\xi_N^2/2}e^{a_N\xi_N}.
$$

By the first identity in (2.6), ξ_i vanishes precisely when μ_i coincides with one of the λ_j. Hence $\xi_i > 0$ for all $i = 1, \dots, N-1$ iff the interlacing $\boldsymbol{\lambda} \succ \boldsymbol{\mu}$ is strict. The weak boundary, where $\xi_i = 0$ for some i, is Lebesgue-null and irrelevant for the density. The same identity gives $\xi_i^{\beta/2-1} \propto \prod_{j=1}^{N}|\mu_i-$

$\lambda_j|^{\beta/2-1}$, where we absorbed $\boldsymbol{\mu}$-dependent terms into the normalization constant. Comparing the coefficients by x^{N-1} in (2.5) gives

$$\xi_N = \sum_{i=1}^{N} \lambda_i - \sum_{i=1}^{N-1} \mu_i, \tag{2.9}$$

so $e^{a_N \xi_N} \propto \exp(a_N \sum_i \lambda_i)$. Finally, comparing the coefficients by x^{N-2} in (2.5) yields the remaining terms:

$$\sum_{i=1}^{N-1} \xi_i = - \sum_{1\le i<j\le N} \lambda_i\lambda_j + \left(\sum_{i=1}^{N} \lambda_i\right)\left(\sum_{i=1}^{N-1} \mu_i\right) + \sum_{1\le i<j\le N-1} \mu_i\mu_j - \left(\sum_{i=1}^{N-1} \mu_i\right)^2. \tag{2.10}$$

Combined with (2.9), we have

$$-\sum_{i=1}^{N-1} \xi_i - \frac{\xi_N^2}{2} = -\frac{1}{2}\sum_{i=1}^{N} \lambda_i^2 + \frac{1}{2}\sum_{i=1}^{N-1} \mu_i^2.$$

Multiplying by the Jacobian $\prod_{i<j}(\lambda_i - \lambda_j)$ from Section 2.4 completes the proof. □

Remark 2.2. Collecting all the terms depending only on $\boldsymbol{\mu}$ in the proof of Proposition 2.1, one can write the conditional density (2.8) more explicitly as

$$h(\boldsymbol{\lambda} \mid \boldsymbol{\mu}) = \frac{e^{-a_N^2/2}}{\Gamma(\beta/2)^{N-1}\sqrt{2\pi}} \prod_{i=1}^{N-1}\prod_{j=1}^{N} |\mu_i - \lambda_j|^{\beta/2-1} \prod_{1\le i<j\le N-1} (\mu_i-\mu_j)^{1-\beta} \prod_{1\le i<j\le N} (\lambda_i - \lambda_j)$$
$$\times \exp\left(a_N \sum_{i=1}^{N} \lambda_i - a_N \sum_{i=1}^{N-1} \mu_i - \frac{1}{2}\sum_{i=1}^{N} \lambda_i^2 + \frac{1}{2}\sum_{i=1}^{N-1} \mu_i^2\right) \mathbf{1}_{\boldsymbol{\lambda}\succ\boldsymbol{\mu}}.$$

3. Perturbed corners process for general β

We now extend the model from Section 2 to general $\beta > 0$ via analytic continuation. The key ingredients are well-known, and include the Gaussian β-ensembles (GβE) [DE02], the β-corners process [GS15], and the β-addition operation on random matrix spectra [GM20], [BGCG22], [KX24], [GXZ24]. For general β, there is no underlying matrix model for which one can consider the corners, so the construction relies on multivariate Bessel generating functions.

3.1. **Multivariable Bessel generating functions.** Here we briefly recall the necessary properties of multivariate Bessel functions which are β-deformations of the Harish-Chandra-Itzykson-Zuber (HCIZ) integrals [HC57], [IZ80].

Remark 3.1. In the literature on Jack polynomials, multivariate Bessel functions, and Dunkl operators it is customary to use the parameters $\theta = \beta/2$ or $\alpha = 2/\beta$ instead of β. In this paper we keep the notation β throughout for consistency with the random matrix formulas.

Let $\mathbf{a} = (a_1, \ldots, a_N)$ be an N-tuple of real numbers. The *multivariate Bessel function*, denoted by $B_{\mathbf{a}}(\mathbf{z};\beta)$, where $\mathbf{z} = (z_1, \ldots, z_N)$ and $\beta > 0$, can be defined as the unique (up to multiplication by a constant) entire eigenfunction, symmetric in $z_1, \ldots, z_N$, of the symmetric Dunkl operators [Opd93]

$$\mathcal{P}_k := \mathcal{D}_1^k + \cdots + \mathcal{D}_N^k, \qquad \mathcal{D}_i := \frac{\partial}{\partial z_i} + \frac{\beta}{2}\sum_{j\ne i} \frac{1}{z_i - z_j}(1 - s_{ij}), \qquad s_{ij} \text{ permutes } z_i \text{ and } z_j,$$

such that

$$\mathcal{P}_k B_{\mathbf{a}}(\mathbf{z};\beta) = \Big(\sum_{i=1}^{N} a_i^k\Big) B_{\mathbf{a}}(\mathbf{z};\beta), \qquad k = 1, 2, \dots.$$

Alternatively, the functions $B_{\mathbf{a}}(\mathbf{z};\beta)$ arise as certain limits of Jack symmetric polynomials, see [OO97], [BGCG22, Section 2.2]. Key properties of the multivariate Bessel functions include:

- Normalization $B_{\mathbf{a}}(\mathbf{0};\beta) = 1$.[2]
- Symmetry in $z_1, \dots, z_N$.
- Extension to an entire function of the $2N$ variables $a_1, \dots, a_N, z_1, \dots, z_N$ jointly; in particular, $B_{\mathbf{a}}(\mathbf{z};\beta)$ is continuous in the label $\mathbf{a}$ at fixed $\mathbf{z}$ and β.
- Duality between the label and the argument:

$$B_{\mathbf{a}}(\mathbf{z};\beta) = B_{\mathbf{z}}(\mathbf{a};\beta). \tag{3.1}$$

  This is manifest from the expansion of $B_{\mathbf{a}}(\mathbf{z};\beta)$ as the ${}_0F_0^{(2/\beta)}$ hypergeometric (Jack) series, which is symmetric in the two sets of variables $\mathbf{a}$ and $\mathbf{z}$ (see [OO97, (4.2)], [For10]); for $\beta = 2$ it also follows from the HCIZ integral below.
- Positivity: $B_{\mathbf{a}}(\mathbf{z};\beta) > 0$ for all real $\mathbf{a}$ and all $\mathbf{z}$ with pairwise distinct coordinates, which is the only case we use. Indeed, by (3.1) it equals $B_{\mathbf{z}}(\mathbf{a};\beta)$, and the corners representation (3.9) in Remark 3.3 below writes the latter as the integral of a positive integrand against a probability density. The duality is what makes this representation applicable: it requires the label to have pairwise distinct coordinates, and in the formulas below the perturbation $\mathbf{a}$ may have ties, while $\mathbf{z}$ does not.

For $\beta = 2$, the Bessel function admits the HCIZ integral representation:

$$B_{\mathbf{a}}(\mathbf{z};2) = \int_{U(N)} \exp\big(\mathrm{Tr}(UAU^*Z)\big)\, dU,$$

where $A = \mathrm{diag}(a_1, \dots, a_N)$, $Z = \mathrm{diag}(z_1, \dots, z_N)$, and dU is the Haar measure on the unitary group $U(N)$ normalized to have total mass 1.

Given a probability measure μ on the ordered N-tuples $\mathbf{b} = (b_1 \geq \dots \geq b_N) \in \mathbb{R}^N$, define the *Bessel generating function of* μ by

$$G_N(\mathbf{z};\beta,\mu) := \int_{b_1 \geq \dots \geq b_N} B_{\mathbf{b}}(\mathbf{z};\beta)\, d\mu(\mathbf{b}). \tag{3.2}$$

By the symmetry and normalization of Bessel functions, G_N is symmetric in $z_1, \dots, z_N$ and satisfies $G_N(\mathbf{0};\beta,\mu) = 1$.

For $\beta = 2$, let $M_A = UAU^*$ and $M_B = VBV^*$ be independent random Hermitian matrices with fixed eigenvalues $\mathbf{a}$ and $\mathbf{b}$, respectively (and random eigenvectors coming from the independent Haar-distributed unitary matrices $U, V \in U(N)$). Let $\mathbf{c}$ denote the eigenvalues of $M_A + M_B$. Then we have

$$\mathbb{E}\big[B_{\mathbf{c}}(\mathbf{z};2) \mid \mathbf{a}, \mathbf{b}\big] = B_{\mathbf{a}}(\mathbf{z};2) \cdot B_{\mathbf{b}}(\mathbf{z};2).$$

For general $\beta \neq 1, 2, 4$, when no underlying invariant matrix model exists, the β-addition $\boxplus_\beta$ is *defined* via the multiplication of Bessel generating functions [GM20], [BGCG22], see (3.5) below.

[2] Here and below, $\mathbf{0}$ stands for the tuple $(0, \dots, 0)$.

3.2. **The GβE and perturbed density.** The Gaussian β-ensemble (GβE) is the symmetric probability measure on ordered N-tuples $\mathbf{g} = (g_1 \geq \cdots \geq g_N) \in \mathbb{R}^N$ with density

$$f(\mathbf{g}) \coloneqq \frac{1}{Z_{N,\beta}} \prod_{1 \leq i < j \leq N} (g_i - g_j)^\beta \prod_{i=1}^{N} \exp\left(-\frac{1}{2} g_i^2\right), \tag{3.3}$$

where $Z_{N,\beta}$ is the probability normalizing constant. The scaling inside the exponent is chosen so that the Bessel generating function (3.2) of the GβE is

$$G_N(\mathbf{x}; \beta, \mathrm{G}\beta\mathrm{E}) = \exp\Big(\frac{1}{2} \sum_{i=1}^{N} x_i^2\Big). \tag{3.4}$$

Consider the β-addition of an N-dimensional GβE random vector $\mathbf{g}$ and a deterministic N-tuple $\mathbf{a} = (a_1, \ldots, a_N) \in \mathbb{R}^N$, denoted by

$$\mathbf{c} = \mathbf{a} \boxplus_\beta \mathbf{g}. \tag{3.5}$$

By definition, the Bessel generating function of $\mathbf{c}$ is the product

$$\mathbb{E}\big[B_{\mathbf{c}}(\mathbf{x}; \beta)\big] \coloneqq B_{\mathbf{a}}(\mathbf{x}; \beta) \cdot \exp\Big(\frac{1}{2} \sum_{i=1}^{N} x_i^2\Big).$$

The operation $\boxplus_\beta$ (3.5) is the β-extension of the matrix addition (2.1) from the classical values $\beta = 1, 2, 4$.

Proposition 3.2. *The probability density of* $\mathbf{c}$ (3.5) *has the form*

$$f_{\mathbf{a}}(\mathbf{x}) \coloneqq \frac{1}{Z_{N,\beta}} \exp\Big(-\frac{1}{2} \sum_{i=1}^{N} a_i^2\Big) \exp\Big(-\frac{1}{2} \sum_{i=1}^{N} x_i^2\Big) \prod_{1 \leq i < j \leq N} (x_i - x_j)^\beta B_{\mathbf{a}}(\mathbf{x}; \beta), \tag{3.6}$$

where $Z_{N,\beta}$ *is the same normalizing constant as in* (3.3).

Proof. The random vector $\mathbf{c}$ is equal in distribution to the marginal at time 1 of the standard β-Dyson Brownian motion started from $\mathbf{a}$. The density formula (3.6) follows from [GXZ24, (23)]. For additional references, see the proof of Lemma 3.8 in [GXZ24]. ☐

The single-level law (3.6) is the Gaussian β-ensemble with external source: after the rescaling $\mathbf{x} = \sqrt{\beta}\,\mathbf{u}$ and $\mathbf{a} = \sqrt{\beta}\,\mathbf{f}$, it coincides with the density studied by Desrosiers and Liu [DL15]; see also [For13]. Our contribution begins with the compatible multilevel corners law of Definition 3.4.

3.3. **The β-corners process.** The Gaussian (Hermite) β-corners process is a probability distribution on interlacing arrays $\mathbf{y}^1 \prec \mathbf{y}^2 \prec \cdots \prec \mathbf{y}^{N-1} \prec \mathbf{y}^N$ with levels

$$\mathbf{y}^k = (y_1^k \geq y_2^k \geq \cdots \geq y_k^k) \in \mathbb{R}^k.$$

For the classical values $\beta = 1, 2, 4$, the corners process describes the joint distribution of eigenvalues of all principal submatrices of a GOE/GUE/GSE matrix, respectively.

Consider first the unperturbed β-corners process (see [GS15]; the general β density already appears in [Ner03]) with fixed top row $\mathbf{y}^N = \mathbf{z} = (z_1 > \cdots > z_N)$.[3] Denote by $\mathbf{Y} = (\mathbf{y}^1, \ldots, \mathbf{y}^{N-1})$

[3] The density requires a strict top row, since the normalizing constant includes $\prod_{i<j}(z_i - z_j)^{\beta-1}$. On the other hand, under the GβE density, the coordinates of $\mathbf{z}$ are distinct with probability one.

the collection of lower levels. The conditional density of $\mathbf{Y}$ given $\mathbf{z}$ is, by definition,

$$\Lambda(\mathbf{Y};\mathbf{z}) = \frac{\mathbf{1}_{\mathbf{y}^1\prec\cdots\prec\mathbf{y}^{N-1}\prec\mathbf{z}}}{\tilde{Z}_{N,\beta,\mathbf{z}}}\prod_{k=1}^{N-1}\left(\prod_{1\le i<j\le k}(y_i^k-y_j^k)^{2-\beta}\prod_{p=1}^{k}\prod_{q=1}^{k+1}|y_p^k-y_q^{k+1}|^{\beta/2-1}\right). \tag{3.7}$$

The normalizing constant is given by the Dixon–Anderson formula [Dix05], [And91] (see also [For10, Chapter 4]):

$$\tilde{Z}_{N,\beta,\mathbf{z}} = \prod_{k=1}^{N}\frac{\Gamma(\beta/2)^k}{\Gamma(k\beta/2)}\cdot\prod_{1\le i<j\le N}(z_i-z_j)^{\beta-1}. \tag{3.8}$$

The joint density of the full unperturbed Gaussian β-corners process thus has the form

$$F(\mathbf{Y},\mathbf{z}) = f(\mathbf{z})\cdot\Lambda(\mathbf{Y};\mathbf{z}),$$

where f is the GβE density (3.3).

Remark 3.3. The β-corners process provides an integral representation for the multivariate Bessel function [GK02]:

$$B_{\mathbf{z}}(\mathbf{x};\beta) = \int\Lambda(\mathbf{Y};\mathbf{z})\cdot\exp\left(\sum_{k=1}^{N}x_k\cdot\left(\sum_{i=1}^{k}y_i^k-\sum_{j=1}^{k-1}y_j^{k-1}\right)\right)d\mathbf{Y}, \tag{3.9}$$

where the integral is with respect to the β-corners process conditioned on having the fixed top row $\mathbf{y}^N=\mathbf{z}$, see (3.7).

3.4. **Perturbed β-corners process.** Combining the perturbed GβE (3.6) with the unperturbed links (3.7), we now give the main definition:

Definition 3.4. The *perturbed Gaussian β-corners process* with perturbation $\mathbf{a}=(a_1,\ldots,a_N)$ is a random interlacing array $\mathbf{y}^1\prec\mathbf{y}^2\prec\cdots\prec\mathbf{y}^{N-1}\prec\mathbf{y}^N=\mathbf{z}$, with joint density

$$F_{\mathbf{a}}(\mathbf{Y},\mathbf{z}) \coloneqq f_{\mathbf{a}}(\mathbf{z})\cdot\Lambda_{\mathbf{a}}(\mathbf{Y};\mathbf{z}), \tag{3.10}$$

where $f_{\mathbf{a}}$ is the perturbed GβE density (3.6), and the perturbed links are defined as

$$\Lambda_{\mathbf{a}}(\mathbf{Y};\mathbf{z}) \coloneqq \Lambda(\mathbf{Y};\mathbf{z})\cdot\frac{1}{B_{\mathbf{a}}(\mathbf{z};\beta)}\exp\left[a_N\sum_{i=1}^{N}z_i+\sum_{k=1}^{N-1}\sum_{i=1}^{k}y_i^k(a_k-a_{k+1})\right]. \tag{3.11}$$

The following statement shows that the perturbed corners process has the desired Bessel generating function:

Proposition 3.5. *The perturbed Gaussian β-corners process satisfies, for every $\mathbf{x}\in\mathbb{R}^N$,*

$$\int\int F_{\mathbf{a}}(\mathbf{Y},\mathbf{z})\exp\Big[x_N\sum_{i=1}^{N}z_i+\sum_{k=1}^{N-1}\sum_{i=1}^{k}y_i^k(x_k-x_{k+1})\Big]\,d\mathbf{Y}\,d\mathbf{z} = \exp\Big(\frac{1}{2}\sum_{i=1}^{N}x_i^2\Big)\exp\Big(\sum_{i=1}^{N}a_ix_i\Big),$$

where the integration is over all interlacing arrays $\mathbf{y}^1\prec\mathbf{y}^2\prec\cdots\prec\mathbf{y}^{N-1}\prec\mathbf{z}$, and we denote $\mathbf{Y}=(\mathbf{y}^1,\ldots,\mathbf{y}^{N-1})$.

Proof. By (3.9) and the duality (3.1) (which lets us write $B_{\mathbf{z}}(\mathbf{a};\beta)=B_{\mathbf{a}}(\mathbf{z};\beta)$), we have

$$\int\Lambda(\mathbf{Y};\mathbf{z})\exp\Big[\sum_{k=1}^{N-1}\sum_{i=1}^{k}y_i^k(a_k-a_{k+1})\Big]\,d\mathbf{Y} = B_{\mathbf{a}}(\mathbf{z};\beta)\exp\Big(-a_N\sum_{i=1}^{N}z_i\Big). \tag{3.12}$$

Therefore,

$$\begin{aligned}\int \Lambda_{\mathbf{a}}(\mathbf{Y};\mathbf{z}) \exp\Big[\sum_{k=1}^{N-1}\sum_{i=1}^{k} y_i^k (x_k - x_{k+1})\Big]\, d\mathbf{Y} \\ = \frac{1}{B_{\mathbf{a}}(\mathbf{z};\beta)} \exp\Big(a_N \sum_{i=1}^{N} z_i\Big) B_{\mathbf{a}+\mathbf{x}}(\mathbf{z};\beta) \exp\Big(-(a_N + x_N)\sum_{i=1}^{N} z_i\Big) \\ = \frac{B_{\mathbf{a}+\mathbf{x}}(\mathbf{z};\beta)}{B_{\mathbf{a}}(\mathbf{z};\beta)} \exp\Big(-x_N \sum_{i=1}^{N} z_i\Big).\end{aligned}$$

Let us multiply by the remaining $\mathbf{z}$-dependent factors and integrate over $\mathbf{z}$. We have

$$\begin{aligned}&\int \frac{B_{\mathbf{a}+\mathbf{x}}(\mathbf{z};\beta)}{B_{\mathbf{a}}(\mathbf{z};\beta)} f_{\mathbf{a}}(\mathbf{z}) d\mathbf{z} \\ &= \int \frac{1}{Z_{N,\beta}} \exp\Big(-\frac{1}{2}\sum_{i=1}^{N} a_i^2\Big) \exp\Big(-\frac{1}{2}\sum_{i=1}^{N} z_i^2\Big) \prod_{1\le i<j\le N} (z_i - z_j)^{\beta} B_{\mathbf{a}+\mathbf{x}}(\mathbf{z};\beta) d\mathbf{z} \\ &= \exp\Big(-\frac{1}{2}\sum_{i=1}^{N} a_i^2\Big) \exp\Big(\frac{1}{2}\sum_{i=1}^{N} (a_i + x_i)^2\Big) = \exp\Big(\frac{1}{2}\sum_{i=1}^{N} x_i^2\Big) \exp\Big(\sum_{i=1}^{N} a_i x_i\Big),\end{aligned}$$

where the last line follows from the duality (3.1) and the Bessel generating function property (3.4) of the unperturbed GβE. This completes the proof. □

3.5. **Matching with matrix models.** Let us verify that Definition 3.4 reduces to the matrix model construction of Section 2 when $\beta = 1, 2$, or 4.

Expanding the joint density (3.10), we have

$$\begin{aligned}F_{\mathbf{a}}(\mathbf{Y},\mathbf{z}) = \frac{1}{Z_{N,\beta}} \Lambda(\mathbf{Y};\mathbf{z}) \exp\Big[a_N \sum_{i=1}^{N} z_i + \sum_{k=1}^{N-1}\sum_{i=1}^{k} y_i^k (a_k - a_{k+1})\Big] \\ \times \exp\Big(-\frac{1}{2}\sum_{i=1}^{N} (a_i^2 + z_i^2)\Big) \prod_{1\le i<j\le N} (z_i - z_j)^{\beta}.\end{aligned} \tag{3.13}$$

Since $\Lambda(\mathbf{Y};\mathbf{z})$ (3.7)–(3.8) contains the $\mathbf{z}$-dependent factor

$$\prod_{1\le i<j\le N} (z_i - z_j)^{1-\beta} \prod_{p=1}^{N-1}\prod_{q=1}^{N} |y_p^{N-1} - z_q|^{\beta/2-1} \mathbf{1}_{\mathbf{z}\succ\mathbf{y}^{N-1}},$$

the conditional density of $\mathbf{z}$ given the lower levels is

$$h(\mathbf{z} \mid \mathbf{Y}) \propto \exp\Big(-\frac{1}{2}\sum_{i=1}^{N} z_i^2\Big) \exp\Big(a_N \sum_{i=1}^{N} z_i\Big) \prod_{1\le i<j\le N} (z_i - z_j) \prod_{p=1}^{N-1}\prod_{q=1}^{N} |y_p^{N-1} - z_q|^{\beta/2-1} \mathbf{1}_{\mathbf{z}\succ\mathbf{y}^{N-1}}. \tag{3.14}$$

This depends on $\mathbf{Y}$ only through $\mathbf{y}^{N-1}$, and it matches the conditional density (2.8) from Proposition 2.1. That identifies the top transition. It remains to check that the levels below the top form a perturbed corners process of size $N-1$.

Lemma 3.6. *Let $N \ge 2$ and write $\mathbf{a}^{(k)} = (a_1, \dots, a_k)$.*

(i) *For $\beta > 0$ and $1 \le k \le N-1$, the conditional density of* $(\mathbf{y}^1, \dots, \mathbf{y}^k)$ *given* $(\mathbf{y}^{k+1}, \dots, \mathbf{y}^N)$ *under Definition 3.4 is* $\Lambda_{\mathbf{a}^{(k+1)}}(\,\cdot\,; \mathbf{y}^{k+1})$.

(ii) *For $\beta = 1, 2, 4$ the marginal density of* $\mathbf{y}^{N-1}$ *is* $f_{\mathbf{a}^{(N-1)}}$, *and* $(\mathbf{y}^1, \dots, \mathbf{y}^{N-1})$ *is the perturbed β-corners process with perturbation* $\mathbf{a}^{(N-1)}$.

Proof. (i) Fix $\mathbf{y}^{k+1}, \dots, \mathbf{y}^N$. In (3.10)–(3.11) the factors $f_{\mathbf{a}}(\mathbf{z})$, $B_{\mathbf{a}}(\mathbf{z}; \beta)$, $\tilde{Z}_{N,\beta,\mathbf{z}}$ and $\exp(a_N \sum_i z_i)$, together with the factors of (3.7) indexed by $k' \ge k+1$, do not involve $\mathbf{y}^1, \dots, \mathbf{y}^k$, and the interlacing indicator splits as $\mathbf{1}_{\mathbf{y}^1 \prec \cdots \prec \mathbf{y}^{k+1}} \mathbf{1}_{\mathbf{y}^{k+1} \prec \cdots \prec \mathbf{z}}$. What remains is

$$\mathbf{1}_{\mathbf{y}^1 \prec \cdots \prec \mathbf{y}^{k+1}} \prod_{k'=1}^{k} \left(\prod_{1 \le i < j \le k'} (y_i^{k'} - y_j^{k'})^{2-\beta} \prod_{p=1}^{k'} \prod_{q=1}^{k'+1} \left| y_p^{k'} - y_q^{k'+1} \right|^{\beta/2-1} \right) \exp\left[\sum_{k'=1}^{k} \sum_{i=1}^{k'} y_i^{k'} (a_{k'} - a_{k'+1}) \right],$$

which is $\Lambda_{\mathbf{a}^{(k+1)}}(\,\cdot\,; \mathbf{y}^{k+1})$ times a factor depending on $\mathbf{y}^{k+1}$ alone. Both are probability densities in $(\mathbf{y}^1, \dots, \mathbf{y}^k)$, so they coincide.

(ii) Write $\boldsymbol{\mu} = \mathbf{y}^{N-1}$ and $\mathbf{Y}' = (\mathbf{y}^1, \dots, \mathbf{y}^{N-2})$. Splitting off the $k = N-1$ factor of (3.7) and integrating out $\mathbf{Y}'$ by the corners representation (3.9) at size $N-1$ with $\mathbf{x} = \mathbf{a}^{(N-1)}$ gives

$$\int \Lambda(\mathbf{Y}'; \boldsymbol{\mu}) \exp\left[\sum_{k=1}^{N-2} \sum_{i=1}^{k} y_i^k (a_k - a_{k+1}) \right] d\mathbf{Y}' = \exp\Big(-a_{N-1} \sum_{i=1}^{N-1} \mu_i \Big) B_{\boldsymbol{\mu}}(\mathbf{a}^{(N-1)}; \beta).$$

Combining the two Vandermonde powers by (3.8), the joint density of $(\boldsymbol{\mu}, \mathbf{z})$ is a constant multiple of

$$\prod_{1 \le i < j \le N-1} (\mu_i - \mu_j) \exp\Big(-a_N \sum_{i=1}^{N-1} \mu_i \Big) B_{\boldsymbol{\mu}}(\mathbf{a}^{(N-1)}; \beta) \mathcal{K}(\boldsymbol{\mu}, \mathbf{z}),$$

where $\mathcal{K}$ is the right-hand side of (2.8) with $\boldsymbol{\lambda} = \mathbf{z}$. Since the density of Remark 2.2 integrates to 1,

$$\int \mathcal{K}(\boldsymbol{\mu}, \mathbf{z})\, d\mathbf{z} = \Gamma(\beta/2)^{N-1} \sqrt{2\pi}\, e^{a_N^2/2} \prod_{1 \le i < j \le N-1} (\mu_i - \mu_j)^{\beta-1} \exp\Big(a_N \sum_{i=1}^{N-1} \mu_i - \frac{1}{2} \sum_{i=1}^{N-1} \mu_i^2 \Big).$$

The two exponentials in a_N cancel and the Vandermonde powers add up to β, so the marginal density of $\boldsymbol{\mu}$ is a constant multiple of $B_{\boldsymbol{\mu}}(\mathbf{a}^{(N-1)}; \beta) \exp(-\frac{1}{2} \sum_i \mu_i^2) \prod_{i<j} (\mu_i - \mu_j)^\beta$, which by the duality (3.1) is (3.6) at size $N-1$. Together with (i) for $k = N-2$, this is the second assertion. $\square$

Both constructions now obey the same recursion in N. On the one hand, (3.14) and Lemma 3.6 give

$$F_{\mathbf{a}}(\mathbf{Y}, \mathbf{z}) = F_{\mathbf{a}^{(N-1)}}\big(\mathbf{Y}', \mathbf{y}^{N-1}\big)\, h(\mathbf{z} \mid \mathbf{y}^{N-1}), \qquad \mathbf{Y}' = (\mathbf{y}^1, \dots, \mathbf{y}^{N-2}). \tag{3.15}$$

On the other hand, the corner C_{N-1} of $C = A + G$ (2.1) is itself a perturbed GOE/GUE/GSE matrix with perturbation $\mathbf{a}^{(N-1)}$, so the eigenvalues of $C_1, \dots, C_{N-1}$ form the corresponding array one size down. Conditionally on the whole of C_{N-1}, the top row $\boldsymbol{\lambda}$ has the density (2.8), since $\boldsymbol{\lambda}$ is determined by $\boldsymbol{\mu}$ and $\boldsymbol{\xi}$ through (2.3), and $\boldsymbol{\xi}$ is independent of C_{N-1} by the discussion after (2.2). So the matrix model array satisfies (3.15) as well. For $N = 1$ both laws are $\mathcal{N}(a_1, 1)$. By induction on N, for $\beta = 1, 2, 4$ the perturbed β-corners process of Definition 3.4, defined through multivariate Bessel functions, agrees with the perturbed corners processes constructed from the matrix models.

3.6. **Down transition density.** The marginal and conditional densities also give the *down transition density* $\Lambda_{\mathbf{a}}(\boldsymbol{\mu},\boldsymbol{\lambda})$, which goes in the opposite direction from $h(\boldsymbol{\lambda}\mid\boldsymbol{\mu})$:

Proposition 3.7. *Let $N\ge 2$. The conditional density of $\mathbf{y}^{N-1}=\boldsymbol{\mu}=(\mu_1\ge\cdots\ge\mu_{N-1})$ given $\mathbf{y}^N=\boldsymbol{\lambda}=(\lambda_1\ge\cdots\ge\lambda_N)$ under the perturbed β-corners process is proportional to*

$$\Lambda_{\mathbf{a}}(\boldsymbol{\mu},\boldsymbol{\lambda})\propto\prod_{1\le i<j\le N-1}(\mu_i-\mu_j)\prod_{i=1}^{N-1}\prod_{j=1}^{N}|\mu_i-\lambda_j|^{\beta/2-1}B_{\mathbf{a}'}(\boldsymbol{\mu};\beta)\exp\Big(-a_N\sum_{i=1}^{N-1}\mu_i\Big)\mathbf{1}_{\boldsymbol{\mu}\prec\boldsymbol{\lambda}},\qquad(3.16)$$

where $\mathbf{a}'=(a_1,\dots,a_{N-1})$ and the proportionality constant depends on $\boldsymbol{\lambda}$ and $\mathbf{a}$.

Proof. The conditional density of $\boldsymbol{\mu}=\mathbf{y}^{N-1}$ given $\boldsymbol{\lambda}=\mathbf{y}^N=\mathbf{z}$ is proportional, as a function of $\boldsymbol{\mu}$, to the joint density of $(\mathbf{y}^{N-1},\mathbf{y}^N)$ obtained from (3.10) by integrating out the lower levels $\mathbf{y}^1,\dots,\mathbf{y}^{N-2}$. Peeling off the $k=N-1$ factor from the links (3.7) and writing $\Lambda'(\mathbf{Y}';\boldsymbol{\mu})$ for the $(N-1)$-level unperturbed links with top row $\boldsymbol{\mu}$, where $\mathbf{Y}'=(\mathbf{y}^1,\dots,\mathbf{y}^{N-2})$, we have

$$\Lambda(\mathbf{Y};\mathbf{z})=\frac{\tilde Z_{N-1,\beta,\boldsymbol{\mu}}}{\tilde Z_{N,\beta,\mathbf{z}}}\Lambda'(\mathbf{Y}';\boldsymbol{\mu})\prod_{1\le i<j\le N-1}(\mu_i-\mu_j)^{2-\beta}\prod_{p=1}^{N-1}\prod_{q=1}^{N}|\mu_p-\lambda_q|^{\beta/2-1}\mathbf{1}_{\boldsymbol{\mu}\prec\boldsymbol{\lambda}}.$$

The exponential in (3.11) splits as $\sum_{k=1}^{N-2}\sum_{i=1}^{k}y_i^k(a_k-a_{k+1})+(a_{N-1}-a_N)\sum_{i=1}^{N-1}\mu_i$. Integrating the lower levels against $\Lambda'(\mathbf{Y}';\boldsymbol{\mu})$ by (3.9) in dimension $N-1$ (exactly as in the proof of Proposition 3.5) gives

$$\int\Lambda'(\mathbf{Y}';\boldsymbol{\mu})\exp\Big[\sum_{k=1}^{N-2}\sum_{i=1}^{k}y_i^k(a_k-a_{k+1})\Big]\,d\mathbf{Y}'=B_{\mathbf{a}'}(\boldsymbol{\mu};\beta)\exp\Big(-a_{N-1}\sum_{i=1}^{N-1}\mu_i\Big).$$

Here $f_{\mathbf{a}}(\mathbf{z})$, $B_{\mathbf{a}}(\mathbf{z};\beta)$, $\tilde Z_{N,\beta,\mathbf{z}}$ and $\exp(a_N\sum_i z_i)$ are constants in $\boldsymbol{\mu}$. Using

$$\tilde Z_{N-1,\beta,\boldsymbol{\mu}}\propto\prod_{1\le i<j\le N-1}(\mu_i-\mu_j)^{\beta-1}$$

from (3.8), the two Vandermonde powers combine as $(\mu_i-\mu_j)^{\beta-1}(\mu_i-\mu_j)^{2-\beta}=(\mu_i-\mu_j)$, and the two exponentials as $(a_{N-1}-a_N)\sum_i\mu_i-a_{N-1}\sum_i\mu_i=-a_N\sum_i\mu_i$. This completes the proof of (3.16). □

3.7. **Random polynomial equation.** The polynomial equation (2.3) derived in Section 2 for $\beta=1,2,4$ extends to general $\beta>0$. This provides an alternative characterization of the perturbed β-corners process, which we use in the zero-temperature analysis below.

Proposition 3.8 (Random polynomial equation for general β)**.** *Let $\boldsymbol{\lambda}=(\lambda_1\ge\cdots\ge\lambda_N)$ and $\boldsymbol{\mu}=(\mu_1\ge\cdots\ge\mu_{N-1})$ be consecutive levels of the perturbed β-corners process with perturbation $\mathbf{a}=(a_1,\dots,a_N)$. Conditionally on $\boldsymbol{\mu}$, the eigenvalues $\boldsymbol{\lambda}$ are the N roots of the random equation*

$$\sum_{i=1}^{N-1}\frac{\xi_i}{x-\mu_i}=x-\xi_N,\qquad(3.17)$$

where the random variables $\xi_1,\dots,\xi_{N-1},\xi_N$ are conditionally independent given $\boldsymbol{\mu}$, with

$$\xi_i\sim\frac{1}{2}\chi^2_\beta,\quad i=1,\dots,N-1,\qquad\text{and}\qquad\xi_N\sim\mathcal{N}(a_N,1).\qquad(3.18)$$

Equivalently, the eigenvalues satisfy the polynomial identity

$$\prod_{i=1}^{N-1}(x-\mu_i)(x-\xi_N)-\prod_{j=1}^{N}(x-\lambda_j)=\sum_{i=1}^{N-1}\xi_i\prod_{k\neq i}(x-\mu_k). \tag{3.19}$$

Proof. The computation repeats the one in Section 2 word for word. There, the classical value of β is used only to identify the law of $\xi_i=\|P_i\|^2$ as $\frac{1}{2}\chi^2_\beta$; the polynomial equation (2.3), the Jacobian of Section 2.4, and the assembly of the conditional density in the proof of Proposition 2.1 are algebraic identities carrying β only through the exponent $\beta/2-1$. For general $\beta>0$ the distributions (3.18) are postulated rather than read off a matrix model, and the same steps turn the density of $\boldsymbol{\xi}$ into the transition kernel (3.14). □

Let us make two concluding remarks about the perturbed β-corners process before turning to asymptotics.

Remark 3.9. Assume $N\geq 2$. The interlacing $\boldsymbol{\mu}\prec\boldsymbol{\lambda}$ follows from (3.17): the left-hand side has poles at each μ_i with positive residues $\xi_i>0$, while the right-hand side is linear in x. By analyzing signs, exactly one root λ_j lies in each interval (μ_{i+1},μ_i), $i=1,\dots,N-2$, plus one root above μ_1 and one below μ_{N-1}. (For $N=1$ equation (3.17) reduces to $\lambda_1=\xi_N$, with no lower row.)

Remark 3.10 (Relation to finite free probability)**.** The polynomial identity (3.19) connects to finite free probability [MSS22], [Mar21], [GM20]. Comparing the coefficients of x^{N-1} and of x^{N-2} in (3.19) returns the identities (2.9) and (2.10), which thus hold for all $\beta>0$. These moment relations encode how the spectrum of a sum of matrices relates to the spectra of the summands.

4. Zero-temperature asymptotics: fixed perturbation

In this section and the next one we analyze the $\beta\to\infty$ asymptotics of the perturbed β-corners process. Here, the perturbation parameters $\mathbf{a}=(a_1,\dots,a_N)$ remain fixed as β grows; a perturbation growing with β is considered in Section 5.

4.1. The ∞-corners lattice.

The $\beta\to\infty$ behavior of the *unperturbed* β-corners process (3.7) was determined in [GM20]: the array freezes onto a deterministic lattice of polynomial-derivative roots, with Gaussian fluctuations of order $\beta^{-1/2}$ on top of it. In this subsection, we recall the results from [GM20] in the form useful to us.

Definition 4.1. Given a top row $\mathbf{z}=(z_1>\cdots>z_N)$, the ∞-*corners lattice* $\mathsf{Y}:=\{\mathsf{y}_i^k\}_{1\leq i\leq k\leq N}$ is defined as follows: $\mathsf{y}_i^N=z_i$ for $i=1,\dots,N$, and for $k=N-1,\dots,1$, the points $\mathsf{y}_1^k>\cdots>\mathsf{y}_k^k$ are the k roots of the polynomial

$$Q^{N\to k}(u)=\frac{1}{N(N-1)\cdots(k+1)}\left(\frac{d}{du}\right)^{N-k}\prod_{j=1}^{N}(u-z_j).$$

Equivalently [GM20, Section 3.4], the lattice positions satisfy the *optimality equations*

$$\sum_{j=1}^{k+1}\frac{1}{\mathsf{y}_i^k-\mathsf{y}_j^{k+1}}=0,\qquad i=1,\dots,k, \tag{4.1}$$

which characterize the critical point of the functional $\prod_{p=1}^{k}\prod_{q=1}^{k+1}|u_p-\mathsf{y}_q^{k+1}|$ in the free variable $\mathbf{u}=(u_1>\cdots>u_k)$, the row above being pinned to the lattice. Here $\mathbf{u}$ ranges over the interlacing chamber $\mathbf{u} \prec \mathsf{y}^{k+1}$, which is bounded and on whose boundary the functional vanishes, so the critical point is an interior maximum. It is unique: the logarithm of the functional splits as $\sum_p \sum_q \log |u_p - \mathsf{y}_q^{k+1}|$, strictly concave in each u_p.

Definition 4.2 (Discrete Gaussian Free Field, [GM20, Definition 1.5])**.** The *discrete Gaussian Free Field (dGFF)* on the ∞-corners lattice $\{\mathsf{y}_i^k\}$ is a centered Gaussian vector $\{\zeta_i^k\}_{1\le i\le k\le N}$ with $\zeta_i^N = 0$ for all $i = 1, \dots, N$ (the top row is pinned), and with the remaining variables having joint density proportional to

$$\exp\left(\sum_{k=1}^{N-1}\left[\sum_{1\le i<j\le k}\frac{(\zeta_i^k-\zeta_j^k)^2}{2(\mathsf{y}_i^k-\mathsf{y}_j^k)^2}-\sum_{p=1}^{k}\sum_{q=1}^{k+1}\frac{(\zeta_p^k-\zeta_q^{k+1})^2}{4(\mathsf{y}_p^k-\mathsf{y}_q^{k+1})^2}\right]\right). \tag{4.2}$$

Remark 4.3. The same-level terms in (4.2) carry a positive sign, so integrability is not automatic. It is proved in [GM20, Section 3.4] by integrating the levels one at a time from the bottom up, each step being an instance of the Gaussian identity [GM20, (64)]. In particular, this computation implies that the quadratic form in the exponent of (4.2) is *negative definite*, as a nonnegative eigenvalue of the form would make the integral diverge.

Theorem 4.4 ([GM20, Theorem 1.6])**.** *Let $\{x_i^k(\beta)\}_{1\le i\le k\le N}$ be the unperturbed β-corners process (3.7) with fixed top row $\mathbf{z} = (z_1 > \cdots > z_N)$, let $\{\mathsf{y}_i^k\}$ be the ∞-corners lattice with the same top row, and let $\{\zeta_i^k\}$ be the dGFF on it. Then, as $\beta \to \infty$, the following hold jointly in all $1 \le i \le k \le N$:*

(i) *$x_i^k(\beta) \to \mathsf{y}_i^k$ in probability;*
(ii) *the array $\{\sqrt{\beta}(x_i^k(\beta) - \mathsf{y}_i^k)\}$ converges in distribution to $\{\zeta_i^k\}$.*

Finally, we record identities tying consecutive levels of the ∞-corners lattice together. They are repeatedly used in Section 5 below.

Lemma 4.5. *For $2 \le k \le N$ and $1 \le i \le k$,*

$$\sum_{j=1}^{k}\prod_{m\ne j}(u-\mathsf{y}_m^k) = k\prod_{q=1}^{k-1}(u-\mathsf{y}_q^{k-1}), \tag{4.3}$$

$$\prod_{j\ne i}(\mathsf{y}_i^k-\mathsf{y}_j^k) = k\prod_{q=1}^{k-1}(\mathsf{y}_i^k-\mathsf{y}_q^{k-1}), \tag{4.4}$$

and consequently,

$$\sum_{j\ne i}\frac{1}{\mathsf{y}_i^k-\mathsf{y}_j^k} = \frac{1}{2}\sum_{q=1}^{k-1}\frac{1}{\mathsf{y}_i^k-\mathsf{y}_q^{k-1}}. \tag{4.5}$$

Proof. By Definition 4.1, the points y_q^{k-1} are the roots of the derivative of $\prod_{j=1}^{k}(u-\mathsf{y}_j^k)$, which is (4.3); setting $u = \mathsf{y}_i^k$ there gives (4.4). Differentiating (4.3) in u and then setting $u = \mathsf{y}_i^k$ gives

$$2\sum_{j\ne i}\prod_{m\ne i,j}(\mathsf{y}_i^k-\mathsf{y}_m^k) = k\sum_{q=1}^{k-1}\prod_{r\ne q}(\mathsf{y}_i^k-\mathsf{y}_r^{k-1}),$$

and dividing this by (4.4) yields (4.5). ∎

4.2. **Crystallization for a fixed perturbation.** We now show that a perturbation which does not grow with β leaves the zero-temperature limit unchanged. This needs no new asymptotic analysis. With the top row fixed, the perturbation enters the law of the array only through the factor

$$\exp\Big[\sum_{k=1}^{N-1}(a_k-a_{k+1})\sum_{i=1}^{k}y_i^k\Big]$$

of (3.11), which does not involve β; the remaining perturbation-dependent factors there do not involve $\mathbf{Y}$ and cancel upon normalization.

Theorem 4.6 (Crystallization for fixed perturbation)**.** *Let $\{y_i^k(\beta)\}_{1\le i\le k\le N}$ be the perturbed β-corners process with fixed perturbation $\mathbf{a}=(a_1,\dots,a_N)$ and fixed top row $\mathbf{z}=(z_1>\cdots>z_N)$. Let $\{\mathsf{y}_i^k\}$ be the deterministic ∞-corners lattice, and let $\{\zeta_i^k\}$ be the dGFF on this lattice. Then as $\beta\to\infty$:*

(i) *$y_i^k(\beta)\to\mathsf{y}_i^k$ in probability;*
(ii) *the array $\big\{\sqrt{\beta}\,\big(y_i^k(\beta)-\mathsf{y}_i^k\big)\big\}$ converges in distribution to $\{\zeta_i^k\}$.*

In particular, the limit is independent of the perturbation $\mathbf{a}$*.*

Proof. The top row is pinned: $y_i^N=z_i=\mathsf{y}_i^N$ and $\zeta_i^N=0$, so the case $k=N$ is trivial. Let us consider the random interlacing array $\mathbf{Y}=(\mathbf{y}^1,\dots,\mathbf{y}^{N-1})$. Set

$$h(\mathbf{Y}):=\exp\left[\sum_{k=1}^{N-1}\sum_{i=1}^{k}y_i^k(a_k-a_{k+1})\right]. \tag{4.6}$$

In the perturbed density (3.10)–(3.11) the three factors $f_{\mathbf{a}}(\mathbf{z})$, $B_{\mathbf{a}}(\mathbf{z};\beta)^{-1}$ and $\exp\big(a_N\sum_i z_i\big)$ do not depend on $\mathbf{Y}$. Hence, at the fixed top row $\mathbf{z}$, the perturbed process is the unperturbed β-corners process (3.7) reweighted by h: writing $\mathbb{E}_{\mathbf{a}}$ and $\mathbb{E}$ for the expectations under the perturbed and the unperturbed law with top row $\mathbf{z}$, we have

$$\mathbb{E}_{\mathbf{a}}\big[G(\mathbf{Y})\big]=\frac{\mathbb{E}\big[G(\mathbf{Y})\,h(\mathbf{Y})\big]}{\mathbb{E}\big[h(\mathbf{Y})\big]}\qquad\text{for every bounded measurable } G. \tag{4.7}$$

Here both densities integrate to 1 in $\mathbf{Y}$: for $\Lambda(\,\cdot\,;\mathbf{z})$ this is the Dixon–Anderson evaluation (3.8), and for $\Lambda_{\mathbf{a}}(\,\cdot\,;\mathbf{z})$ it follows from (3.12).

Note that h does not depend on β. Moreover, interlacing confines $\mathbf{Y}$ to the bounded Gelfand–Tsetlin polytope $\{\mathbf{y}^1\prec\cdots\prec\mathbf{y}^{N-1}\prec\mathbf{z}\}$, on which every coordinate obeys $z_N\le y_i^k\le z_1$, so that

$$\big|\log h(\mathbf{Y})\big|\le N^2\max_{1\le k\le N-1}|a_k-a_{k+1}|\cdot\max_{1\le i\le N}|z_i|=:C_{\mathbf{a},\mathbf{z}}\qquad\text{for every }\beta>0. \tag{4.8}$$

Thus h is bounded above and below by positive constants depending only on $\mathbf{a}$ and $\mathbf{z}$. (For $N=1$ the array $\mathbf{Y}$ is empty and $h\equiv 1$, so both assertions are vacuous; we take $N\ge 2$ below.)

(i) Let G be bounded and continuous. By part (i) of Theorem 4.4, $\mathbf{Y}\to\mathsf{Y}$ in probability under the unperturbed law; since G and h are continuous, $G(\mathbf{Y})h(\mathbf{Y})\to G(\mathsf{Y})h(\mathsf{Y})$ and $h(\mathbf{Y})\to h(\mathsf{Y})$ in probability as well. Both sequences are bounded, by hypothesis on G and by (4.8), so the expectations converge: $\mathbb{E}[Gh]\to G(\mathsf{Y})h(\mathsf{Y})$ and $\mathbb{E}[h]\to h(\mathsf{Y})>0$. Now (4.7) gives $\mathbb{E}_{\mathbf{a}}[G]\to G(\mathsf{Y})$ for every bounded continuous G, which implies convergence in probability under the perturbed law.

(ii) Put $\Delta\mathbf{Y} := \sqrt{\beta}\,(\mathbf{Y} - \mathsf{Y})$ and

$$\varepsilon(\mathbf{Y}) := \frac{h(\mathbf{Y})}{h(\mathsf{Y})} - 1 = \exp\left[\sum_{k=1}^{N-1}\sum_{i=1}^{k}\big(y_i^k - \mathsf{y}_i^k\big)(a_k - a_{k+1})\right] - 1.$$

By (4.8) we have $|\varepsilon| \le e^{2C_{\mathbf{a},\mathbf{z}}}$ on the polytope, and by part (i) of Theorem 4.4 the unperturbed array satisfies $\mathbf{Y} \to \mathsf{Y}$ in probability. Since h is continuous, $\varepsilon(\mathbf{Y}) \to 0$ in probability, and being bounded it converges to 0 in L^1 as well. Therefore, for bounded continuous G,

$$\mathbb{E}_{\mathbf{a}}\big[G(\Delta\mathbf{Y})\big] = \frac{\mathbb{E}\big[G(\Delta\mathbf{Y})(1+\varepsilon)\big]}{\mathbb{E}\big[1+\varepsilon\big]} = \frac{\mathbb{E}\big[G(\Delta\mathbf{Y})\big] + O\big(\|G\|_\infty \mathbb{E}|\varepsilon|\big)}{1 + O\big(\mathbb{E}|\varepsilon|\big)} \xrightarrow[\beta\to\infty]{} \mathbb{E}\big[G(\zeta)\big],$$

because $\Delta\mathbf{Y} \to \zeta$ in distribution under the unperturbed law by part (ii) of Theorem 4.4. This completes the proof. ☐

Remark 4.7 (Statistical mechanics interpretation)**.** The β-corners process can be viewed as particles at inverse temperature β subject to logarithmic repulsion and a confining potential. As $\beta \to \infty$ (zero temperature), particles freeze to their ground state positions, which are the roots of polynomial derivatives. The perturbation $\mathbf{a}$ acts as an external field of strength $O(1)$, while the repulsion grows as $O(\beta)$; the uniform bound (4.8) is the quantitative form of this mismatch. Thus, the field is too weak to move the ground state. The dGFF (Definition 4.2) captures thermal fluctuations around the ground state, at the scale $O(\beta^{-1/2})$ set by the energy barrier of order β. Again, the perturbation leaves the fluctuations unaffected.

5. Zero-temperature asymptotics: linearly growing perturbation

We now consider the regime where the perturbation grows linearly with β:

$$a_i = \frac{\beta}{2}\mathsf{a}_i, \qquad i = 1, \dots, N, \tag{5.1}$$

for fixed parameters $\mathsf{a} = (\mathsf{a}_1, \dots, \mathsf{a}_N) \in \mathbb{R}^N$. In this scaling, the external field competes with the eigenvalue repulsion. Throughout the subsection, we refer to the special case $\mathsf{a}_1 = \cdots = \mathsf{a}_N$ as the *constant perturbation*.

5.1. The deformed lattice: existence and uniqueness.

In the growing perturbation regime, the limiting crystal lattice is *deformed* compared to the unperturbed case:

Definition 5.1. Given a top row $\mathbf{z} = (z_1 > \cdots > z_N)$ and parameters $\mathsf{a} = (\mathsf{a}_1, \dots, \mathsf{a}_N)$, the *deformed* ∞*-corners lattice* $\{\bar{\mathsf{y}}_i^k\}_{1\le i\le k\le N}$ is defined by $\bar{\mathsf{y}}_i^N = z_i$ for $i = 1, \dots, N$, and the remaining positions $\{\bar{\mathsf{y}}_i^k\}_{1\le i\le k\le N-1}$ are the unique solution of the *coupled optimality equations*:

$$\sum_{\substack{j=1\\ j\ne i}}^{k} \frac{1}{\bar{\mathsf{y}}_i^k - \bar{\mathsf{y}}_j^k} - \frac{1}{2}\sum_{q=1}^{k+1}\frac{1}{\bar{\mathsf{y}}_i^k - \bar{\mathsf{y}}_q^{k+1}} - \frac{1}{2}\sum_{p=1}^{k-1}\frac{1}{\bar{\mathsf{y}}_i^k - \bar{\mathsf{y}}_p^{k-1}} = \frac{\mathsf{a}_k - \mathsf{a}_{k+1}}{2}, \qquad i = 1, \dots, k, \tag{5.2}$$

for $k = 1, \dots, N-1$, subject to *strict* interlacing. That is, $\bar{\mathsf{y}}^k \prec \bar{\mathsf{y}}^{k+1}$, and $\bar{\mathsf{y}}_i^k \ne \bar{\mathsf{y}}_j^{k+1}$ for all $1 \le k \le N-1$, $1 \le i \le k$ and $1 \le j \le k+1$. Equivalently, $\bar{\mathsf{Y}} := \{\bar{\mathsf{y}}_i^k\}_{1\le i\le k\le N-1}$ lies in the open Gelfand–Tsetlin polytope, where every denominator of (5.2) is nonzero. Existence and uniqueness of the solution to (5.2) are established in Proposition 5.6 below.

Remark 5.2. The system (5.2) couples all levels simultaneously: the equation for position (i,k) involves levels $k-1$, k, and $k+1$. For the constant perturbation, the right-hand side vanishes, and (5.2) holds for the unperturbed lattice Y (Definition 4.1), thanks to (4.1)–(4.5). By uniqueness, the deformed lattice $\bar{\mathsf{Y}}$ then coincides with Y.

The coupled optimality equations (5.2) arise as the critical point conditions for the *effective Hamiltonian*[4]

$$H_{\mathsf{a}}(\mathbf{Y}) \coloneqq \sum_{k=1}^{N-1}\left[\sum_{1\le i<j\le k}\log(y_i^k-y_j^k)-\frac{1}{2}\sum_{p=1}^{k}\sum_{q=1}^{k+1}\log|y_p^k-y_q^{k+1}|-\frac{\mathsf{a}_k-\mathsf{a}_{k+1}}{2}\sum_{i=1}^{k}y_i^k\right], \tag{5.3}$$

in which the top row $\mathbf{y}^N=\mathbf{z}$ is fixed and $\mathbf{Y}=(\mathbf{y}^1,\dots,\mathbf{y}^{N-1})$ ranges over the Gelfand–Tsetlin polytope $\{\mathbf{y}^1\prec\cdots\prec\mathbf{y}^{N-1}\prec\mathbf{z}\}$. It is read off from the joint density: at the fixed top row, and under the scaling (5.1), the $\mathbf{Y}$-dependent part of (3.10) (see also (3.13)) is exactly $e^{-\beta H_{\mathsf{a}}(\mathbf{Y})}$ times a factor not involving β.

In the rest of this subsection, we establish two properties of H_{a}: it tends to $+\infty$ at every boundary point of the polytope, and it is strictly convex in the interior. From these properties we deduce the existence and uniqueness of the deformed lattice, and later in Section 5.2 we use them for the zero-temperature asymptotic analysis.

Lemma 5.3. *Let*

$$\rho(\mathbf{Y}) \coloneqq \min_{1\le k\le N-1}\ \min_{1\le p\le k}\ \min_{1\le q\le k+1}\left|y_p^k-y_q^{k+1}\right|, \qquad \mathbf{y}^N=\mathbf{z}, \tag{5.4}$$

denote the smallest cross-level gap of the array. There is a constant $C=C(N,\mathbf{z},\mathsf{a})$ such that

$$H_{\mathsf{a}}(\mathbf{Y}) \ \ge\ \frac{1}{2}\log\frac{1}{\rho(\mathbf{Y})}-C \tag{5.5}$$

at every point $\mathbf{Y}$ of the open Gelfand–Tsetlin polytope. In particular $H_{\mathsf{a}}(\mathbf{Y})\to+\infty$ as $\mathbf{Y}$ approaches any boundary point of the polytope, and every sublevel set of H_{a} is a compact subset of its interior.

Proof. The linear term of (5.3) is bounded on the bounded polytope, so it suffices to bound its logarithmic part, which is $-\log\Psi(\mathbf{Y})$ for

$$\Psi(\mathbf{Y}) \coloneqq \prod_{k=1}^{N-1}\frac{\prod_{p=1}^{k}\prod_{q=1}^{k+1}\left|y_p^k-y_q^{k+1}\right|^{1/2}}{\prod_{1\le i<j\le k}(y_i^k-y_j^k)}. \tag{5.6}$$

For a strictly decreasing $\mathbf{w}\in\mathbb{R}^n$ write $\gamma(\mathbf{w})=\min_p(w_p-w_{p+1})$ for its smallest gap. We also display the top row, writing $\Psi(\mathbf{Y};\mathbf{w})$ and $\rho(\mathbf{Y};\mathbf{w})$ for the quantities (5.6) and (5.4) formed from an array $\mathbf{y}^1\prec\cdots\prec\mathbf{y}^{n-1}\prec\mathbf{w}$, so that $\Psi(\mathbf{Y})=\Psi(\mathbf{Y};\mathbf{z})$ and $\rho(\mathbf{Y})=\rho(\mathbf{Y};\mathbf{z})$. We prove the scale invariant bound

$$\Psi(\mathbf{Y};\mathbf{w}) \ \le\ V(\mathbf{w})\sqrt{\rho(\mathbf{Y};\mathbf{w})/\gamma(\mathbf{w})}, \qquad n\ge 2, \tag{5.7}$$

by induction on the number of levels; taking $n=N$, $\mathbf{w}=\mathbf{z}$ and logarithms gives (5.5).

Let $\mathbf{u}\in\mathbb{R}^{n-1}$ interlace $\mathbf{w}\in\mathbb{R}^n$ and put $\Pi=\prod_{p=1}^{n-1}\prod_{q=1}^{n}|u_p-w_q|$. For $1\le p\le n-1$ and $1\le q\le n$ set $B_{p,q}=w_q-w_{p+1}$ if $q\le p$ and $B_{p,q}=w_p-w_q$ if $q\ge p+1$. Each $B_{p,q}$ is a difference w_i-w_j with $i<j$, so $B_{p,q}\ge\gamma(\mathbf{w})$, and $w_{p+1}<u_p<w_p$ gives $|u_p-w_q|<B_{p,q}$ in both cases. As

[4] For $N=1$ the array $\mathbf{Y}$ is empty, so $H_{\mathsf{a}}\equiv 0$, and the deformed lattice is the top row itself. In the rest of this subsection we thus take $N\ge 2$.

(p,q) runs over its range, the pairs $(q,p+1)$ with $q \le p$ and the pairs (p,q) with $q \ge p+1$ each run exactly once through all pairs $i<j$ in $\{1,\dots,n\}$, so that $\prod_{p,q} B_{p,q} = V(\mathbf{w})^2$. Bounding all factors of Π but one by their $B_{p,q}$, and the omitted one from below by $\gamma(\mathbf{w})$, therefore gives, for every (p_0,q_0),

$$\Pi \;\le\; |u_{p_0} - w_{q_0}|\,\frac{V(\mathbf{w})^2}{\gamma(\mathbf{w})}.$$

Minimizing over (p_0,q_0) yields

$$\Pi \;\le\; \frac{V(\mathbf{w})^2}{\gamma(\mathbf{w})}\,\min_{p,q}|u_p - w_q|, \tag{5.8}$$

while choosing p_0 with $u_{p_0} - u_{p_0+1} = \gamma(\mathbf{u})$ and $q_0 = p_0+1$, for which $u_{p_0} - w_{p_0+1} < u_{p_0} - u_{p_0+1}$ because $u_{p_0+1} < w_{p_0+1}$, yields

$$\Pi \;\le\; \frac{V(\mathbf{w})^2}{\gamma(\mathbf{w})}\,\gamma(\mathbf{u}) \qquad (n\ge 3). \tag{5.9}$$

For $n=2$ the array is a single point, and (5.7) says that the larger of its two gaps to $\mathbf{w}$ is at most their sum. Let $n\ge 3$, assume (5.7) for $n-1$, and peel off the top bond, writing $\mathbf{u} = \mathbf{y}^{n-1}$ and $\mathbf{Y}' = (\mathbf{y}^1,\dots,\mathbf{y}^{n-2})$:

$$\Psi(\mathbf{Y};\mathbf{w}) = \Psi(\mathbf{Y}';\mathbf{u})\,\frac{\Pi^{1/2}}{V(\mathbf{u})} \;\le\; \sqrt{\frac{\rho(\mathbf{Y}';\mathbf{u})\,\Pi}{\gamma(\mathbf{u})}}, \qquad \rho(\mathbf{Y};\mathbf{w}) = \min\bigl\{\rho(\mathbf{Y}';\mathbf{u}),\ \min_{p,q}|u_p - w_q|\bigr\}.$$

If the minimum is the first term, (5.9) bounds the display by $V(\mathbf{w})\sqrt{\rho(\mathbf{Y};\mathbf{w})/\gamma(\mathbf{w})}$; otherwise (5.8) bounds it by the same quantity times $\sqrt{\rho(\mathbf{Y}';\mathbf{u})/\gamma(\mathbf{u})}$. The latter factor is at most 1 because the point of $\mathbf{y}^{n-2}$ inside the shortest gap of $\mathbf{u}$ lies within $\gamma(\mathbf{u})$ of both its ends. This proves (5.7).

Finally, $\rho(\mathbf{Y}) \to 0$ exactly when $\mathbf{Y}$ approaches the boundary of the polytope, and $\{H_{\mathsf{a}} \le M\}$ is contained in $\{\rho \ge e^{-2(M+C)}\}$, a compact subset of the interior. $\square$

To address the convexity of H_{a}, we need the following lemma:

Lemma 5.4. *Let $n \ge 2$, let $t_1 > \cdots > t_n$ or $t_1 < \cdots < t_n$, and let $u \in \mathbb{R}^n$. Then*

$$\sum_{1\le i<j\le n} (-1)^{i+j}\,\frac{(u_i-u_j)^2}{(t_i-t_j)^2} \;\le\; 0, \tag{5.10}$$

with equality if and only if $u_1 = \cdots = u_n$.

Observe that the sum in (5.10) is not invariant under permutations of the t_i's due to the signs $(-1)^{i+j}$, so we assume the t_i's to be monotone. In fact, the statement becomes false if the t_i's are not ordered.

Proof of Lemma 5.4. Assume $t_1 < \cdots < t_n$, the other case is equivalent. Let

$$\Phi(t) \coloneqq \sum_{1\le i<j\le n} (-1)^{i+j}\log(t_j - t_i),$$

and move the points along $t_i(s) = t_i + s u_i$, the ordering persisting for small $|s|$. Since

$$\frac{d^2}{ds^2}\log\bigl(t_j(s) - t_i(s)\bigr)\big|_{s=0} = -(u_j-u_i)^2/(t_j-t_i)^2,$$

we have

$$\frac{d^2}{ds^2}\Phi\big(t(s)\big)\Big|_{s=0} = -\sum_{1\le i<j\le n}(-1)^{i+j}\frac{(u_i-u_j)^2}{(t_i-t_j)^2}. \tag{5.11}$$

Step 1. Put $m := \lfloor n/2\rfloor$ and $\epsilon := n-2m \in \{0,1\}$, and separate the odd and even positions,

$$a_i := t_{2i-1},\quad \alpha_i := u_{2i-1}\quad (1\le i\le m+\epsilon),\qquad b_j := t_{2j},\quad \beta_j := u_{2j}\quad (1\le j\le m),$$

so that $a_1 < b_1 < a_2 < \cdots$. Two indices have equal parity exactly when they lie in the same family, so (recall that $V(\cdot)$ denotes the Vandermonde)

$$\Phi = \log V(a) + \log V(b) - \log\prod_{i,j}|a_i-b_j|.$$

Step 2. Let $P(z) := \prod_i(z-a_i)$, $Q(z) := \prod_j(z-b_j)$, $R := P/Q$, and set

$$c_j := \frac{|P(b_j)|}{|Q'(b_j)|} > 0,\qquad d_i := \frac{|Q(a_i)|}{|P'(a_i)|} > 0.$$

Counting negative factors in each product gives $P(b_j)/Q'(b_j) = (-1)^\epsilon c_j$ and $Q(a_i)/P'(a_i) = (-1)^{\epsilon+1}d_i$. Since $\deg P - \deg Q = \epsilon$, the partial fraction expansion of R reads

$$R(z) = \epsilon z + \mathrm{const} + (-1)^\epsilon\sum_{j=1}^m \frac{c_j}{z-b_j}, \tag{5.12}$$

and by definition we have

$$\frac{1}{d_i} = (-1)^{\epsilon+1}R'(a_i). \tag{5.13}$$

Let $i\neq k$. Both a_i and a_k are roots of P, hence of R, so the divided difference $\frac{R(a_i)-R(a_k)}{a_i-a_k}$ vanishes. We take the difference $R(a_i)-R(a_k)$ using (5.12) and $\frac{1}{a_i-b_j}-\frac{1}{a_k-b_j} = \frac{a_k-a_i}{(a_i-b_j)(a_k-b_j)}$, and get

$$0 = \epsilon - (-1)^\epsilon\sum_{j=1}^m\frac{c_j}{(a_i-b_j)(a_k-b_j)}.$$

For $i=k$, differentiating the same expansion at a_i gives

$$R'(a_i) = \epsilon - (-1)^\epsilon\sum_{j=1}^m\frac{c_j}{(a_i-b_j)^2}.$$

Multiplying both by $(-1)^{\epsilon+1}$, and using $(-1)^{\epsilon+1}\epsilon = \epsilon$ (recall $\epsilon\in\{0,1\}$), together with (5.13), the two cases combine into

$$\epsilon + \sum_{j=1}^m\frac{c_j}{(a_i-b_j)(a_k-b_j)} = \frac{\mathbf{1}_{i=k}}{d_i}. \tag{5.14}$$

Multiplying (5.14) by $\sqrt{d_id_k}$ and setting

$$O_{i0} := \sqrt{\epsilon d_i},\qquad O_{ij} := \frac{\sqrt{d_ic_j}}{a_i-b_j}\quad (1\le j\le m), \tag{5.15}$$

turns it into $OO^\mathsf{T} = I$. For $\epsilon=1$ the matrix O is square of size $m+1$; for $\epsilon=0$ its zeroth column vanishes and the remaining $m\times m$ block is orthogonal. In either case the matrix formed by the nonvanishing columns is square and orthogonal, so with $w_{ij} := O_{ij}^2 = d_i c_j/(a_i - b_j)^2 > 0$ ($1 \le j \le m$) both the rows and the columns give

$$\epsilon d_i + \sum_{j=1}^{m} w_{ij} = 1, \qquad \sum_{i=1}^{m+\epsilon} w_{ij} = 1. \tag{5.16}$$

Finally,

$$\prod_j |P(b_j)| = \prod_i |Q(a_i)| = \prod_{i,j} |a_i - b_j|, \qquad \prod_j |Q'(b_j)| = V(b)^2, \qquad \prod_i |P'(a_i)| = V(a)^2,$$

so that

$$\prod_j c_j = \frac{\prod_{i,j} |a_i - b_j|}{V(b)^2}, \qquad \prod_i d_i = \frac{\prod_{i,j} |a_i - b_j|}{V(a)^2}, \qquad \Phi = -\frac{1}{2}\sum_i \log d_i - \frac{1}{2}\sum_j \log c_j. \tag{5.17}$$

Step 3. Let the points move as above, so that a_i, b_j, c_j, d_i, w_{ij} become functions of s; dots denote d/ds at $s = 0$. Since (5.14) and (5.16) were derived at an arbitrary point of the chamber, and the chamber is preserved for small $|s|$, they hold as identities in s and may be differentiated. Put

$$X_{ij} := \frac{\alpha_i - \beta_j}{a_i - b_j}, \quad \sigma_i := \frac{d}{ds}\log d_i, \quad \kappa_j := \frac{d}{ds}\log c_j, \quad Y_{ij} := \frac{\sigma_i + \kappa_j}{2} - X_{ij}.$$

From $\log|O_{ij}| = \frac{1}{2}\log d_i + \frac{1}{2}\log c_j - \log|a_i - b_j|$ (where O is defined in (5.15)) and $\frac{d}{ds}X_{ij} = -X_{ij}^2$ we get, for $1 \le j \le m$,

$$\frac{d}{ds}\log|O_{ij}| = Y_{ij}, \qquad \frac{d^2}{ds^2}\log|O_{ij}| = \tfrac{1}{2}\dot\sigma_i + \tfrac{1}{2}\dot\kappa_j + X_{ij}^2,$$

while for $\epsilon = 1$, where $O_{i0} = \sqrt{d_i}$, one has $\frac{d}{ds}\log O_{i0} = \frac{1}{2}\sigma_i$ and $\frac{d^2}{ds^2}\log O_{i0} = \frac{1}{2}\dot\sigma_i$; for $\epsilon = 0$ the zeroth column vanishes identically and contributes nothing. Since $\frac{d^2}{ds^2}e^{2g} = 2e^{2g}\big(g'' + 2(g')^2\big)$, differentiating the first relation of (5.16) twice, dividing by 2 and summing over i, and then using both relations of (5.16) to collapse the resulting weighted sums of $\dot\sigma_i$ and $\dot\kappa_j$, gives

$$0 = \frac{1}{2}\sum_i \dot\sigma_i + \frac{1}{2}\sum_j \dot\kappa_j + \frac{\epsilon}{2}\sum_i d_i\sigma_i^2 + \sum_{i,j} w_{ij}X_{ij}^2 + 2\sum_{i,j} w_{ij}Y_{ij}^2.$$

Differentiating the last identity in (5.17) twice yields $\frac{d^2}{ds^2}\Phi = -\frac{1}{2}\sum_i \dot\sigma_i - \frac{1}{2}\sum_j \dot\kappa_j$, so that

$$\frac{d^2}{ds^2}\Phi\big(t(s)\big)\Big|_{s=0} = \sum_{i,j} w_{ij}X_{ij}^2 + 2\sum_{i,j} w_{ij}Y_{ij}^2 + \frac{\epsilon}{2}\sum_i d_i\sigma_i^2. \tag{5.18}$$

All the w_{ij} and d_i are positive, so the right-hand side of (5.18) is nonnegative; by (5.11) this is the asserted inequality.

Step 4. Finally, if the left-hand side of (5.18) vanishes, then $\sum_{i,j} w_{ij}X_{ij}^2 = 0$, and since every w_{ij} is positive, $X_{ij} = 0$, that is $\alpha_i = \beta_j$, for all i and j. Both families are nonempty because $n \ge 2$, so varying i at fixed j and then j at fixed i shows that all the α_i and all the β_j equal one common value, that is, $u_1 = \cdots = u_n$. Conversely, a constant u makes every difference $u_i - u_j$ vanish. $\square$

Let us denote the Hessian of the effective Hamiltonian H_{a} (5.3) by

$$Q_{\mathbf{Y}}(\mathbf{v}) = -\frac{1}{2}\sum_{i,k}\sum_{j,\ell} v_i^k \,\frac{\partial^2 H_{\mathsf{a}}}{\partial y_i^k\,\partial y_j^\ell}\, v_j^\ell = \sum_{k=1}^{N-1}\left[\sum_{1\le i<j\le k}\frac{(v_i^k-v_j^k)^2}{2(y_i^k-y_j^k)^2} - \sum_{p=1}^{k}\sum_{q=1}^{k+1}\frac{(v_p^k-v_q^{k+1})^2}{4(y_p^k-y_q^{k+1})^2}\right], \tag{5.19}$$

acting on fields $\mathbf{v} = \{v_i^k\}$ with $v^N \equiv 0$. Here $\mathbf{Y}$ is an interior point of the Gelfand–Tsetlin polytope, so that all denominators are nonzero. The second formula in (5.19) readily follows from (5.3). In particular, the Hessian of H_{a} does not depend on a.

Proposition 5.5. *The quadratic form $Q_{\mathbf{Y}}$ is negative definite, so H_{a} is strictly convex on the Gelfand–Tsetlin polytope.*

Proof. For a bond $(k, k+1)$, merge the two levels into the single decreasing sequence $t_1 = y_1^{k+1} > t_2 = y_1^k > t_3 = y_2^{k+1} > \cdots > t_{2k+1} = y_{k+1}^{k+1}$, and let $u \in \mathbb{R}^{2k+1}$ carry the values of $\mathbf{v}$ at the corresponding positions, that is, $u_{2r-1} = v_r^{k+1}$ for $r = 1, \dots, k+1$ and $u_{2r} = v_r^k$ for $r = 1, \dots, k$. Pairs of equal parity are exactly the same-level pairs, and pairs of opposite parity the cross-level ones, so Lemma 5.4 with $n = 2k+1$ states that

$$B_k \coloneqq \sum_{1\le i<j\le k}\frac{(v_i^k-v_j^k)^2}{(y_i^k-y_j^k)^2} + \sum_{1\le i<j\le k+1}\frac{(v_i^{k+1}-v_j^{k+1})^2}{(y_i^{k+1}-y_j^{k+1})^2} - \sum_{p=1}^{k}\sum_{q=1}^{k+1}\frac{(v_p^k-v_q^{k+1})^2}{(y_p^k-y_q^{k+1})^2} \;\le\; 0,$$

with equality only if the entries of v^k and v^{k+1} are all equal to one another. Level 1 carries no same-level pair and $v^N \equiv 0$, so each same-level sum with $2 \le k \le N-1$ appears in exactly two of the B_k and the boundary sums vanish; summing over the bonds therefore telescopes to

$$\sum_{k=1}^{N-1} B_k = 2\sum_{k=1}^{N-1}\sum_{1\le i<j\le k}\frac{(v_i^k-v_j^k)^2}{(y_i^k-y_j^k)^2} - \sum_{k=1}^{N-1}\sum_{p=1}^{k}\sum_{q=1}^{k+1}\frac{(v_p^k-v_q^{k+1})^2}{(y_p^k-y_q^{k+1})^2} = 4Q_{\mathbf{Y}}(\mathbf{v}) \le 0.$$

If $Q_{\mathbf{Y}}(\mathbf{v}) = 0$, then every B_k vanishes, so for each k the entries of v^k and v^{k+1} are all equal to one another; consecutive bonds overlap in a level, so all entries equal a single constant, which is 0 because $v^N \equiv 0$. Hence $Q_{\mathbf{Y}}$ is negative definite and, by (5.19), H_{a} is strictly convex. $\square$

Putting together Lemma 5.3 and Proposition 5.5, we obtain:

Proposition 5.6. *The deformed lattice $\bar{\mathsf{Y}} = \{\bar{\mathsf{y}}_i^k\}$ of Definition 5.1 exists, is unique, and is the unique critical point, as well as the global minimizer, of the effective Hamiltonian H_{a} (5.3) in the Gelfand–Tsetlin polytope.*

Remark 5.7. Let us compare our variational argument with the level-by-level one of [GM20, Section 3.4], which in the unperturbed case established that the dGFF of Definition 4.2 is well defined, that is, that its density (4.2) is integrable.

At $\mathbf{Y} = \mathsf{Y}$, the ∞-corners lattice of Definition 4.1, the negative definiteness of $Q_{\mathbf{Y}}$ is already contained in Remark 4.3. That argument integrates the levels one at a time, and each step uses that the level being integrated consists of the critical points of the polynomial with roots at the level above. Neither the deformed lattice nor a general point $\mathbf{Y}$ of the interlacing polytope admits such a description, and strict convexity requires the Hessian condition at every point rather than at a lattice inside the polytope. Proposition 5.5 provides such a convexity statement.

5.2. **Law of Large Numbers and Central Limit Theorem.** Under the scaling (5.1), the $\mathbf{Y}$-dependent part of the perturbation in the joint density (3.10) becomes:

$$\exp\left[a_N\sum_{i=1}^{N} z_i + \sum_{k=1}^{N-1}\sum_{i=1}^{k} y_i^k(a_k - a_{k+1})\right] = \exp\left[\frac{\beta}{2}\left(\mathsf{a}_N\sum_{i=1}^{N} z_i + \sum_{k=1}^{N-1}\sum_{i=1}^{k} y_i^k(\mathsf{a}_k - \mathsf{a}_{k+1})\right)\right].$$

This is now the same order as the repulsion terms. The remaining perturbation-dependent factors of (3.10), namely $e^{-\frac{1}{2}\sum_i a_i^2}$ and $B_{\mathbf{a}}(\mathbf{z};\beta)$, do not depend on $\mathbf{Y}$ and drop out of the conditional density below.

By the definition of the perturbed corners process (Definition 3.4), the factor $B_{\mathbf{a}}(\mathbf{z};\beta)$ in the perturbed GβE density (3.6) cancels with $1/B_{\mathbf{a}}(\mathbf{z};\beta)$ in the perturbed links (3.11). Since the top row is fixed, we pass to the conditional density of $\mathbf{Y}$ given $\mathbf{z}$, discarding the factors that depend on $\mathbf{z}$ alone. Extracting the β-dependent terms from the unperturbed links (3.7) and the perturbation, the conditional density takes the exact form

$$p_\beta(\mathbf{Y}\mid\mathbf{z}) = \frac{1}{Z_\beta}\,R(\mathbf{Y})\,\exp\left[-\beta\cdot H_{\mathsf{a}}(\mathbf{Y})\right], \qquad R(\mathbf{Y}) := \prod_{k=1}^{N-1}\frac{\prod_{1\le i<j\le k}(y_i^k - y_j^k)^2}{\prod_{p=1}^{k}\prod_{q=1}^{k+1}|y_p^k - y_q^{k+1}|}, \tag{5.20}$$

for $\mathbf{Y}$ in the Gelfand–Tsetlin polytope, where H_{a} is given by (5.3), the residual factor R does not depend on β, and the normalizing constant Z_β depends on β, $\mathbf{z}$, and a. Clearly, on compact subsets of the open Gelfand–Tsetlin polytope, R is bounded between two positive constants.

Differentiating H_{a} with respect to y_i^k and setting the result to zero recovers the coupled optimality equations (5.2) from Definition 5.1.

Theorem 5.8 (Crystallization for scaled perturbation)**.** *Let $\{y_i^k(\beta)\}_{1\le i\le k\le N}$ be the perturbed β-corners process with scaled perturbation $a_i = \frac{\beta}{2}\mathsf{a}_i$ and fixed top row $\mathbf{z} = (z_1 > \cdots > z_N)$. Let $\{\bar{\mathsf{y}}_i^k\}$ be the deformed ∞-corners lattice from Definition 5.1. Then as $\beta\to\infty$:*

$$y_i^k(\beta) \to \bar{\mathsf{y}}_i^k \qquad \textit{in probability.}$$

Both the theorem and the central limit theorem below follow from the following tail bound at the fluctuation scale $\beta^{-1/2}$.

Lemma 5.9. *In the setting of Theorem 5.8 there are $C, c, \beta_1 > 0$, depending only on N, $\mathbf{z}$ and a, such that*

$$\mathbb{P}\Big(\sqrt{\beta}\,\max_{i,k}|y_i^k(\beta) - \bar{\mathsf{y}}_i^k| \ge s\Big) \le C\,e^{-cs^2} \qquad \textit{for all } s>0 \textit{ and } \beta\ge\beta_1. \tag{5.21}$$

Proof. Write $H_0 = H_{\mathsf{a}}(\bar{\mathsf{Y}})$, and for $\varepsilon>0$ put $K_\varepsilon = \{\max_{i,k}|y_i^k - \bar{\mathsf{y}}_i^k|\ge\varepsilon\}$ and $\delta(\varepsilon) = \inf_{K_\varepsilon} H_{\mathsf{a}} - H_0$, which is positive because the infimum is attained (Lemma 5.3) and the minimizer is unique (Proposition 5.6). Fix $\beta_1>0$. On K_ε we have $e^{-\beta H_{\mathsf{a}}} \le e^{-(\beta-\beta_1)(H_0+\delta(\varepsilon))}\,e^{-\beta_1 H_{\mathsf{a}}}$, and $\int R e^{-\beta_1 H_{\mathsf{a}}} < \infty$ because the integrand is a constant multiple of the perturbed link (3.11) at β_1; near $\bar{\mathsf{Y}}$, where R is bounded below and $H_{\mathsf{a}} \le H_0 + \delta(\varepsilon)/2$, we get $Z_\beta \ge c e^{-\beta(H_0+\delta(\varepsilon)/2)}$. Together these give

$$\mathbb{P}\big(K_\varepsilon\big) \le C(\varepsilon)\,e^{-\beta\,\delta(\varepsilon)/2}, \qquad \beta\ge 2\beta_1. \tag{5.22}$$

By Proposition 5.5 the Hessian of H_{a} at $\bar{\mathsf{Y}}$ is positive definite, so on a ball $B_{r_0}(\bar{\mathsf{Y}})$ inside the Gelfand–Tsetlin polytope the difference $H_{\mathsf{a}} - H_0$ is comparable to $|\mathbf{Y} - \bar{\mathsf{Y}}|^2$, and there R is bounded above and below. Restricting the integral defining Z_β to $B_{r_0}(\bar{\mathsf{Y}})$ and rescaling by $\sqrt{\beta}$ gives $Z_\beta \ge c\beta^{-N(N-1)/4}e^{-\beta H_0}$. Comparing the numerator and the normalization of (5.20) with Gaussian integrals gives (5.21) for $s\le r_0\sqrt{\beta}$. For larger s we use (5.22) with $\varepsilon = r_0$; since the

Gelfand–Tsetlin polytope is bounded, the event in (5.21) is empty unless $s^2 = O(\beta)$, so that bound is again of the form Ce^{-cs^2}. ☐

Proof of Theorem 5.8. Apply Lemma 5.9 with $s = \varepsilon\sqrt{\beta}$: for every $\varepsilon > 0$ one gets

$$\mathbb{P}\big(\max_{i,k} |y_i^k(\beta) - \bar{\mathsf{y}}_i^k| \geq \varepsilon\big) \leq Ce^{-c\varepsilon^2\beta} \to 0,$$

which is the desired convergence in probability (moreover, with an exponential rate). ☐

The fluctuations of the perturbed β-corners process $\mathbf{Y}$ around the limiting lattice $\bar{\mathsf{Y}}$ are again Gaussian, but with a perturbed covariance structure. We need the following object.

Definition 5.10 (Deformed discrete Gaussian Free Field)**.** The *deformed dGFF* on the lattice $\{\bar{\mathsf{y}}_i^k\}$ is the centered Gaussian vector $\{\bar{\zeta}_i^k\}_{1\leq i\leq k\leq N}$ with $\bar{\zeta}_i^N = 0$ (pinned top row) and joint density proportional to

$$\exp\left(-\frac{1}{2}\sum_{i,k}\sum_{j,\ell}\bar{\zeta}_i^k \cdot \mathcal{H}_{(i,k),(j,\ell)} \cdot \bar{\zeta}_j^\ell\right), \tag{5.23}$$

where $\mathcal{H}$ is the Hessian matrix of the effective Hamiltonian (5.3) evaluated at the deformed lattice:

$$\mathcal{H}_{(i,k),(j,\ell)} = \frac{\partial^2 H_{\mathsf{a}}}{\partial y_i^k \partial y_j^\ell}\bigg|_{\mathbf{Y}=\bar{\mathsf{Y}}}.$$

The matrix $\mathcal{H} = -2Q_{\bar{\mathsf{Y}}}$ given by (5.19) is positive definite by Proposition 5.5, so the density (5.23) defines a centered Gaussian vector.

Remark 5.11. The quadratic form in the exponent of (5.23) is the same function of the underlying lattice as in the unperturbed case (4.2). What changes is the lattice at which it is evaluated, namely, $\bar{\mathsf{Y}}$ in place of Y, and so $Q_{\bar{\mathsf{Y}}}$ depends on the perturbation parameters a through the deformed lattice $\bar{\mathsf{Y}}$.

Theorem 5.12 (CLT for scaled perturbation)**.** *Under the assumptions of Theorem 5.8, we have*

$$\sqrt{\beta}\,(y_i^k(\beta) - \bar{\mathsf{y}}_i^k) \xrightarrow{d} \bar{\zeta}_i^k, \qquad \beta \to \infty,$$

jointly for all $1 \leq i \leq k \leq N$, where $\{\bar{\zeta}_i^k\}$ is the deformed dGFF from Definition 5.10.

Proof. Substituting $y_i^k = \bar{\mathsf{y}}_i^k + \frac{\Delta y_i^k}{\sqrt{\beta}}$ into (5.3) and expanding as $\beta \to \infty$, we get

$$H_{\mathsf{a}}(\mathbf{Y}) = H_{\mathsf{a}}(\bar{\mathsf{Y}}) + \frac{1}{2\beta}\sum_{i,k,j,\ell}\mathcal{H}_{(i,k),(j,\ell)}\Delta y_i^k \Delta y_j^\ell + O(\beta^{-3/2}),$$

uniformly for $\Delta\mathbf{Y} := \{\Delta y_i^k\}_{1\leq i\leq k\leq N-1}$ in compact sets (the top row is fixed, so $\Delta y_i^N = 0$). Here the linear term vanishes because $\bar{\mathsf{Y}}$ is a critical point of H_{a} (Proposition 5.6), and the remainder comes from the third-order Taylor terms of the finitely many logarithmic factors of (5.3), whose arguments stay bounded away from zero because $\bar{\mathsf{Y}}$ lies in the open Gelfand–Tsetlin polytope. Hence, by (5.20), the density of $\Delta\mathbf{Y}$, up to a factor independent of $\Delta\mathbf{Y}$, converges to the deformed dGFF density (5.23) uniformly on compact sets. Indeed, the residual factor contributes $R(\bar{\mathsf{Y}} + \Delta\mathbf{Y}/\sqrt{\beta})/R(\bar{\mathsf{Y}}) \to 1$, since R is independent of β and is continuous and positive at $\bar{\mathsf{Y}}$.

The tail bound (Lemma 5.9) upgrades this local convergence to convergence in distribution. Indeed, for a bounded continuous $\varphi\colon \mathbb{R}^{N(N-1)/2} \to \mathbb{R}$ and $s > 0$, split the expectation $\mathbb{E}[\varphi(\Delta\mathbf{Y})]$ and the normalization at the ball $\{\max_{i,k}|\Delta y_i^k| \leq s\}$: the exterior contributes $O(\|\varphi\|_\infty e^{-cs^2})$

by (5.21), while inside the ball the unnormalized density converges uniformly, and the unknown normalizing constant of the conditional density (5.20) cancels in the ratio over the common ball. Letting $\beta \to \infty$ and then $s \to \infty$ identifies the limit of $\mathbb{E}[\varphi(\boldsymbol{\Delta}\mathbf{Y})]$ as the expectation of φ under the deformed dGFF. This completes the proof. $\square$

5.3. **The deformed lattice: shifted derivatives for a single spike.** In the unperturbed case, the ∞-corners lattice is built level by level starting from the top row $\mathbf{z}$, and each subsequent level consists of the roots of the derivative of the polynomial vanishing at the level above (Definition 4.1). The *shifted derivative* is defined for $c \in \mathbb{R}$ by

$$D_c f(x) := f'(x) + c\, f(x) = e^{-cx} \frac{d}{dx}\big[e^{cx} f(x)\big]. \tag{5.24}$$

We prove that for a single spike in the last coordinate, that is, for perturbations of the form $\mathsf{a}_1 = \cdots = \mathsf{a}_{N-1}$, with a_N unrestricted, the deformed lattice is built exactly as the unperturbed one, except that the first step, from the top row to level $N-1$, applies D_{c} with $\mathsf{c} = \mathsf{a}_{N-1} - \mathsf{a}_N$ in place of the derivative (Proposition 5.14). For all other a, including a single spike at any other coordinate, the natural analogue of this construction fails (Remark 5.15).

We first record the root structure of D_c on polynomials with distinct real roots.

Proposition 5.13. *Let $P(x) = \prod_{j=1}^N (x - \lambda_j)$ with $\lambda_1 > \cdots > \lambda_N$, and let $c \in \mathbb{R}$; parts (i)–(iv) below assume $c \neq 0$, while part (v) treats the case $c = 0$.*

(i) *The polynomial $D_c P(x) = P'(x) + c\,P(x)$ has degree N with leading coefficient c. Its roots are precisely the critical points of $g(x) = e^{cx} P(x)$.*
(ii) *Exactly $N-1$ of the N roots of $D_c P$ lie in the interlacing intervals $(\lambda_{j+1}, \lambda_j)$, $j = 1, \ldots, N-1$, one root per interval.*
(iii) *The remaining root r_* satisfies*

$$r_* < \lambda_N \quad \text{if } c > 0, \qquad r_* > \lambda_1 \quad \text{if } c < 0.$$

(iv) *The sum of all N roots equals $\sum_{j=1}^N \lambda_j - N/c$.*
(v) *When $c = 0$, the polynomial $D_0 P = P'$ has degree $N-1$, and all its roots lie in the interlacing intervals, one root per interval; there is no root outside $[\lambda_N, \lambda_1]$.*

Of the N roots of $D_c P$ for $c \neq 0$, we call the $N-1$ roots in the interlacing intervals the *interlacing roots*, and the remaining root r_* the *spurious root*. The constructions of this subsection use the interlacing roots and discard the spurious one; in the zero-temperature up transition of Section 6.2, the spurious root plays the role of one of the limiting positions.

Proof of Proposition 5.13. (i) Since P is monic of degree N, $D_c P = P' + cP$ has degree N with leading coefficient c. The identity $D_c P = e^{-cx} \frac{d}{dx}[e^{cx} P]$ follows from the product rule; since $e^{-cx} \neq 0$, the roots of $D_c P$ coincide with the zeros of $(e^{cx} P)' = g'(x)$.

(ii) The function $g(x) = e^{cx} P(x)$ vanishes at each λ_j (since $P(\lambda_j) = 0$) and $e^{cx} > 0$. Between consecutive nodes λ_{j+1} and λ_j, the polynomial P changes sign at each simple root, so g changes sign as well. By Rolle's theorem, g' has at least one root in every interval $(\lambda_{j+1}, \lambda_j)$, giving at least $N-1$ interlacing critical points, that is, roots of $D_c P$. One further real root lies outside $[\lambda_N, \lambda_1]$. Indeed, $D_c P(\lambda_j) = P'(\lambda_j) = \prod_{i \neq j} (\lambda_j - \lambda_i)$, whose sign is $(-1)^{j-1}$. If $c > 0$, then $D_c P(x) \sim c x^N$ has sign $(-1)^N$ as $x \to -\infty$, opposite to the sign $(-1)^{N-1}$ of $D_c P(\lambda_N)$, so $D_c P$ has a root in $(-\infty, \lambda_N)$; if $c < 0$, then $D_c P(x) \to -\infty$ as $x \to +\infty$ while $D_c P(\lambda_1) > 0$, so $D_c P$ has a root in (λ_1, ∞). This accounts for N real roots; since $D_c P$ has degree N, there are no others, and thus exactly one root lies in each interval $(\lambda_{j+1}, \lambda_j)$ and exactly one outside $[\lambda_N, \lambda_1]$.

(iii) For $c > 0$: on $(-\infty, \lambda_N)$, $g(x) = e^{cx}P(x)$ has constant sign $(-1)^N$ and satisfies $g(x) \to 0$ as $x \to -\infty$ (since $e^{cx} \to 0$ dominates) while $g(\lambda_N) = 0$. Thus g achieves an extremum in $(-\infty, \lambda_N)$, placing the spurious critical point r_* there. The case $c < 0$ is analogous, with $r_* > \lambda_1$.

(iv)–(v) follow from the classical Vieta's formulas and the Rolle's theorem argument. □

Proposition 5.14. *Let $N \geq 2$, let $\mathsf{a}_1 = \cdots = \mathsf{a}_{N-1}$ with a_N arbitrary, and set $\mathsf{c} := \mathsf{a}_{N-1} - \mathsf{a}_N$. Let $P_N(x) = \prod_{j=1}^{N}(x - z_j)$ be the top-row polynomial, and let P_{N-1} be the monic polynomial of degree $N-1$ whose roots are the $N-1$ interlacing roots of $D_{\mathsf{c}}P_N$. Then, for each $k = 1, \ldots, N-1$, the level-k positions $\bar{\mathsf{y}}_1^k > \cdots > \bar{\mathsf{y}}_k^k$ of the deformed lattice are the roots of $\left(\frac{d}{dx}\right)^{N-1-k} P_{N-1}$. Equivalently, the levels $N-1, \ldots, 1$ of the deformed lattice form the ∞-corners lattice of Definition 4.1 whose top row consists of the roots of P_{N-1}. For $\mathsf{c} = 0$ we have $P_{N-1} = P_N'/N$, and the deformed lattice is the unperturbed one.*

Proof. By Proposition 5.13(ii) for $\mathsf{c} \neq 0$, and (v) for $\mathsf{c} = 0$, the roots of P_{N-1} are distinct and strictly interlace with the top row, one in each gap (z_{j+1}, z_j). The roots of the successive derivatives interlace strictly by Rolle's theorem. Hence the array lies in the open Gelfand–Tsetlin polytope. By the uniqueness in Proposition 5.6, it suffices to check that the array solves the coupled equations (5.2).

The levels $N-1, \ldots, 1$ of the array form, by construction, the unperturbed ∞-corners lattice of Definition 4.1 with top row the roots of P_{N-1}, so Lemma 4.5 applies to them. To identify the array with the deformed lattice, we compare (4.5) with (5.2): the same-level sum of (5.2) cancels against the lower-level one at every level (at $k = 1$ both are empty). The equation at level k thus reduces to

$$-\frac{1}{2}\sum_{q=1}^{k+1} \frac{1}{\bar{\mathsf{y}}_i^k - \bar{\mathsf{y}}_q^{k+1}} = \frac{\mathsf{a}_k - \mathsf{a}_{k+1}}{2}, \qquad i = 1, \ldots, k.$$

For $k \leq N-2$ both sides vanish: the right-hand side because $\mathsf{a}_k = \mathsf{a}_{k+1}$, and the left-hand side because the level-k points are the critical points of the monic polynomial P_{k+1} with roots $\bar{\mathsf{y}}^{k+1}$, so that $\sum_q (\bar{\mathsf{y}}_i^k - \bar{\mathsf{y}}_q^{k+1})^{-1} = (P_{k+1}'/P_{k+1})(\bar{\mathsf{y}}_i^k) = 0$. For $k = N-1$ the reduced equation reads $\sum_{j=1}^{N} (\bar{\mathsf{y}}_i^{N-1} - z_j)^{-1} = -\mathsf{c}$, that is, $P_N'(\bar{\mathsf{y}}_i^{N-1}) + \mathsf{c}P_N(\bar{\mathsf{y}}_i^{N-1}) = 0$, which holds because $\bar{\mathsf{y}}_i^{N-1}$ is a root of $D_{\mathsf{c}}P_N$. □

Remark 5.15. For all other a — those with $\mathsf{a}_k \neq \mathsf{a}_{k+1}$ for some $k \leq N-2$, for instance a single spike at a coordinate other than the last (for $N \geq 3$) — the natural analogue of this construction fails. The candidate is the top-down recursion that takes, at each level, the interlacing roots of $D_{\mathsf{a}_k - \mathsf{a}_{k+1}}$ applied to the polynomial vanishing at the level above. It produces an interlacing array $\hat{\mathsf{Y}} = \{\hat{\mathsf{y}}_i^k\}$, but $\hat{\mathsf{Y}} \neq \bar{\mathsf{Y}}$. The latter may be verified by a direct computation, which we omit.

6. Asymptotics of up transitions

The random polynomial equation (3.17) determining the conditional distribution of the upper row $\boldsymbol{\lambda}$ given the lower row $\boldsymbol{\mu}$, $\boldsymbol{\lambda} \succ \boldsymbol{\mu}$, runs in the direction opposite to Theorem 4.6 and Theorem 5.8. In this section, we discuss the zero-temperature limit $\beta \to \infty$ of this conditional density in various regimes, in the fixed and scaled perturbation settings. Throughout the section $P_{\boldsymbol{\mu}}(x) = \prod_{i=1}^{N-1}(x - \mu_i)$, and we assume $N \geq 2$.

6.1. Fixed perturbation.

Proposition 6.1. *Fix $a_N \in \mathbb{R}$, and let random $\boldsymbol{\lambda}$ be obtained from a fixed $\boldsymbol{\mu}$ through (3.17). Then, as $\beta \to \infty$:*

(i) *If $\boldsymbol{\mu}$ does not depend on β, then the $N-2$ inner roots $\lambda_2, \dots, \lambda_{N-1}$ converge in probability to the roots of $P'_{\boldsymbol{\mu}}$,*

$$\sum_{i=1}^{N-1} \frac{1}{x-\mu_i} = 0, \tag{6.1}$$

one per gap (μ_{i+1}, μ_i), while the two outer roots escape (both limits in probability):

$$\frac{\lambda_1}{\sqrt{\beta}} \to \sqrt{\frac{N-1}{2}}, \qquad \frac{\lambda_N}{\sqrt{\beta}} \to -\sqrt{\frac{N-1}{2}}.$$

(ii) *If instead $\mu_i = \sqrt{\beta} m_i$ with $\mathbf{m} = (m_1 > \cdots > m_{N-1})$ fixed, then no root escapes: the whole rescaled row $\boldsymbol{\lambda}/\sqrt{\beta}$ converges in probability to the N roots of*

$$P'_{\mathbf{m}}(x) - 2x\,P_{\mathbf{m}}(x) = e^{x^2} \frac{d}{dx}\Big[e^{-x^2} P_{\mathbf{m}}(x)\Big] = 0, \tag{6.2}$$

and these N roots interlace with $\mathbf{m}$.

On Hermite polynomials H_n, the up transition of (ii) is the Rodrigues (raising) step

$$e^{x^2} \frac{d}{dx}\big[e^{-x^2} H_{n-1}\big] = -H_n,$$

taking the roots of H_{n-1} to the roots of H_n. This agrees with the ∞-corners lattice of [GM20]: for the top row at the roots of H_N, the lowering relation $H'_n = 2nH_{n-1}$ places level k at the roots of H_k for every k, and the up transition climbs the same tower.

Proof of Proposition 6.1. Since $\xi_i \sim \frac{1}{2}\chi^2_\beta$ has mean $\beta/2$ and variance $\beta/2$, Chebyshev's inequality gives $\xi_i/(\beta/2) \to 1$ in probability for $i = 1, \dots, N-1$; and $\xi_N \sim \mathcal{N}(a_N, 1)$ is $O_P(1)$. Both statements are unconditional, the law of $\boldsymbol{\xi}$ in (3.18) being independent of $\boldsymbol{\mu}$.

(i) Dividing (3.17) by $\beta/2$ gives

$$\sum_{i=1}^{N-1} \frac{2\xi_i/\beta}{x-\mu_i} = \frac{2}{\beta}\big(x - \xi_N\big).$$

On compact sets at positive distance from $\mu_1, \dots, \mu_{N-1}$ the left-hand side becomes $\sum_i (x-\mu_i)^{-1}$, while the right-hand side goes to 0, both uniformly and in probability. This gives (6.1). Its solutions are the $N-2$ roots of $P'_{\boldsymbol{\mu}}$, one in each gap (μ_{i+1}, μ_i) by Rolle's theorem. Multiplying the last display by $P_{\boldsymbol{\mu}}(x)$ turns it into a polynomial equation of degree N whose roots are $\lambda_1, \dots, \lambda_N$. The coefficients of this polynomial converge in probability to those of $P'_{\boldsymbol{\mu}}$, in particular the two leading ones to 0. So $N-2$ roots converge to the roots of $P'_{\boldsymbol{\mu}}$, and the other two leave every compact set. By the interlacing $\boldsymbol{\mu} \prec \boldsymbol{\lambda}$, the former are $\lambda_2, \dots, \lambda_{N-1}$. For the remaining two, solve (3.19) for $\prod_{j=1}^N (x-\lambda_j)$ and substitute $x = \sqrt{\beta}t$: the monic polynomial $\prod_{j=1}^N \big(t - \lambda_j/\sqrt{\beta}\big)$ converges in probability, coefficientwise, to $t^{N-2}\big(t^2 - \frac{N-1}{2}\big)$, so exactly two roots reach scale $\sqrt{\beta}$, with limits $\pm\sqrt{(N-1)/2}$.

(ii) With $\mu_i = \sqrt{\beta} m_i$ and $x = \sqrt{\beta}\,\ell$, equation (3.17) reads

$$\frac{1}{\sqrt{\beta}} \sum_{i=1}^{N-1} \frac{\xi_i}{\ell - m_i} = \sqrt{\beta}\,\ell - \xi_N.$$

Dividing by $\sqrt{\beta}$ and using $\xi_i/(\beta/2) \to 1$ together with $\xi_N/\sqrt{\beta} \to 0$ turns the last display into $\frac{1}{2}\sum_i (\ell - m_i)^{-1} = \ell$, which is (6.2) after multiplying by $2P_{\mathbf{m}}(\ell)$. The limiting polynomial equation

(6.2) has degree N with leading coefficient -2, so it has N roots. They interlace with $\mathbf{m}$: the function $e^{-x^2}P_{\mathbf{m}}(x)$ vanishes at $m_1, \dots, m_{N-1}$ and tends to 0 at $\pm\infty$, so by Rolle's theorem its derivative vanishes once in each of the $N-2$ gaps, once above m_1 and once below m_{N-1}. Convergence of the roots follows from that of the coefficients, since the leading coefficient is bounded away from 0. This completes the proof. □

The fixed perturbation is invisible in both regimes of Proposition 6.1, for the same reason as in Theorem 4.6: a_N enters only through $\xi_N \sim \mathcal{N}(a_N, 1)$, which is small compared to other coefficients.

6.2. **Linearly growing perturbation.** The difference from Section 6.1 is that ξ_N now contributes at leading order, and the ordinary derivative of (6.1) is replaced by the shifted derivative D_{a_N} (5.24), whose root structure is recorded in Proposition 5.13.

Proposition 6.2. *Let $a_N = \frac{\beta}{2}\mathsf{a}_N$ with $\mathsf{a}_N \neq 0$ fixed, let $\boldsymbol{\mu} = (\mu_1 > \cdots > \mu_{N-1})$ be fixed and not depend on β, and let random $\boldsymbol{\lambda}$ be obtained from $\boldsymbol{\mu}$ through (3.17). Then, as $\beta \to \infty$:*

(i) *$N-1$ of the roots stay bounded and converge in probability to the roots of*

$$\sum_{i=1}^{N-1} \frac{1}{x-\mu_i} = -\mathsf{a}_N, \qquad \text{that is,} \qquad D_{\mathsf{a}_N} P_{\boldsymbol{\mu}}(x) = P'_{\boldsymbol{\mu}}(x) + \mathsf{a}_N P_{\boldsymbol{\mu}}(x) = 0. \tag{6.3}$$

Proposition 5.13, applied to $P_{\boldsymbol{\mu}}$, locates these $N-1$ roots: exactly $N-2$ of them interlace with $\boldsymbol{\mu}$, one in each gap (μ_{i+1}, μ_i), and the last lies outside the range of $\boldsymbol{\mu}$ — below μ_{N-1} if $\mathsf{a}_N > 0$, above μ_1 if $\mathsf{a}_N < 0$.

(ii) *The remaining root escapes on the scale of the perturbation: $\dfrac{2}{\beta}\max_j |\lambda_j| \to |\mathsf{a}_N|$, the unbounded root having the sign of a_N.*

We call the root in (ii) the *escaping root* and denote it by λ_{esc}. The root of (6.3) outside the range of $\boldsymbol{\mu}$ in (i) is referred to as the *spurious root*, as in Proposition 5.13.

Proof of Proposition 6.2. As in the proof of Proposition 6.1, $\xi_i/(\beta/2) \to 1$ in probability for $i = 1, \dots, N-1$; and $\xi_N \sim \mathcal{N}(\frac{\beta}{2}\mathsf{a}_N, 1)$, so $\frac{2}{\beta}\xi_N = \mathsf{a}_N + O_P(\beta^{-1})$.

(i) Dividing (3.17) by $\beta/2$ gives

$$\sum_{i=1}^{N-1} \frac{2\xi_i/\beta}{x-\mu_i} = \frac{2x}{\beta} - \frac{2\xi_N}{\beta}.$$

On any compact set at positive distance from $\mu_1, \dots, \mu_{N-1}$, the left-hand side converges to $\sum_i \frac{1}{x-\mu_i}$ in probability, while the right-hand side converges to $-\mathsf{a}_N$; multiplying by $P_{\boldsymbol{\mu}}(x)$ turns (6.3) into $D_{\mathsf{a}_N} P_{\boldsymbol{\mu}} = 0$. By Proposition 5.13, exactly $N-1$ of the roots of this equation stay bounded. That they converge in probability to the deterministic roots of (6.3) follows similarly to the proof of Proposition 6.1 (i).

(ii) The row $\boldsymbol{\lambda}$ has N points and (i) accounts for $N-1$ of them, which stay bounded in probability. We denoted the remaining root by λ_{esc}. From the relation (2.9) between $|\boldsymbol{\lambda}|$, $|\boldsymbol{\mu}|$, and ξ_N, we see that the $N-1$ bounded roots and $|\boldsymbol{\mu}|$ contribute $O_P(1)$, while $\frac{2}{\beta}\xi_N \to \mathsf{a}_N$; hence $\frac{2}{\beta}\lambda_{\text{esc}} \to \mathsf{a}_N$ in probability, which gives the claimed scale together with the sign of the escaping root. □

Remark 6.3. The two extreme roots λ_1 and λ_N — the two lying outside the range of $\boldsymbol{\mu}$ — are governed in Proposition 6.1 (i) and Proposition 6.2 by a common approximate equation. Indeed, replacing $\sum_i \xi_i/(x-\mu_i)$ by $\frac{\beta(N-1)}{2x}$ and ξ_N by $\frac{\beta}{2}\mathsf{a}_N$ in (3.17) yields

$$x^2 - \frac{\beta \mathsf{a}_N}{2}x - \frac{\beta(N-1)}{2} = 0.$$

At $\mathsf{a}_N = 0$ its roots form the symmetric pair $\pm\sqrt{\beta(N-1)/2}$, matching the limits of λ_1 and λ_N in Proposition 6.1 (i); for fixed $\mathsf{a}_N \neq 0$ they are approximately $\frac{\beta \mathsf{a}_N}{2}$ and $-\frac{N-1}{\mathsf{a}_N}$: the former matches the escaping root of Proposition 6.2 (ii), and the latter approaches the spurious root of Proposition 6.2 (i) as $\mathsf{a}_N \to 0$. The two regimes cross over at $\mathsf{a}_N \asymp \beta^{-1/2}$, where both extreme roots are at scale $\sqrt{\beta}$.

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L. Petrov, University of Virginia, Charlottesville, VA, USA

e-mail: `lenia.petrov@gmail.com`

J. Xu, The Ohio State University, Columbus, OH, USA

e-mail: `jxu0800@gmail.com`