A Borodin–Okounkov–Geronimo–Case identity for tilted Toeplitz minors

Leonid Petrov

Abstract

We prove a Fredholm determinantal identity for the tilted Toeplitz minor

$$D_N^{\xi,\theta}(\varphi) \coloneqq \det\big[(\theta_i \xi_j \varphi)_{i-j}\big]_{i,j=1}^{N},$$

generalizing the Borodin–Okounkov–Geronimo–Case (BOGC) identity to oblique splittings of the Hardy space. The tilts ξ_j, θ_i enter only through an oblique projection that multiplies the trace-class kernel K inside the Fredholm determinant; the BOGC operator $A = I - K$ constructed from φ is unchanged.

Baik–Liao–Liu [BLL26] and Liu–Tripathi [LT26] have recently shown that the same tilted Toeplitz minor admits a contour Fredholm-determinantal representation, in connection with the periodic Totally Asymmetric Simple Exclusion Process (TASEP). In the periodic TASEP application of Baik–Liao–Liu, the formula plays an important role in identifying the periodic KPZ fixed point with general initial data. Our formula is a companion to their Fredholm determinant and readily reduces to the original BOGC identity.

The one-sided tilted Toeplitz minor (that is, when all $\theta_i = 1$) admits a bialternant form recovering Schur and Grothendieck polynomials as special cases. A Cauchy–Binet expansion realizes $D_N^{\xi,\theta}$ as a restricted sum over partitions of products of Jacobi–Trudi type determinants, generalizing Gessel's theorem. In the pure-shift setting this specializes to a skew Schur expansion. Finally, for finite Laurent exponential symbols, we record explicit resolvent-block flow identities and formulate the associated finite-dimensional closure problem. We also illustrate a possible asymptotic application leading to finite-rank perturbations of the Airy kernel.

1 Introduction

Section 1.1 recalls the Borodin–Okounkov–Geronimo–Case (BOGC) identity [GC79], [BO00], [BW00], [Böt01]. Section 1.2 introduces the tilted Toeplitz minors and the Hardy space framework. The main results are stated in Section 1.3; the relation to [BLL26], [LT26], and other works is discussed in Section 1.4.

1.1 Background

Let $\varphi\colon \mathbb{T} \to \mathbb{C} \setminus \{0\}$ be a function on the unit circle (called the *symbol*). The $N \times N$ Toeplitz determinant

$$D_N(\varphi) \coloneqq \det\big[\widehat{\varphi}(i-j)\big]_{i,j=0}^{N-1}, \qquad \widehat{\varphi}(k) \coloneqq \int_{\mathbb{T}} \varphi(z)\, z^{-k}\, \frac{dz}{2\pi i z},$$

is built from the Fourier coefficients of φ. Such determinants have been studied since Szegő [Sze15], [Sze52], mainly for the role of their asymptotic behavior as $N \to \infty$ in analysis, probability, and mathematical physics, starting with Onsager's [Ons44] solution of the two-dimensional Ising model.

For symbols of zero winding number, the strong Szegő limit theorem [Sze52] gives the leading-order asymptotic $D_N(\varphi) \sim G(\varphi)^N Z$ as $N \to \infty$, with $G(\varphi)$ and Z explicit scalars built from the Fourier coefficients of $\log \varphi$. The Borodin–Okounkov–Geronimo–Case identity (BOGC, for short) sharpens this into an exact formula valid at every N:

$$D_N(\varphi) = G(\varphi)^N \cdot Z \cdot \det\big[\,\delta_{ij} - K_{ij}\,\big]_{i,j \geq N}. \tag{1.1}$$

The right-hand side is the Fredholm determinant of the infinite matrix $I - K$ restricted to rows and columns $i, j \geq N$. The operator K is an N-independent product of two Hankel matrices,

$$K_{ij} = \big(\mathcal{H}(b)\,\mathcal{H}(\widetilde{c})\big)_{ij}, \qquad \mathcal{H}(\psi)_{ij} := \widehat{\psi}(i + j + 1),$$

built from the Wiener–Hopf factorization of the symbol into analytic factors $\varphi = \varphi_-\, G(\varphi)\, \varphi_+$ (with φ_- extending into $|z| > 1$ with $\varphi_-(\infty) = 1$, and φ_+ extending into $|z| < 1$ with $\varphi_+(0) = 1$) via $b = \varphi_-/\varphi_+$, $c = 1/b$, and $\widetilde{c}(z) = c(z^{-1})$.

Identity (1.1) originated in Geronimo–Case [GC79] and was rediscovered in the Toeplitz form by Borodin–Okounkov [BO00]. It was given two further proofs (including a block-Toeplitz extension) by Basor–Widom [BW00] and Böttcher [Böt01]. The route most directly relevant here is another proof by Böttcher–Widom [BW06], who recognized (1.1) as the trace-class infinite-dimensional incarnation of the classical *Jacobi complementary minor identity*: the minor of A^{-1} on a subspace U equals $\det(A)^{-1}$ times the minor of A on a complementary subspace V. In this way, the BOGC identity corresponds to the special case of the most natural orthogonal splitting of the Hardy space $H = \ell^2(\mathbb{Z}_{\geq 0})$ (the Hardy space $H^2(\mathbb{D})$ of analytic functions on the unit disk) into $U = \operatorname{span}\{e_0, \ldots, e_{N-1}\}$ and $V = \ell^2(\mathbb{Z}_{\geq N})$. Our starting point is a straightforward extension of this Jacobi identity to arbitrary (oblique) splittings $H = U \dotplus V$ into closed subspaces.

1.2 Tilted Toeplitz minors

We now introduce our main object, the tilted Toeplitz minor. It depends on two collections of analytic functions $\xi_1, \ldots, \xi_N$ (extending into $|z| < 1$) and $\theta_1, \ldots, \theta_N$ (extending into $|z| > 1$), and a symbol φ, as above. The *tilted Toeplitz minor* attached to this data is

$$D_N^{\xi,\theta}(\varphi) := \det\big[(\theta_i\, \xi_j\, \varphi)_{i-j}\big]_{i,j=1}^N, \qquad (f)_k := \widehat{f}(k), \tag{1.2}$$

where $\widehat{f}(k)$ is the k-th Fourier coefficient of f on $\mathbb{T}$. Assemble ξ_j and θ_i into operators $\Xi\colon H \to H$ and $\Theta\colon H \to \mathbb{C}^N$ on the Hardy space $H = \ell^2(\mathbb{Z}_{\geq 0}) \simeq H^2(\mathbb{D})$ (the convention of [BO00], [Böt01]), and form the finite-dimensional pair

$$R := \Theta\, T(\varphi_+), \qquad C := T(\varphi_-)\, \Xi\, P_N, \tag{1.3}$$

where $T(\varphi_\pm)$ are the Toeplitz operators of the Wiener–Hopf factors which are triangular with respect to the standard basis of H, and P_N is the orthogonal projection onto $\operatorname{span}\{e_0, \ldots, e_{N-1}\}$. By a mild abuse of notation, we also write P_N for the canonical isomorphism $P_N H \simeq \mathbb{C}^N$ where needed, so that maps such as $T(\varphi_-)\, \Xi\, P_N$ in (1.3) are read as $\mathbb{C}^N \to H$. The pair (R, C)

generalizes the orthogonal pair $R = C = P_N$ of BOGC to an oblique one; when the "Gram" matrix $\Gamma_{\xi,\theta} = RC$ is invertible (equivalently, $\mathrm{Ran}(C) \cap \mathrm{Ker}(R) = \{0\}$), this produces the (in general non-orthogonal) splitting $H = \mathrm{Ran}(C) \dotplus \mathrm{Ker}(R)$.

In our generalizations, the symbol φ fixes the operator $A = I - K$. The tilt (ξ, θ) is recorded only in the row and column maps (R, C) (1.3), which define the oblique projection $\Pi_V = I - C(RC)^{-1}R$ onto the space $V = \mathrm{Ker}(R)$.

1.3 Main results

The main results of the note are the following identities.

Standing assumptions. Throughout the rest of the paper, $\varphi\colon \mathbb{T} \to \mathbb{C} \setminus \{0\}$ is of zero winding number with $\log \varphi \in C^\alpha(\mathbb{T})$ for some $\alpha > \frac{1}{2}$, and admits the canonical Wiener–Hopf factorization $\varphi = \varphi_-\, G(\varphi)\, \varphi_+$ as in Section 1.1. Under this assumption $\mathcal{H}(b)$ and $\mathcal{H}(\widetilde{c})$ are Hilbert–Schmidt, so $K = \mathcal{H}(b)\mathcal{H}(\widetilde{c})$ is trace class and the right-hand side of (1.1) is well-defined. The tilt functions $\xi_1, \dots, \xi_N$ and $\theta_1, \dots, \theta_N$ are bounded analytic in $|z| < 1$ and $|z| > 1$ respectively, so that Ξ and Θ are bounded on H. Rank and transversality of the resulting chart are not automatic for degenerate tilts; the needed Gram-invertibility hypotheses are stated alongside each theorem below.

(1) Fredholm determinantal identity for tilted Toeplitz minors (Theorem 4.4). For any tilts (ξ_j, θ_i) subject to the invertibility of the $N \times N$ Gram matrix $\Gamma_{\xi,\theta} = \Theta\, T(\varphi_+) T(\varphi_-)\, \Xi\, P_N$, we have the identity for the determinant (1.2):

$$D_N^{\xi,\theta}(\varphi) = G(\varphi)^N \cdot Z \cdot \det\nolimits_{\mathbb{C}^N} \Gamma_{\xi,\theta} \cdot \det\nolimits_{\mathrm{Ker}(R)}\big(I_{\mathrm{Ker}(R)} - \Pi_V K|_{\mathrm{Ker}(R)}\big). \tag{1.4}$$

The deformed kernel $\Pi_V K = K - C\Gamma_{\xi,\theta}^{-1} R K$ differs from K by a correction of rank at most N. At fixed N this realizes $\Pi_V K$ as a finite rank perturbation of the BOGC operator K; in asymptotic regimes where N varies, the identity separates $D_N^{\xi,\theta}(\varphi)$ into the finite-dimensional Gram determinant $\det_{\mathbb{C}^N} \Gamma_{\xi,\theta}$ and the Fredholm determinant on the oblique tail $\mathrm{Ker}(R)$.

When $RP_N\colon P_N H \to \mathbb{C}^N$ is also invertible, we identify this oblique determinant with an ordinary Fredholm determinant on the fixed tail $Q_N H = \overline{\mathrm{span}}(e_N, e_{N+1}, \dots)$ (see Proposition 4.6); for polynomial tilts of bounded degree, the resulting kernel is a bounded rank perturbation of $Q_N K Q_N$. This formulation may be useful for soft edge asymptotics, as we discuss in Appendix A.

(2) Bialternant representation (Theorem 4.11). When the negative part is exactly rank N, $\varphi_-(z) = \prod_{l=1}^N (1 - y_l/z)^{-1}$ (with pairwise distinct $|y_l| < 1$), and the row tilts are trivial, $\theta_i \equiv 1$, the column tilted minor (1.2) has the bialternant form

$$D_N^{\xi,\mathbf{1}}(\varphi) = \det\big[(\xi_j \varphi)_{i-j}\big]_{i,j=1}^N = G(\varphi)^N \cdot \frac{\det\big[y_i^{N-j}\, \xi_{N-j+1}(y_i)\big]_{i,j=1}^N}{\Delta(Y)} \cdot \prod_{l=1}^N \varphi_+(y_l),$$

where $\Delta(Y) = \prod_{1 \le i < j \le N} (y_i - y_j)$ is the Vandermonde determinant. Particular choices of the analytic functions ξ_j recover the Schur and symmetric Grothendieck polynomials (see the end of Section 4.3 for details).

(3) Two-sided Cauchy–Binet expansion (Theorem 4.16). Under analyticity of $\xi_j\varphi_+$ and $\theta_i\varphi_-$ on overlapping annuli, the tilted Toeplitz minor (1.2) admits the absolutely convergent expansion

$$D_N^{\xi,\theta}(\varphi) = G(\varphi)^N \sum_{\mu:\,\ell(\mu)\le N} \mathrm{JT}_\mu^{(N)}(\mathbf{a}^{\leftarrow})\ \mathrm{JT}_\mu^{(N)}(\mathbf{b}^{\leftarrow}),$$

where $\mathrm{JT}_\mu^{(N)}(\mathbf{c}) := \det\big[c^{(i)}_{\mu_j-j+i}\big]_{i,j=1}^N$ is a Jacobi–Trudi type determinant, and $\mathbf{a}^{\leftarrow}$, $\mathbf{b}^{\leftarrow}$ are the column and row Fourier coefficient sequences of $\xi_j\varphi_+$ and $\theta_i\varphi_-$, respectively, indexed in reverse. This identity generalizes Gessel's theorem [Ges90] (recovered at $\xi=\theta=1$). The Jacobi–Trudi determinants become skew Schur polynomials (Corollary 4.21) in a particular case of the pure-shift tilts $\xi_j(z)=z^{a_j}$, $\theta_i(z)=z^{-b_i}$.

The main technical input for the proof of (1.4) is the Grassmannian Jacobi identity (Theorem 2.4), which extends the classical finite-dimensional Jacobi complementary minor identity to arbitrary closed splittings $H = U \dotplus V$ in the trace-class setting. Its finite rank specialization leads to (1.4). The bialternant and Cauchy–Binet identities are derived directly from the tilted Toeplitz minor.

1.4 Related work

Let us discuss the relation of our Fredholm determinantal identity (1.4) to previous work (the history of the original BOGC identity was already outlined in Section 1.1).

(i) Grassmannian language and Sato–Segal–Wilson tau functions. There is a well-known Grassmannian interpretation of certain Toeplitz and block Toeplitz determinants as tau functions. With respect to the Hardy polarization $L^2(\mathbb{T}) = H_+ \oplus H_-$ (where H_+ is the same as our H, and H_- is the L^2 closure of the span of z^{-m}, $m>0$), the Sato–Segal–Wilson construction [SW85], [CW15, Definition 2.6] associates to a big-cell point $W = \mathrm{graph}(h_W)$, $h_W\colon H_+\to H_-$, and to a positive loop g the tau function obtained as follows. Write

$$g^{-1} = \begin{pmatrix} d & 0 \\ b & a \end{pmatrix}$$

for the inverse of g. The block matrix is written with rows and columns ordered as $H_-\oplus H_+$, so that $a\colon H_+\to H_+$, $b\colon H_-\to H_+$, and $d\colon H_-\to H_-$. Then

$$\tau_{\mathrm{SSW}}(g;W) = \det\nolimits_{H_+}\big(I + a^{-1}b\,h_W\big).$$

For the loop/Riemann–Hilbert data considered in [CW15], this tau function is identified with the large-size Szegő–Widom limit $D_\infty(g^{-1}\gamma)$ of block Toeplitz determinants [CW15, Theorem 3.4]. We refer to the latter paper for further details.

Our use of Grassmannian language is different. On the open set where A is invertible, and hence $A^{-1}-I$ is trace class in our setting, choose a unitary identification $\jmath\colon H_+ \xrightarrow{\sim} H_-$ and associate to A the graph

$$W_A = \big\{h + \jmath(A^{-1}-I)h \colon h\in H_+\big\}.$$

It is a point in the big cell of the restricted Grassmannian associated with the polarization $H_+\oplus H_-$, but it is not generally a point of the SSW/KP Grassmannian, since the invariance condition $zW_A\subset W_A$ need not hold.

The finite maps R, C (discussed in Section 1.2) turn the point W_A into a finite exterior matrix coefficient. Let $c_j = Ce_j$ be the columns of C, and let r_i be the coordinate row functionals of R. Extend r_i to a functional on $H_+ \oplus H_-$ by

$$\widetilde{r}_i(u+v) = r_i\big(u + \jmath^{-1}v\big).$$

Then

$$\widetilde{r}_i\big(c_j + \jmath(A^{-1} - I)c_j\big) = r_i(A^{-1}c_j),$$

and hence

$$\Delta_{R,C}(A) := \det\nolimits_{\mathbb{C}^N}(RA^{-1}C) = \det\big[\widetilde{r}_i\big(c_j + \jmath(A^{-1} - I)c_j\big)\big]_{i,j=1}^N.$$

This is the precise sense in which $\Delta_{R,C}(A)$ is a finite coordinate of W_A: for coordinate choices of R and C it is a Plücker coordinate, while for general finite maps it is the exterior matrix coefficient obtained by pairing the Plücker vector of W_A against $\widetilde{r}_1 \wedge \cdots \wedge \widetilde{r}_N$ and $c_1 \wedge \cdots \wedge c_N$. In Sections 3 and 4 we prove that $\Delta_{R,C}(A) = G(\varphi)^{-N} D_N^{\xi,\theta}(\varphi)$. Our identity (1.4) is the complementary Fredholm form of this finite coordinate. We emphasize that the tilt is encoded in the finite maps R, C, not in a positive loop g acting on a Grassmannian point as in the SSW setting.

This coordinate viewpoint is used again in Section 5. When the symbol φ depends on times $\mathbf{t} = (t_1, t_2, \dots)$ via the flow $\varphi(z;\mathbf{t}) = \exp\big(\sum_{k=1}^M t_k(z^k + z^{-k})\big)$, the operator $A = A(\mathbf{t}) = I - K_{\mathbf{t}}$ and hence the graph point $W_{A(\mathbf{t})}$ move in the restricted Grassmannian. For polynomial tilts, the corresponding exterior matrix coefficient is represented by the finite resolvent block

$$Y^{m,n}_{\varphi(\cdot;\mathbf{t})} = R_m(I - K_{\mathbf{t}})^{-1}C_n.$$

The identities in Section 5 compute the $\mathbf{t}$-evolution of these finite matrix coefficients directly.

(ii) Shifted Toeplitz minors and Jacobi–Trudi expansions. The particular case of the pure-shift tilt $\xi_j(z) = z^{a_j}$, $\theta_i(z) = z^{-b_i}$ reduces $D_N^{\xi,\theta}(\varphi)$ to the shifted (lacunary) Toeplitz determinant $\det[\widehat{\varphi}(p_i - q_j)]_{i,j=1}^N$ with $p_i = i + b_i$, $q_j = j + a_j$. This shifted determinant was considered by Bump–Diaconis [BD02] and Tracy–Widom [TW02b] in connection with averages over the unitary group, and further analyzed by Kozlowski [Koz14] via Riemann–Hilbert techniques in the regime where the shifts a, b grow with N. Its closed combinatorial expansion as a finite sum of products of skew Schur polynomials was given by Maximenko–Moctezuma-Salazar [MMS17] and García-García–Tierz [GGT20].

(iii) The Liu–Tripathi and Baik–Liao–Liu identities. In a recent preprint, Liu–Tripathi [LT26, Proposition 1.2] prove a contour Fredholm determinant identity which, under the matching described below, gives a different formula for the tilted Toeplitz minor:

$$D_N^{\xi,\theta}(\varphi) = \det\nolimits_{L^2(\Gamma)}\big(I + K^{\mathrm{LT}}\big),$$

where K^{LT} is an explicit double contour integral kernel on $L^2(\Gamma)$, of rank at most N, built from the same data $(\xi_j, \theta_i, \varphi_\pm)$. The earlier identity of Baik–Liao–Liu [BLL26, Proposition 4.3] is a particular case of [LT26] connected to the periodic TASEP. In their work, this contour formula plays an important role in identifying the periodic KPZ fixed point with general initial data.

The starting matrix entries in [LT26] are sums of two contour integrals,

$$\oint_0 v^{i-j-1}\, p_i(v) f_j(v)\, \frac{dv}{2\pi i} + \int_\Gamma q_i(u) g_j(u)\, \frac{du}{2\pi i}, \tag{1.5}$$

with p_i a polynomial of degree $\le i$, f_j, g_j analytic functions, Γ a contour enclosing the origin, and the four objects tied by a reproducing relation $\oint_0 v^{-i} f_i(v) H(v,u)\, dv/(2\pi i) = g_i(u)$. This sum form comes from an *additive* splitting of the symbol $\varphi = \varphi_+ + (\varphi - \varphi_+)$: the inner loop at 0 captures the Taylor part φ_+ (analytic in $|z| < 1$), and the outer contour Γ captures the remainder. Matching $f_j(v) = \widetilde{\xi}_j(v)\, \varphi_+(v)$, $q_i(u) = u^{i-1} p_i(u)$, $g_j(u) = u^{-j} \widetilde{\xi}_j(u)\, (\varphi - \varphi_+)(u)$, and

$$H(v,u) = \frac{\varphi(u) - \varphi_+(u)}{(u-v)\, \varphi_+(v)}$$

(with $\widetilde{\xi}_j$ a column tilt and p_i a row polynomial of degree $\le i$), both contours deform to the unit circle, and the entry of (1.5) collapses to $(p_i\, \widetilde{\xi}_j\, \varphi)_{j-i}$. After transposing the matrix and identifying $p_j \leftrightarrow \xi_j$ (column tilt) and $\widetilde{\xi}_i \leftrightarrow \theta_i$ (row tilt), this is exactly the tilted Toeplitz minor (1.2). The contour form of [LT26] is amenable to steepest descent asymptotic analysis, while ours is more directly connected to the original BOGC identity. It would be interesting to understand the precise relation between the two Fredholm determinants, and to see if one can be transformed into the other without going through the original determinant (1.2).

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2 Abstract Grassmannian Jacobi identity

Let H be a separable Hilbert space, and let K be a trace class operator (notation: $K \in \mathfrak{S}_1(H)$). Assume that

$$A := I - K$$

is invertible. Moreover, the Fredholm determinant $\det_H(A) = \det_H(I-K)$ is well-defined, and $\det_H(A^{-1}) = \det_H(A)^{-1}$.

Let

$$H = U \dotplus V \tag{2.1}$$

be a direct sum decomposition into closed subspaces (not necessarily orthogonal), and let

$$\Pi_U \colon H \to U, \qquad \Pi_V \colon H \to V$$

be the corresponding projections. The space of splittings $H = U \dotplus V$ into closed subspaces is the *Grassmannian* of the section title. The identity below describes how the observable $\tau_A(U,V)$ of Definition 2.1 depends on an element of this Grassmannian, for fixed A. We have

$$\Pi_U + \Pi_V = I, \qquad \Pi_U\Pi_V = \Pi_V\Pi_U = 0, \qquad \Pi_U^2 = \Pi_U, \qquad \Pi_V^2 = \Pi_V.$$

Definition 2.1. The *Grassmannian observable* attached to A and the Hilbert space splitting (2.1) is defined as

$$\tau_A(U,V) := \det_U\big(\Pi_U A^{-1}|_U\big) = \det_U\big(I_U + \Pi_U(A^{-1} - I)|_U\big).$$

The second expression is an identity plus a trace class operator on U. Indeed, because $A^{-1} - I = A^{-1}K \in \mathfrak{S}_1(H)$, the compression

$$\Pi_U(A^{-1} - I)|_U : U \to U$$

is trace class. Hence $\tau_A(U,V)$ is well-defined.

Equivalently, writing A^{-1} in block form with respect to $H = U \dotplus V$, we have

$$A^{-1} = \begin{pmatrix} A^{-1}_{UU} & A^{-1}_{UV} \\ A^{-1}_{VU} & A^{-1}_{VV} \end{pmatrix}, \qquad A^{-1}_{UU} : U \to U, \qquad \tau_A(U,V) = \det_U(A^{-1}_{UU}). \tag{2.2}$$

Remark 2.2. The trace-class condition is invariant under bounded change of basis on U inherited from a bounded automorphism of H that preserves the splitting $H = U \dotplus V$, so $\tau_A(U,V)$ depends only on the pair (U,V), independently of chosen bases or coordinates within U and V.

The proof of Theorem 2.4 below rests on the following block-triangular factorization of Fredholm determinants, adapted to oblique splittings.

Lemma 2.3. *Let H be a separable Hilbert space and $H = U \dotplus V$ a topological direct sum of closed subspaces, with associated (oblique) projections Π_U, Π_V. Let $T = I + S$ with $S \in \mathfrak{S}_1(H)$, and suppose T is block triangular with respect to this splitting, i.e. either $\Pi_V T \Pi_U = 0$ (upper) or $\Pi_U T \Pi_V = 0$ (lower). Then*

$$\det_H(T) = \det_U\big(\Pi_U T|_U\big)\, \det_V\big(\Pi_V T|_V\big). \tag{2.3}$$

Proof. Identify the Banach direct sum $U \oplus V$ with H via the bounded bijection $J\colon (u,v) \mapsto u+v$; the inverse $J^{-1}h = (\Pi_U h, \Pi_V h)$ is bounded by the closed graph theorem and the topological-sum assumption. Conjugation by J preserves $\mathfrak{S}_1$ and Fredholm determinants. After conjugation, the diagonal blocks $\Pi_U T|_U - I_U$ and $\Pi_V T|_V - I_V$ and the off-diagonal block (either $\Pi_U T|_V$ in the upper case or $\Pi_V T|_U$ in the lower case) all lie in $\mathfrak{S}_1$. Approximate them in $\mathfrak{S}_1$ by finite rank operators of the same block form; for the resulting finite rank perturbations of the identity, (2.3) is the standard linear-algebra identity for the determinant of a block-triangular matrix. Pass to the limit using continuity of det in $\mathfrak{S}_1$ [Sim05b, Theorem 3.5]. □

Theorem 2.4 (Grassmannian Jacobi complementary minor identity)**.** *With the notation above, we have*

$$\tau_A(U,V) = \det_H(A^{-1})\,\det_V\big(\Pi_V A|_V\big) = \det_H(A^{-1})\,\det_V\big(I_V - \Pi_V K|_V\big). \tag{2.4}$$

Remark 2.5. In the finite-dimensional case, identity (2.4) reduces to a classical linear algebra fact known as *Jacobi's complementary minor identity*: for an invertible matrix A, the minor of A^{-1} indexed by U on both sides equals $\det(A)^{-1}$ times the minor of A indexed by the complementary subspace V. Theorem 2.4 extends this to the infinite-dimensional trace class setting.

Remark 2.6. In Section 3.4 below we reduce Theorem 2.4 to the classical Borodin–Okounkov–Geronimo–Case identity.

Proof of Theorem 2.4. Define

$$B := \Pi_U + A\Pi_V = I - K\Pi_V,$$

where the second equality uses $\Pi_U + \Pi_V = I$ and $A = I - K$. We will compute $\det_H(B)$ and $\det_H(A^{-1}B)$ in two ways and compare. In the finite-dimensional case this is the standard textbook proof of Jacobi's complementary minor identity; we follow that route here, using Lemma 2.3 to handle the block-triangular Fredholm determinant factorizations in the trace-class setting.

We first check that all three operators A^{-1}, B, and $A^{-1}B$ are of the form "identity plus trace class", so that their Fredholm determinants on H are well-defined. For A^{-1}, write

$$A^{-1} = I + A^{-1}K,$$

and note that $A^{-1}K \in \mathfrak{S}_1(H)$ by the trace-class ideal property (product of bounded A^{-1} and trace-class K). For B, the second equality in the definition gives $B - I = -K\Pi_V$, again trace class. Multiplying the two,

$$A^{-1}B = (I + A^{-1}K)(I - K\Pi_V) = I + A^{-1}K - K\Pi_V - A^{-1}KK\Pi_V,$$

so $A^{-1}B - I$ is a sum of three trace-class terms and is therefore in $\mathfrak{S}_1(H)$. (One can equally rewrite this as $A^{-1}B - I = A^{-1}K\Pi_U$ using $A^{-1} = I + A^{-1}K$ and $\Pi_U + \Pi_V = I$, but only trace-classness matters here.)

The Fredholm determinant is multiplicative [Sim05b, Theorem 3.5]:

$$\det_H\big((I+S)(I+T)\big) = \det_H(I+S)\det_H(I+T),$$

where $S, T \in \mathfrak{S}_1(H)$. This implies

$$\det_H(A^{-1}B) = \det_H(A^{-1})\det_H(B). \tag{2.5}$$

Let us identify the factors in (2.5) with those in the desired identity (2.4). In the splitting $H = U \dotplus V$, we have the block representations

$$B = \begin{pmatrix} I_U & \Pi_U A|_V \\ 0 & \Pi_V A|_V \end{pmatrix}, \qquad A^{-1}B = \begin{pmatrix} \Pi_U A^{-1}|_U & 0 \\ \Pi_V A^{-1}|_U & I_V \end{pmatrix},$$

cf. (2.2). Both matrices are block-triangular (B upper, $A^{-1}B$ lower), so Lemma 2.3 gives

$$\det_H(B) = \det_V\big(\Pi_V A|_V\big), \qquad \det_H(A^{-1}B) = \det_U\big(\Pi_U A^{-1}|_U\big) = \tau_A(U,V). \tag{2.6}$$

This completes the proof of Theorem 2.4. ☐

3 Specialization to finite rank

Here we specialize Section 2 to the classical setup of $H = \ell^2(\mathbb{Z}_{\geq 0})$ (the Hardy space of the unit circle) with a finite-dimensional subspace $U \subset H$. The latter is encoded by a column map $C\colon \mathbb{C}^N \to H$ with $\operatorname{Ran}(C) = U$, coupled with a row map $R\colon H \to \mathbb{C}^N$. In particular, the Grassmannian observable $\tau_A(U,V)$ becomes an $N \times N$ determinant. The classical Borodin–Okounkov–Geronimo–Case (BOGC) identity for $N \times N$ Toeplitz determinants arises as the orthogonal ("vacuum") case when U is the span of the first N basis vectors.

3.1 Toeplitz operator setup

Take $H = \ell^2(\mathbb{Z}_{\geq 0})$ with the orthonormal basis $\{e_k\}_{k\geq 0}$, identified with the holomorphic basis $e_k \leftrightarrow z^k$ on the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, with boundary circle $\mathbb{T} = \{|z| = 1\}$. Let $P_N\colon H \to H$ be the orthogonal projection onto the subspace $\operatorname{span}(e_0, \ldots, e_{N-1})$, and let $Q_N \coloneqq I - P_N$ be the complementary projection. Set

$$P_N H = \operatorname{span}(e_0, \ldots, e_{N-1}), \qquad Q_N H = \overline{\operatorname{span}}(e_N, e_{N+1}, \ldots), \tag{3.1}$$

so that $H = P_N H \oplus Q_N H$ is a closed orthogonal direct sum decomposition.

Equip $\mathbb{T}$ with the normalized Lebesgue measure. For a function $\varphi \in L^1(\mathbb{T})$ with Fourier expansion $\varphi(z) = \sum_{k\in\mathbb{Z}} \widehat{\varphi}(k)\, z^k$, define the Toeplitz operator $T(\varphi)\colon H \to H$ and the Hankel operator $\mathcal{H}(\varphi)\colon H \to H$ by their matrix entries

$$T(\varphi)_{ij} \coloneqq \widehat{\varphi}(i-j), \qquad \mathcal{H}(\varphi)_{ij} \coloneqq \widehat{\varphi}(i+j+1), \qquad i,j \geq 0.$$

The associated $N \times N$ *Toeplitz determinant* is

$$D_N(\varphi) \coloneqq \det\nolimits_{P_N H}\big(P_N T(\varphi) P_N\big) = \det\big[\widehat{\varphi}(i-j)\big]_{i,j=0}^{N-1}. \tag{3.2}$$

We assume $\varphi\colon \mathbb{T} \to \mathbb{C}^\times$ is nonvanishing with zero winding number around the origin, so that $\log\varphi$ is well-defined as a continuous function on $\mathbb{T}$. We also assume that

$$\log\varphi \in C^\alpha(\mathbb{T}) \quad \text{for some } \alpha > \tfrac{1}{2}. \tag{3.3}$$

Set

$$G(\varphi) \coloneqq \exp \widehat{\log\varphi}(0),$$

this is often called the geometric mean of φ. Then φ admits a *canonical Wiener–Hopf factorization*

$$\varphi(z) = G(\varphi)\varphi_-(z)\varphi_+(z),$$

where φ_+ extends analytically to $|z| < 1$ with $\varphi_+(0) = 1$, and φ_- extends analytically to $|z| > 1$ with $\varphi_-(\infty) = 1$. Set

$$b(z) \coloneqq \frac{\varphi_-(z)}{\varphi_+(z)}, \qquad c(z) \coloneqq \frac{1}{b(z)} = \frac{\varphi_+(z)}{\varphi_-(z)}, \qquad \widetilde{c}(z) \coloneqq c(z^{-1}),$$

and define

$$K = K^\varphi \coloneqq \mathcal{H}(b)\mathcal{H}(\widetilde{c})\colon H \to H. \tag{3.4}$$

Under (3.3), $\mathcal{H}(b)$ and $\mathcal{H}(\widetilde{c})$ are Hilbert–Schmidt, so $K \in \mathfrak{S}_1(H)$ is trace class. (Indeed, $C^\alpha(\mathbb{T}) \subset H^s(\mathbb{T})$ for every $s < \alpha$; choosing $s > \frac{1}{2}$ gives $\sum_{n\geq 1} n\,|\widehat{b}(n)|^2 < \infty$ and likewise for $\widetilde{c}$, whence the Hankel operators are Hilbert–Schmidt by the standard criterion; see, e.g., [Pel03], [BS06].) We use two classical inputs from [BW00], [Böt01], [Sim05a]:

1. The *operator Wiener–Hopf identity*
$$T(\varphi) = G(\varphi)T(\varphi_+)(I-K)^{-1}T(\varphi_-); \tag{3.5}$$

2. The *strong Szegő relation*
$$\det{}_H(I-K)^{-1} = Z, \qquad \log Z = \sum_{k\geq 1} k\,\widehat{\log\varphi}(k)\,\widehat{\log\varphi}(-k), \tag{3.6}$$

   where $\widehat{\log\varphi}(k)$ denotes the k-th Fourier coefficient of $\log\varphi$. Under (3.3), the series in (3.6) converges, and Z is a nonzero complex number.

Throughout the rest of the paper, we denote
$$A = A^\varphi \coloneqq I - K \colon H \to H, \tag{3.7}$$
so that $Z = \det_H(A^{-1})$.

3.2 Finite rank chart

Fix $N \geq 1$. A *finite rank chart* on H is a pair of operators
$$R\colon H \to \mathbb{C}^N \quad \text{(bounded, surjective)}, \qquad C\colon \mathbb{C}^N \to H \quad \text{(injective)},$$
together with the *Gram operator* $\Gamma \coloneqq RC \in \operatorname{End}(\mathbb{C}^N)$, which we assume invertible. Granted the injectivity of C and surjectivity of R, the invertibility of Γ is equivalent to the transversality condition $\operatorname{Ran}(C) \cap \operatorname{Ker}(R) = \{0\}$. Set
$$J \coloneqq C\Gamma^{-1}\colon \mathbb{C}^N \to H, \qquad \Pi_U \coloneqq JR = C(RC)^{-1}R, \qquad \Pi_V \coloneqq I - \Pi_U. \tag{3.8}$$
both viewed as operators $H \to H$. Then $RJ = I_{\mathbb{C}^N}$, so J is a bounded right-inverse of R with $\operatorname{Ran}(J) = \operatorname{Ran}(C)$; Π_U projects onto $\operatorname{Ran}(C)$ along $\operatorname{Ker}(R)$; and Π_V is the complementary projection onto $\operatorname{Ker}(R)$ along $\operatorname{Ran}(C)$. The splitting
$$H = U \dotplus V, \qquad U \coloneqq \operatorname{Ran}(C), \qquad V \coloneqq \operatorname{Ker}(R), \tag{3.9}$$
satisfies the hypotheses of Theorem 2.4, with $\dim U = N$.

3.3 Finite rank Grassmannian Jacobi identity

Theorem 3.1. *For K, A as in (3.4), (3.7) depending on a symbol φ as in Section 3.1, and (R, C) a finite rank chart with $\Gamma = RC$ invertible, we have*
$$\det{}_{\mathbb{C}^N}\big(RA^{-1}C\big) = Z \cdot \det{}_{\mathbb{C}^N}\Gamma \cdot \det{}_{\operatorname{Ker}(R)}\big(I_{\operatorname{Ker}(R)} - \Pi_V K|_{\operatorname{Ker}(R)}\big), \tag{3.10}$$
where Z is the strong Szegő normalization (3.6).

Proof. Factor the left-hand side of (3.10) as

$$RA^{-1}C = (RC) \cdot (RC)^{-1}RA^{-1}C,$$

giving

$$\det_{\mathbb{C}^N}\big(RA^{-1}C\big) = \det_{\mathbb{C}^N}\Gamma \cdot \det_{\mathbb{C}^N}\big(\Gamma^{-1}RA^{-1}C\big). \tag{3.11}$$

The bijection $C\colon \mathbb{C}^N \to U = \mathrm{Ran}(C)$ has inverse $\Gamma^{-1}R|_U\colon U \to \mathbb{C}^N$. Thus, under the isomorphism $C\colon \mathbb{C}^N \xrightarrow{\sim} U$, the operator $(RC)^{-1}RA^{-1}C$ on $\mathbb{C}^N$ is conjugate to

$$\Pi_U A^{-1} \Pi_U = C\Gamma^{-1}RA^{-1}C\Gamma^{-1}R$$

on the N-dimensional space U. Thus, their determinants are equal:

$$\det_{\mathbb{C}^N}\big((RC)^{-1}RA^{-1}C\big) = \det_U\big(\Pi_U A^{-1}|_U\big) = \tau_A(U,V).$$

Applying Theorem 2.4 yields the desired result. □

Lemma 3.2 (Finite rank oblique correction)**.** *Let (R,C) be a finite rank chart with $\Gamma = RC$ invertible, and let $f_1,\ldots,f_N$ be the standard basis of $\mathbb{C}^N$. Set*

$$c_\alpha := Cf_\alpha \in H, \qquad \psi_\alpha(j) := f_\alpha^\top \Gamma^{-1}RKe_j, \quad 1 \le \alpha \le N.$$

Then, before restriction to $V = \mathrm{Ker}(R)$, we have

$$(\Pi_V K)(i,j) = K(i,j) - \sum_{\alpha=1}^{N} c_\alpha(i)\,\psi_\alpha(j).$$

Thus, the oblique kernel is the BOGC kernel plus at most N rank-one corrections.

Proof. From $\Pi_V = I - C\Gamma^{-1}R$ in (3.8),

$$\Pi_V K = K - C\Gamma^{-1}RK = K - \sum_{\alpha=1}^{N}(Cf_\alpha)\,(f_\alpha^\top\Gamma^{-1}RK),$$

which is the stated coordinate identity. □

3.4 Reduction to Borodin–Okounkov–Geronimo–Case identity

The classical Borodin–Okounkov–Geronimo–Case (BOGC) identity [GC79], [BO00], [BW00], [Böt01] is the orthogonal ("vacuum") case $R = C = P_N$ of Theorem 3.1. The same Jacobi-based reduction in special case was carried out in [BW06]. For completeness, let us record how our general finite rank identity (Theorem 3.1) specializes to the BOGC identity:

Corollary 3.3 (Borodin–Okounkov–Geronimo–Case identity)**.** *With the notation (3.1)–(3.6), we have*

$$D_N(\varphi) = G(\varphi)^N \cdot Z \cdot \det_{Q_N H}\big(I_{Q_N H} - Q_N K|_{Q_N H}\big). \tag{3.12}$$

Proof. Identify U with $\mathbb{C}^N$ in a canonical way, that is, set $R = C = P_N$ in Theorem 3.1, where P_N is the orthogonal projection onto the first N basis vectors. Then Γ is the identity on $\mathbb{C}^N$, and $\det_{\mathbb{C}^N} \Gamma = 1$. We have $U = P_N H$ and $V = Q_N H$, the orthogonal splitting of H along the first N basis vectors. The right-hand side of (3.10) becomes

$$Z \cdot \det_{Q_N H}\big(I_{Q_N H} - Q_N K|_{Q_N H}\big), \tag{3.13}$$

where Z is the strong Szegő normalization (3.6). For the left-hand side, we use the Wiener–Hopf identity (3.5) together with the triangular nature of the Toeplitz operators $T(\varphi_\pm)$. Namely, since φ_+ is analytic in $|z| < 1$ with $\varphi_+(0) = 1$, the matrix $T(\varphi_+)$ is uni-lower-triangular, hence $P_N T(\varphi_+) Q_N = 0$, so

$$P_N T(\varphi_+) = P_N T(\varphi_+) P_N. \tag{3.14}$$

Similarly, $T(\varphi_-)$ is uni-upper-triangular, $Q_N T(\varphi_-) P_N = 0$, so

$$T(\varphi_-) P_N = P_N T(\varphi_-) P_N. \tag{3.15}$$

Notice that $P_N T(\varphi_\pm) P_N$ are unitriangular $N \times N$ matrices on $P_N H$, with determinant 1. Applying (3.14)–(3.15), we get

$$P_N T(\varphi) P_N = G(\varphi)\big(P_N T(\varphi_+) P_N\big)(I - K)^{-1}\big(P_N T(\varphi_-) P_N\big),$$

an equality of operators on H that act trivially on $Q_N H$, equivalently, of operators on $P_N H$ after compression. The outer factors are unitriangular and act within $P_N H$, so restricting to $P_N H$ and taking determinants yields

$$D_N(\varphi) = \det_{P_N H}\big(P_N T(\varphi) P_N\big) = G(\varphi)^N \det_{P_N H}\big(P_N (I - K)^{-1} P_N\big) = G(\varphi)^N \det_{\mathbb{C}^N}\big(R A^{-1} C\big),$$

where the factor $G(\varphi)^N$ comes from the scalar $G(\varphi)$ in (3.5) acting on the N-dimensional space $P_N H$. Combined with (3.10) and (3.13), this gives the desired identity. □

4 Tilted Toeplitz minors

Starting here, we make the operators R, C defining the finite rank chart in Theorem 3.1 explicit via families of analytic "tilt" functions ξ_j and θ_i. The resulting Toeplitz-like determinants are interpreted as Weyl-type bialternant expressions, and also admit Cauchy–Binet expansions. The latter extend Gessel's theorem [Ges90] representing a restricted sum of products of Schur functions as a Toeplitz determinant.

4.1 Tilted Toeplitz minors setup

Fix two collections of functions ξ_j and θ_i, $1 \le i, j \le N$: each ξ_j is bounded analytic on $|z| < 1$, and each θ_i is bounded analytic on $|z| > 1$ and regular at $z = \infty$. We impose no normalization at $z = 0$ or $z = \infty$, so monomials and arbitrary analytic profiles are both allowed. Recall the basis identification $e_k \leftrightarrow z^k$ for the Hardy space H, under which a vector $h \in H$ is identified with the power series $h(z) = \sum_{k \ge 0} h_k z^k$.

Definition 4.1 (Tilt operators)**.** The *column tilt operator* $\Xi\colon H \to H$ acts column by column: it sends the j-th basis vector e_{j-1} (corresponding to the monomial z^{j-1}) to the vector identified with the function $z^{j-1}\xi_j(z)$, and leaves the basis vectors e_k for $k \geq N$ untouched. In matrix form, the j-th column of Ξ for $1 \leq j \leq N$ is the Fourier coefficient sequence of $z^{j-1}\xi_j(z)$, padded with zeros at indices $< j-1$:

$$\Xi\, e_{j-1} = \sum_{k\geq 0} \widehat{\xi}_j(k)\, e_{j-1+k} \quad (1 \leq j \leq N), \qquad \Xi\, e_k = e_k \quad (k \geq N). \tag{4.1}$$

This sum is finite whenever each ξ_j is a polynomial in z.

The *row tilt operator* $\Theta\colon H \to \mathbb{C}^N$ acts row by row: its i-th component $(\Theta h)_i \in \mathbb{C}$, for $1 \leq i \leq N$, is the z^{i-1}-Fourier coefficient of the product $\theta_i(z)\, h(z)$. Since θ_i is bounded analytic on $|z| > 1$ and regular at infinity, its Fourier expansion $\theta_i(z) = \sum_{m\geq 0} \widehat{\theta}_i(-m)\, z^{-m}$ has only nonpositive modes, and so

$$(\Theta h)_i = \sum_{m\geq 0} \widehat{\theta}_i(-m)\, h_{i-1+m} \quad (1 \leq i \leq N). \tag{4.2}$$

The sum is finite whenever each θ_i is a polynomial in z^{-1}.

Both $\Xi - I$ and Θ have rank at most N.

The tilt operators combine with the Wiener–Hopf factors of φ into the finite-dimensional chart maps

$$R \coloneqq \Theta\, T(\varphi_+)\colon H \to \mathbb{C}^N, \qquad C \coloneqq T(\varphi_-)\, \Xi\, P_N\colon \mathbb{C}^N \to H. \tag{4.3}$$

The associated Gram operator is

$$\Gamma_{\xi,\theta} \coloneqq R\, C = \Theta\, T(\varphi_+)\, T(\varphi_-)\, \Xi\, P_N \in \mathrm{End}(\mathbb{C}^N). \tag{4.4}$$

When $\Gamma_{\xi,\theta}$ is invertible, these maps give a finite rank chart in the sense of Section 3.2.

Definition 4.2. The $N \times N$ *tilted Toeplitz minor* associated to the symbol φ and the tilts ξ_j, θ_i is defined as

$$D_N^{\xi,\theta}(\varphi) \coloneqq \det\Big[\big(\theta_i\, \xi_j\, \varphi\big)_{i-j}\Big]_{i,j=1}^{N}, \tag{4.5}$$

where $(f)_k \coloneqq \widehat{f}(k)$ denotes the k-th Fourier coefficient.

Lemma 4.3. *The tilted Toeplitz minor can be written as*

$$D_N^{\xi,\theta}(\varphi) = G(\varphi)^N\, \det\nolimits_{\mathbb{C}^N}\big(R\, A^{-1}\, C\big).$$

Proof. By the operator Wiener–Hopf identity (3.5),

$$\Theta\, T(\varphi)\, \Xi\, P_N = G(\varphi)\, \Theta\, T(\varphi_+)\, (I-K)^{-1}\, T(\varphi_-)\, \Xi\, P_N = G(\varphi)\, R\, A^{-1}\, C,$$

where R, C are as in (4.3) and $A = I - K$.

Let us compute the matrix entries of the operator $\Theta\, T(\varphi)\, \Xi\, P_N$ on $\mathbb{C}^N$ by tracking Fourier coefficients. Throughout the calculation, φ is a general symbol with Fourier coefficients $\widehat{\varphi}(c)$ defined at all integers $c \in \mathbb{Z}$; note that ξ_j and θ_i have constrained Fourier support.

Fix $1 \le j \le N$. Since $j-1 < N$, the projection P_N acts as the identity on e_{j-1}, so

$$\Xi\, P_N\, e_{j-1} = \sum_{n\ge 0} \widehat{\xi}_j(n)\, e_{j-1+n},$$

a vector in H whose ℓ-th coordinate equals $\widehat{\xi}_j(\ell-j+1)$ for $\ell \ge j-1$ and zero otherwise. Applying $T(\varphi)$ to the vector above, we obtain a vector whose ℓ-th coordinate is given by

$$\big(T(\varphi)\,\Xi\, P_N\, e_{j-1}\big)_\ell = \sum_{k\ge 0} \widehat{\varphi}(\ell-k)\,\big(\Xi\, P_N\, e_{j-1}\big)_k = \sum_{n\ge 0} \widehat{\varphi}(\ell-j+1-n)\,\widehat{\xi}_j(n).$$

Finally, the operator Θ extracts row Fourier coefficients. Substituting the vector above with $\ell = i-1+m$, we get by (4.2):

$$\big(\Theta\, T(\varphi)\,\Xi\, P_N\big)_{ij} = \sum_{m\ge 0}\sum_{n\ge 0} \widehat{\theta}_i(-m)\,\widehat{\varphi}(i-j+m-n)\,\widehat{\xi}_j(n). \tag{4.6}$$

We compare (4.6) with the convolution

$$\big(\theta_i\, \xi_j\, \varphi\big)_{i-j} = \sum_{a+b+c=i-j} \widehat{\theta}_i(a)\,\widehat{\xi}_j(b)\,\widehat{\varphi}(c),$$

where $a, b, c \in \mathbb{Z}$. Note that $\widehat{\theta}_i(a) = 0$ for $a > 0$ and $\widehat{\xi}_j(b) = 0$ for $b < 0$. Setting $a = -m$ ($m \ge 0$), $b = n$ ($n \ge 0$), and $c = i-j+m-n$, we see that the convolution above is exactly the double sum in (4.6). This completes the proof. ☐

Theorem 4.4. *Assume that* $\Gamma_{\xi,\theta}$ (4.4) *is invertible on* $\mathbb{C}^N$*. Then the tilted Toeplitz minor admits the Fredholm representation*

$$D_N^{\xi,\theta}(\varphi) = G(\varphi)^N \cdot Z \cdot \det_{\mathbb{C}^N}(\Gamma_{\xi,\theta}) \cdot \det_{\mathrm{Ker}(R)}\big(I_{\mathrm{Ker}(R)} - \Pi_V K|_{\mathrm{Ker}(R)}\big), \tag{4.7}$$

where $R = \Theta\, T(\varphi_+)$ *and* $C = T(\varphi_-)\,\Xi\, P_N$ *are the chart maps* (4.3)*, and* $\Pi_V = I - C\Gamma_{\xi,\theta}^{-1}R$ *is the oblique projection onto* $\mathrm{Ker}(R)$ *along* $\mathrm{Ran}(C)$*.*

Proof. Immediately follows from Lemma 4.3 and Theorem 3.1. ☐

Remark 4.5. Setting $\xi_j = \theta_i = 1$ for all i, j recovers the BOGC identity (Corollary 3.3) from Theorem 4.4. Moreover, the pure shift tilts $\xi_j(z) = z^{a_j}$, $\theta_i(z) = z^{-b_i}$ for nonnegative integers a_j, b_i recovers the shifted Toeplitz minors of [BD02]: the matrix entries in Definition 4.2 reduce to $\widehat{\varphi}(p_i - q_j)$, where $p_i \coloneqq i + b_i$ and $q_j \coloneqq j + a_j$.

4.2 Properties of the tilted kernel

The kernel $\Pi_V K$ in the tilted Fredholm determinant in (4.7) differs from the underlying operator $K = \mathcal{H}(b)\mathcal{H}(\widetilde{c})$ (3.4) of the BOGC identity (Corollary 3.3) by an operator of rank at most N. Indeed, since $\Pi_V = I - C\Gamma_{\xi,\theta}^{-1}R$, we have

$$\Pi_V K = K - C\,\Gamma_{\xi,\theta}^{-1}\,R\,K.$$

Since $C\,\Gamma_{\xi,\theta}^{-1}\,R\,K$ factors through $\mathbb{C}^N$, it has rank at most N.

The kernel $\Pi_V K|_{\mathrm{Ker}(R)}$ lives on the oblique tail space $\mathrm{Ker}(R)$, which depends on the tilts ξ_j, θ_i and, as a result, may nontrivially depend on N. In the BOGC identity, the tail space $Q_N H = \overline{\mathrm{span}}(e_N, e_{N+1}, \ldots)$ is canonical. Let us record a version of our tilted identity when the oblique tail space is brought back to the canonical form $Q_N H$ by a change of basis.

For shorter notation, write $P = P_N$, $Q = Q_N$, and set

$$B \coloneqq RP|_{PH} \colon PH \longrightarrow \mathbb{C}^N.$$

When B is invertible, we view B^{-1} as a map $\mathbb{C}^N \to PH$ and define the graph parametrization

$$\mathcal{J}_N \colon QH \longrightarrow H, \qquad \mathcal{J}_N y \coloneqq y - B^{-1}Ry,$$

where the second term lies in PH. Equivalently,

$$\mathcal{J}_N = (Q - PB^{-1}RQ)|_{QH}.$$

On QH set

$$\mathsf{K}_N^{\xi,\theta} \coloneqq Q\big(I - C\Gamma_{\xi,\theta}^{-1}R\big)K\mathcal{J}_N \colon QH \to QH, \tag{4.8}$$

and

$$\mathsf{F}_N^{\xi,\theta} \coloneqq -QKPB^{-1}RQ - QC\Gamma_{\xi,\theta}^{-1}RK\mathcal{J}_N. \tag{4.9}$$

Proposition 4.6. *Keep the assumptions of Theorem 4.4, and assume that B is invertible. Then $\mathcal{J}_N$ is a bounded isomorphism from QH onto $\mathrm{Ker}(R)$, with inverse $Q|_{\mathrm{Ker}(R)}$. Consequently,*

$$D_N^{\xi,\theta}(\varphi) = G(\varphi)^N \cdot Z \cdot \det_{\mathbb{C}^N}(\Gamma_{\xi,\theta}) \cdot \det_{Q_N H}\big(I_{Q_N H} - \mathsf{K}_N^{\xi,\theta}\big). \tag{4.10}$$

Moreover, as operators $QH \to QH$, we have

$$\mathsf{K}_N^{\xi,\theta} = QKQ - QKPB^{-1}RQ - QC\Gamma_{\xi,\theta}^{-1}RK\mathcal{J}_N = QKQ + \mathsf{F}_N^{\xi,\theta}. \tag{4.11}$$

In particular, the rank of the correction $\mathsf{F}_N^{\xi,\theta}$ between the tilted kernel $\mathsf{K}_N^{\xi,\theta}$ and the BOGC kernel QKQ satisfies

$$\mathrm{rank}\big(\mathsf{K}_N^{\xi,\theta} - QKQ\big) \le \mathrm{rank}(RQ) + \mathrm{rank}(QC).$$

Proof. For $y \in QH$, we have

$$R\mathcal{J}_N y = Ry - RB^{-1}Ry = Ry - BB^{-1}Ry = 0,$$

so $\mathcal{J}_N(QH) \subseteq \mathrm{Ker}(R)$. Conversely, if $v \in \mathrm{Ker}(R)$, then $v = Pv + Qv$ and

$$0 = Rv = R(Pv) + R(Qv) = B(Pv) + R(Qv).$$

Hence $Pv = -B^{-1}R(Qv)$, and therefore $v = \mathcal{J}_N(Qv)$. This proves that $\mathcal{J}_N \colon QH \to \mathrm{Ker}(R)$ is bijective, and its inverse is $Q|_{\mathrm{Ker}(R)}$.

Conjugating the operator $I_{\mathrm{Ker}(R)} - \Pi_V K|_{\mathrm{Ker}(R)}$ by this isomorphism gives, on QH,

$$Q\big(I_{\mathrm{Ker}(R)} - \Pi_V K|_{\mathrm{Ker}(R)}\big)\mathcal{J}_N = I_{QH} - Q\Pi_V K\mathcal{J}_N,$$

because $Q\mathcal{J}_N = I_{QH}$. The operator $Q\Pi_V K\mathcal{J}_N$ is trace class on QH, since K is trace class on H and all other maps in the composition are bounded. Since Fredholm determinants are invariant under bounded conjugation, (4.10) follows from Theorem 4.4.

The expansion (4.11) follows by substituting $\Pi_V = I - C\Gamma_{\xi,\theta}^{-1}R$ and $\mathcal{J}_N = Q - PB^{-1}RQ$ into (4.8). The first correction term in (4.9) factors through $RQ\colon QH \to \mathbb{C}^N$, while the second factors through $QC\colon \mathbb{C}^N \to QH$; this yields the rank bound. ☐

Proposition 4.7. *Assume the hypotheses of Proposition 4.6. If the tilts are polynomial of uniformly bounded degrees,*

$$\deg \xi_j \le d_\xi, \qquad \deg_{z^{-1}} \theta_i \le d_\theta,$$

then

$$\operatorname{rank}\big(\mathsf{K}_N^{\xi,\theta} - QKQ\big) \le d_\xi + d_\theta.$$

For $\xi_j = \theta_i = 1$, this correction vanishes because $d_\xi = d_\theta = 0$, and so we have $\mathsf{K}_N^{\mathbf{1},\mathbf{1}} = QKQ$, the original BOGC kernel.

Proof of Proposition 4.7. Because $T(\varphi_+)$ is lower triangular, $T(\varphi_+)QH \subseteq QH$. If each θ_i has degree at most d_θ in z^{-1}, then Θ restricted to QH only sees the coordinates $e_N, \dots, e_{N+d_\theta-1}$. Thus $RQ = \Theta T(\varphi_+)Q$ factors through a space of dimension d_θ, so $\operatorname{rank}(RQ) \le d_\theta$. Similarly, ΞP_N has image in $\operatorname{span}(e_0, \dots, e_{N+d_\xi-1})$, and the upper triangularity of $T(\varphi_-)$ preserves this finite span. Therefore, QC has image in $\operatorname{span}(e_N, \dots, e_{N+d_\xi-1})$, and $\operatorname{rank}(QC) \le d_\xi$. Combining these two estimates with the last statement of Proposition 4.6 yields the desired rank bound. ☐

Remark 4.8. In the column-only case $\theta_i \equiv 1$, identify $\mathbb{C}^N$ with PH. Then $R = PT(\varphi_+)$, and the lower triangularity of $T(\varphi_+)$ gives $RQ = PT(\varphi_+)Q = 0$. Since $PT(\varphi_+)P|_{PH}$ is triangular with nonzero diagonal, it is invertible; hence $\operatorname{Ker}(R) = QH$ and $\mathcal{J}_N$ is the identity map on QH. Thus the fixed-tail conjugation is trivial on the R-side, and

$$\mathsf{K}_N^{\xi,\mathbf{1}} = QKQ - QC\Gamma_{\xi,\mathbf{1}}^{-1}RKQ.$$

If only k of the tilts ξ_j differ from 1, then the correction $QC = QT(\varphi_-)(\Xi P - P)$ has rank at most k.

Remark 4.9. The preceding formulas take a transparent form for a rank-one chart. Take a column vector $\alpha \in H$ and a row covector $\beta^* \in H^*$, and interpret them as $C = \alpha\colon \mathbb{C} \to H$ and $R = \beta^*\colon H \to \mathbb{C}$. Assume that $\beta^*\alpha \neq 0$, and write $\beta^\perp := \ker \beta^*$. Then $\Gamma = \beta^*\alpha$, and the notation (3.8) becomes

$$J = \frac{\alpha}{\beta^*\alpha}, \qquad \Pi_U = \frac{\alpha\beta^*}{\beta^*\alpha}, \qquad \Pi_V = I - \frac{\alpha\beta^*}{\beta^*\alpha}.$$

The rank-one identity is

$$\beta^* A^{-1}\alpha = Z\,\beta^*\alpha \cdot \det_{\beta^\perp}\big(I_{\beta^\perp} - \Pi_V K|_{\beta^\perp}\big).$$

Varying α and β, one recovers the matrix coefficients of the inverse operator A^{-1} through these codimension-one Fredholm determinants.

If, in addition, $\beta^* e_0 \neq 0$, the fixed-tail change of basis above is available. It is

$$\mathcal{J}_1 \colon Q_1 H \longrightarrow H, \qquad \mathcal{J}_1 y = y - \frac{\beta^* y}{\beta^* e_0}\, e_0,$$

and it maps $Q_1 H$ isomorphically onto $\beta^\perp$, with inverse $Q_1|_{\beta^\perp}$. The fixed-tail kernel is therefore

$$\mathsf{K}_1^{\alpha,\beta} := Q_1 \left(I - \frac{\alpha \beta^*}{\beta^* \alpha} \right) K \mathcal{J}_1 \colon Q_1 H \to Q_1 H,$$

and the determinant becomes

$$\beta^* A^{-1} \alpha = Z\, \beta^* \alpha \cdot \det_{Q_1 H}\big(I_{Q_1 H} - \mathsf{K}_1^{\alpha,\beta}\big).$$

In this form the finite-rank correction is explicit:

$$\mathsf{K}_1^{\alpha,\beta} = Q_1 K Q_1 - \frac{Q_1 K e_0\, \beta^*|_{Q_1 H}}{\beta^* e_0} - \frac{Q_1 \alpha\, \beta^* K \mathcal{J}_1}{\beta^* \alpha}.$$

Thus, the difference from the ordinary tail kernel $Q_1 K Q_1$ is the sum of at most two rank-one operators on $Q_1 H$.

4.3 Bialternant form of one-sided tilted Toeplitz minors

Throughout this subsection we write $D_N^{\xi,\mathbf{1}}$ and $D_N^{\mathbf{1},\theta}$ for the column-only and row-only specializations of the tilted Toeplitz minor of Definition 4.2, and we assume $G(\varphi) = 1$ in each case.

Definition 4.10 (Rank-N rational Wiener–Hopf factors)**.** Fix tuples $X = (x_1, \ldots, x_N)$ and $Y = (y_1, \ldots, y_N)$ of pairwise distinct complex numbers with $|x_k| < 1$ and $|y_l| < 1$. We say a symbol φ has rank-N rational factor φ_+ or φ_- if, respectively,

$$\varphi_+(z) = \prod_{k=1}^{N} \frac{1}{1 - x_k z}, \quad \text{or} \quad \varphi_-(z) = \prod_{l=1}^{N} \frac{1}{1 - y_l / z}.$$

When one of these rational conditions is imposed, the other factor is arbitrary, subject to the usual assumptions of Section 3.1 on the full symbol φ.

Theorem 4.11 (Bialternant factorization, column-side)**.** *Let φ have rank-N rational φ_- with alphabet Y as in Definition 4.10. Let $\xi_1, \ldots, \xi_N$ be analytic in a neighborhood of $|z| < 1$. Then the column-only tilted Toeplitz minor factorizes as*

$$D_N^{\xi,\mathbf{1}}(\varphi) = \mathcal{S}_\xi(Y) \cdot \prod_{l=1}^{N} \varphi_+(y_l), \qquad \mathcal{S}_\xi(Y) := \frac{\det\big[y_i^{N-j}\, \xi_{N-j+1}(y_i)\big]_{i,j=1}^{N}}{\Delta(Y)}, \tag{4.12}$$

where $\Delta(Y) = \det[y_i^{N-j}]_{i,j=1}^{N} = \prod_{i<j}(y_i - y_j)$ is the Vandermonde determinant of Y.

Proof. Set $P(z) \coloneqq \prod_{\ell=1}^{N}(z-y_\ell)$. Since

$$\varphi_-(z) = \prod_{\ell=1}^{N} \frac{1}{1-y_\ell/z} = \frac{z^N}{P(z)},$$

we have $\varphi(z) = \varphi_+(z)\, z^N/P(z)$ under the normalization $G(\varphi) = 1$. The (i,j)-entry of the tilted Toeplitz minor is the Fourier coefficient

$$M_{ij} = \frac{1}{2\pi i} \oint_{|z|=\rho} \xi_j(z)\varphi(z)\, z^{j-i-1}\, dz = \frac{1}{2\pi i} \oint_{|z|=\rho} \xi_j(z)\varphi_+(z)\, \frac{z^{N+j-i-1}}{P(z)}\, dz,$$

where we choose $\max_\ell |y_\ell| < \rho < 1$, so $\xi_j\varphi_+$ is analytic inside the contour. Since $1 \le i,j \le N$, the exponent $N+j-i-1 \ge 0$, so the integrand has no pole at $z=0$; the only poles inside $|z| = \rho$ are the simple poles at $z = y_\ell$. By the residue theorem,

$$M_{ij} = \sum_{\ell=1}^{N} \xi_j(y_\ell)\, \varphi_+(y_\ell)\, \frac{y_\ell^{N+j-i-1}}{P'(y_\ell)} = \sum_{\ell=1}^{N} y_\ell^{N-i} \cdot \frac{\varphi_+(y_\ell)}{P'(y_\ell)} \cdot y_\ell^{j-1}\xi_j(y_\ell).$$

This implies the decomposition $M = ADB_\xi$ with $A_{i\ell} = y_\ell^{N-i}$, $D = \operatorname{diag}\big(\varphi_+(y_\ell)/P'(y_\ell)\big)$, $(B_\xi)_{\ell j} = y_\ell^{j-1}\xi_j(y_\ell)$. The first factor gives $\Delta(Y)$ in the numerator, but $\prod_\ell P'(y_\ell) = (-1)^{N(N-1)/2}\Delta(Y)^2$, which leaves a single Vandermonde in the denominator. The determinant of B_ξ becomes, up to the sign $(-1)^{N(N-1)/2}$ from the denominator, the numerator in the bialternant $\mathcal{S}_\xi(Y)$. This completes the proof. ☐

Corollary 4.12. *Specializing $\xi_j \equiv 1$ in Theorem 4.11 gives unshifted Toeplitz minor $D_N(\varphi) = \prod_{l=1}^{N} \varphi_+(y_l)$, so that (4.12) can be rewritten as $D_N^{\xi,\mathbf{1}}(\varphi) = \mathcal{S}_\xi(Y) \cdot D_N(\varphi)$.*

Remark 4.13. A similar bialternant factorization holds for the row-only tilted Toeplitz minor $D_N^{\mathbf{1},\theta}(\varphi)$ when φ has rank-N rational φ_+ with alphabet $X = (x_1, \dots, x_N)$. We omit the statement here, but only record the corresponding bialternant

$$\mathcal{S}_\theta^{\#}(X) \coloneqq \frac{\det\big[x_k^{-(N-i)}\, \theta_{N-i+1}(1/x_k)\big]_{i,k=1}^{N}}{\Delta(X^{-1})}.$$

Remark 4.14 (Smaller alphabet on the residue side)**.** The hypotheses of Theorem 4.11 require the alphabet Y to have exactly N entries. If φ_- has $|Y| < N$ poles, formula (4.12) can be specialized to setting the extra entries of Y to zero and using the l'Hôpital rule to cancel the resulting zeros in the numerator and denominator of $\mathcal{S}_\xi(Y)$.

Let us record a special case of the bialternant expression $\mathcal{S}_\xi(Y)$ arising in Theorem 4.11. First, if $\lambda = (\lambda_1 \ge \dots \ge \lambda_N \ge 0)$ is a partition with at most N parts, $\widehat{\lambda}_j = \lambda_{N+1-j}$, and the tilt functions are binomials $\xi_j(z) = z^{\widehat{\lambda}_j}(1+\beta z)^{N-j}$, then

$$\mathcal{S}_\xi(Y) = \frac{\det\big[y_i^{\lambda_j+N-j}(1+\beta y_i)^{j-1}\big]_{i,j=1}^{N}}{\Delta(Y)} =: G_\lambda^\beta(y_1, \dots, y_N),$$

which is the symmetric Grothendieck polynomial, [LS82],[FK94],[Buc02],[Yel17]. Further taking $\beta = 0$ recovers the Schur polynomial $s_\lambda(y_1, \dots, y_N)$.

An alternative choice

$$\xi_j(z) = (1 + \beta z)^{\widehat{\lambda}_j}$$

yields a different Grothendieck-type bialternant which we denote as

$$\mathcal{S}_\xi(Y) = \frac{\det\big[y_i^{N-j}(1+\beta y_i)^{\lambda_j}\big]_{i,j=1}^N}{\Delta(Y)} =: \tilde{G}_\lambda^\beta(y_1, \dots, y_N).$$

This determinant essentially appears in [BLL26, (3.16)], and is related to the "dual" Grothendieck-type polynomials from [MS13]. Let us emphasize that the "dual" term used in [MS13] conflicts with the more standard dual Grothendieck polynomials g_λ arising from the original ones through the Hall inner product on the algebra of symmetric functions [Mac95]. We refer to [Yel17], [HJK+24] for details on the g_λ's and their combinatorial interpretations.

4.4 Cauchy–Binet expansion

Here we expand the two-sided tilted Toeplitz minor $D_N^{\xi,\theta}(\varphi)$ into a restricted sum over partitions, generalizing Gessel's theorem [Ges90] that was a key ingredient in the original proof of the BOGC identity in [BO00].

Throughout this subsection we assume there exists $R > 1$ such that $\xi_j\varphi_+$ is holomorphic in $|z| < R$ for each j, and $\theta_i\varphi_-$ is holomorphic in $|z| > R^{-1}$ and regular at ∞ for each i. Expand

$$\xi_j(z)\,\varphi_+(z) = \sum_{r\ge 0} a_r^{(j)}\, z^r \qquad \theta_i(z)\,\varphi_-(z) = \sum_{r\ge 0} b_r^{(i)}\, z^{-r},$$

where $1 \le i, j \le N$. Set $a_r^{(j)} := 0$, $b_r^{(i)} := 0$ for $r < 0$. Denote the coefficient sequences by $\mathbf{a}^{(j)} = (a_r^{(j)})_{r\ge 0}$ and $\mathbf{b}^{(i)} = (b_r^{(i)})_{r\ge 0}$, and write

$$\mathbf{a} = (\mathbf{a}^{(1)}, \dots, \mathbf{a}^{(N)}), \quad \mathbf{b} = (\mathbf{b}^{(1)}, \dots, \mathbf{b}^{(N)}), \quad \mathbf{a}^{\leftarrow} = (\mathbf{a}^{(N)}, \dots, \mathbf{a}^{(1)}), \quad \mathbf{b}^{\leftarrow} = (\mathbf{b}^{(N)}, \dots, \mathbf{b}^{(1)}).$$

Definition 4.15. Let $\mathbf{c} = (\mathbf{c}^{(1)}, \dots, \mathbf{c}^{(N)})$ be a family of one-sided sequences $\mathbf{c}^{(i)} = (c_r^{(i)})_{r\ge 0}$ (with $c_r^{(i)} := 0$ for $r < 0$), and let $\mu = (\mu_1 \ge \dots \ge \mu_N \ge 0)$ be a partition with at most N parts. The associated *Jacobi–Trudi type determinant* is

$$\mathrm{JT}_\mu^{(N)}(\mathbf{c}) := \det\big[c_{\mu_j - j + i}^{(i)}\big]_{i,j=1}^N.$$

Theorem 4.16 (Cauchy–Binet expansion)**.** *Under the above assumptions, we have the absolutely convergent identity*

$$D_N^{\xi,\theta}(\varphi) = G(\varphi)^N \sum_{\mu:\ \ell(\mu)\le N} \mathrm{JT}_\mu^{(N)}(\mathbf{a}^{\leftarrow})\ \mathrm{JT}_\mu^{(N)}(\mathbf{b}^{\leftarrow}). \tag{4.13}$$

Proof. Set $A_j(z) := \xi_j(z)\varphi_+(z)$ and $B_i(z) := \theta_i(z)\varphi_-(z)$. We have

$$[z^{i-j}]\, B_i(z)A_j(z) = \sum_{s\ge 0} b_s^{(i)}\, a_{s+i-j}^{(j)} = \sum_{k\ge 0} b_{k-i+1}^{(i)}\, a_{k-j+1}^{(j)} \qquad (k := s + i - 1),$$

using $a_r^{(j)} = b_r^{(i)} = 0$ for $r < 0$. Set $\mathsf{B}_{i,k} \coloneqq b^{(i)}_{k-i+1}$, $\mathsf{A}_{k,j} \coloneqq a^{(j)}_{k-j+1}$ $(k \geq 0)$. Then the matrix of the tilted minor $D_N^{\xi,\theta}(\varphi)$ factors as $G(\varphi)\,\mathsf{BA}$ with B of size $N \times \infty$ and A of size $\infty \times N$.

Cauchy–Binet applied to finite truncations $\mathsf{B}^{(M)}\mathsf{A}^{(M)}$ in the k-index gives

$$\det\bigl(\mathsf{B}^{(M)}\mathsf{A}^{(M)}\bigr) = \sum_{0 \leq k_1 < \cdots < k_N \leq M} \det\bigl[b^{(i)}_{k_\ell - i+1}\bigr]_{i,\ell=1}^{N} \det\bigl[a^{(j)}_{k_\ell - j+1}\bigr]_{\ell,j=1}^{N}.$$

Fix any $\rho \in (1, R)$. Cauchy's estimate on $|z| = \rho$ for $\xi_j \varphi_+$ and on $|z| = \rho^{-1}$ for $\theta_i \varphi_-$ (holomorphic at ∞ via $w = z^{-1}$) yields $|a_r^{(j)}| + |b_r^{(i)}| \leq C\rho^{-r}$ for all $r \geq 0$, uniformly in $1 \leq i, j \leq N$. Hence each $N \times N$ minor in the sum is bounded by $N!\, C^N \rho^{N(N-1)/2} \rho^{-(k_1+\cdots+k_N)}$, and the product of the two minors on the right is $O(\rho^{-2(k_1+\cdots+k_N)})$. Thus, the limit $M \to \infty$ exists and yields, after multiplication by $G(\varphi)^N$, the absolutely convergent identity

$$D_N^{\xi,\theta}(\varphi) = G(\varphi)^N \sum_{0 \leq k_1 < \cdots < k_N} \det\bigl[b^{(i)}_{k_\ell - i+1}\bigr]_{i,\ell=1}^{N} \det\bigl[a^{(j)}_{k_\ell - j+1}\bigr]_{\ell,j=1}^{N}.$$

Reparameterizing $k_\ell = \mu_{N-\ell+1} + \ell - 1$ yields partitions μ with at most N parts, and the minors in the right-hand side above become the Jacobi–Trudi determinants $\mathrm{JT}^{(N)}_\mu(\mathbf{b}^{\leftarrow})$ and $\mathrm{JT}^{(N)}_\mu(\mathbf{a}^{\leftarrow})$. This completes the proof. □

Remark 4.17. The sum (4.13) is a partition function of a N-point *biorthogonal ensemble* in the sense of [Bor98], since the Jacobi–Trudi determinants have the form

$$\mathrm{JT}^{(N)}_\mu(\mathbf{c}) = \det[f_i(m_j)]_{i,j=1}^{N}, \qquad f_i(r) = c^{(i)}_{r+i-N}, \qquad m_j = \mu_j + N - j,\ j = 1, \ldots, N.$$

Remark 4.18. The biorthogonal weight in (4.13) should be distinguished from the Grothendieck measure on partitions (and from the tilted biorthogonal ensemble) introduced in [GP24]. In that construction, each particle j carries a finite-difference operator chain of length $j-1$, column-flagged by the particle position. Here the row tilt θ_i is absorbed into the row-indexed sequence $\mathbf{b}^{(i)}$ inside the Jacobi–Trudi determinant. The two ensembles share the feature that their probability weights are given by products of two determinants, but the tilt is placed in different data. It would be interesting to find a common generalization of the two objects.

4.5 Schur symbols

Let us consider the case when the symbol $\varphi(z)$ corresponds to two generic Schur-positive specializations $\rho^{\pm} = (\alpha^{\pm}; \beta^{\pm}; \gamma^{\pm})$ of the algebra of symmetric functions associated with the factors φ_+ and φ_-. We refer to [BO16] for background on Schur-positive specializations and their connection to Toeplitz determinants and representations of the infinite symmetric group. We set

$$\varphi(z) = e^{\gamma^+ z} \prod_{k=1}^{\infty} \frac{1 + \beta_k^+ z}{1 - \alpha_k^+ z} \cdot e^{\gamma^- z^{-1}} \prod_{l=1}^{\infty} \frac{1 + \beta_l^- z^{-1}}{1 - \alpha_l^- z^{-1}},$$

where

$$\sum_{k=1}^{\infty} (\alpha_k^+ + \beta_k^+) < \infty, \qquad \sum_{l=1}^{\infty} (\alpha_l^- + \beta_l^-) < \infty, \qquad \gamma^+, \gamma^- \geq 0.$$

In the case of the trivial tilt $\xi_j = \theta_i = 1$ for all i, j, Theorem 4.16 reduces to Gessel's theorem [Ges90]. Indeed, then the coefficients $a_r^{(i)}, b_r^{(i)}$ are independent of i, and equal to the complete homogeneous symmetric functions under the specializations $\rho^\pm$.

Corollary 4.19 (Gessel's theorem [Ges90])**.** *With the above assumptions, we have the following identity for the non-tilted Toeplitz determinants:*

$$D_N(\varphi) = \sum_{\mu\colon \ell(\mu)\le N} s_\mu(\rho^+)\, s_\mu(\rho^-). \tag{4.14}$$

Remark 4.20. When one of the specializations is a pure alpha specialization with at most N nonzero α-parameters, the restriction $\ell(\mu) \le N$ in the sum above is automatically satisfied. Therefore, in this case, $D_N(\varphi)$ has a product form thanks to the Cauchy summation identity for Schur functions [Mac95, I.(4.3)]. If ρ^- is such a finite alpha specialization, the same product form follows by setting $\xi_j \equiv 1$ in Theorem 4.11; if ρ^+ is finite alpha, it follows from the analogous row-side bialternant of Remark 4.13.

In the general case, the biorthogonal ensemble arising from the right-hand side of (4.14) is a *conditional Schur measure*, conditioned on the length of the partition being at most N. Asymptotic analysis of such conditional ensembles is delicate and usually calls for methods other than computing the correlation kernel in the conditional setting.

In the case of two-sided pure shift tilts $\xi_j(z) = z^{\lambda_{N+1-j}}$, $\theta_i(z) = z^{-\nu_{N+1-i}}$ for partitions λ, ν with at most N parts, the Jacobi–Trudi determinants in Theorem 4.16 reduce to skew Schur polynomials under the specializations $\rho^\pm$.

Corollary 4.21 (Skew Schur expansion)**.** *With the above assumptions and notation, we have*

$$D_N^{\xi,\theta}(\varphi) = \sum_{\eta\colon \ell(\eta)\le N} s_{\eta/\lambda}(\rho^+)\, s_{\eta/\nu}(\rho^-). \tag{4.15}$$

Under Schur-positive specializations $\rho^\pm$, the summands in (4.15) are nonnegative. Thus, $D_N^{\xi,\theta}(\varphi)$ is the normalizing constant for a probability measure on partitions with at most N parts, supported on η's with $\eta \supseteq \lambda, \nu$.

5 Resolvent flows and a finite-dimensional closure problem

Specialize the symbol $\varphi(z)$ to the family depending on finitely many times $\mathbf{t} = (t_1, \ldots, t_M)$ given by

$$\varphi(z; \mathbf{t}) := \exp\Bigl(\sum_{r=1}^{M} t_r (z^r + z^{-r})\Bigr). \tag{5.1}$$

In the case $M = 1$, this is the Bessel symbol $\varphi_t(z) = e^{t(z+z^{-1})}$. The goal of this section is to describe the dependence of the tilted Toeplitz minors on the times $\mathbf{t}$, inspired by the differential equation method for integrable Fredholm determinants of Its–Izergin–Korepin–Slavnov [IIKS90]; the Airy-kernel F_2/Painlevé II formula of Tracy–Widom [TW02a] is the model example. For the Bessel Toeplitz determinants, related Painlevé equations and recurrences for orthogonal polynomials on the unit circle (OPUC) are part of the classical scalar theory, recalled below in Remark 5.11. Here we record the finite resolvent flow identities, and formulate the separate finite-dimensional closure problem.

5.1 Universal resolvent matrix and polynomial tilts

For an arbitrary symbol φ satisfying the assumptions of Section 3.1, define

$$R_m\colon H \to \mathbb{C}^m, \quad e_p^\top R_m = e_p^\top T(\varphi_+), \qquad C_n\colon \mathbb{C}^n \to H, \quad C_n e_q = T(\varphi_-) e_q,$$

where m, n are arbitrary positive integers, $0 \le p < m$, $0 \le q < n$. The pair (R_m, C_n) is determined by the symbol φ, and does not depend on the tilt.

Definition 5.1. The *universal resolvent matrix* of (R_m, C_n) at the symbol φ is

$$Y_\varphi^{m,n} := R_m\, A^{-1}\, C_n \in \mathrm{Mat}_{m,n}(\mathbb{C}), \qquad A = I - K^\varphi.$$

Let $A_\xi \in \mathbb{C}^{n\times N}$ and $A_\theta \in \mathbb{C}^{N\times m}$, and define

$$R = A_\theta\, R_m\colon H \to \mathbb{C}^N, \qquad C = C_n\, A_\xi\colon \mathbb{C}^N \to H. \tag{5.2}$$

Then, by associativity,

$$\det\nolimits_{\mathbb{C}^N}\bigl(R\, A^{-1}\, C\bigr) = \det\nolimits_{\mathbb{C}^N}\bigl(A_\theta\, Y_\varphi^{m,n}\, A_\xi\bigr). \tag{5.3}$$

Let us now specialize to polynomial tilts.

Proposition 5.2. *Let $\xi_j(z)$ be a polynomial of degree at most $d_\xi \le n - N$ in z and $\theta_i(z)$ be a polynomial of degree at most $d_\theta \le m - N$ in z^{-1} for all i, j. Then the chart $R = \Theta\, T(\varphi_+)$, $C = T(\varphi_-)\, \Xi\, P_N$ of (4.3) has the factorization (5.2) with banded matrices*

$$(A_\theta)_{i,p} = \widehat{\theta}_i(i-1-p), \qquad (A_\xi)_{q,j} = \widehat{\xi}_j(q-j+1), \tag{5.4}$$

where $1 \le i, j \le N$, $0 \le p < m$, $0 \le q < n$. Then

$$G(\varphi)^{-N}\, D_N^{\xi,\theta}(\varphi) = \det\nolimits_{\mathbb{C}^N}\bigl(A_\theta\, Y_\varphi^{m,n}\, A_\xi\bigr). \tag{5.5}$$

Proof. Expand the row tilt operator of (4.2): $(\Theta h)_i = \sum_{a\ge 0} \widehat{\theta}_i(-a)\, h_{i-1+a}$. Since θ_i has degree at most d_θ in z^{-1}, the i-th row of $\Theta T(\varphi_+)$ is

$$\sum_{a=0}^{d_\theta} \widehat{\theta}_i(-a)\, e_{i-1+a}^\top T(\varphi_+).$$

The coefficient formula for A_θ in (5.4) therefore gives $\Theta T(\varphi_+) = A_\theta R_m$; the degree bound $d_\theta \le m - N$ ensures that all rows used by this sum lie among $0, \dots, m-1$. Similarly, from (4.1), the formula for A_ξ gives $T(\varphi_-)\Xi P_N = C_n A_\xi$, and the bound $d_\xi \le n - N$ ensures that all columns used lie among $0, \dots, n-1$. Thus the tilted chart has the factorization (5.2). Applying (5.3) and then Lemma 4.3 gives (5.5). □

Remark 5.3. For (5.3), the matrices A_ξ, A_θ can be arbitrary. However, to connect them to a finite chart (R, C) as in Section 3, the matrices $A_\theta R_m$ and $C_n A_\xi$ must have rank N, and the Gram matrix $A_\theta R_m C_n A_\xi$ must be invertible. The polynomial tilts with bounded degrees provide one such family of examples.

Definition 5.4. For a symbol φ and chart maps R, C (fixed, or built from φ as in (4.3)) with $\Gamma := RC$ invertible on $\mathbb{C}^N$, the *oblique Fredholm tau* is

$$\mathcal{T}_{R,C}(\varphi) := \det_{\mathbb{C}^N}(\Gamma)\ \det_{\mathrm{Ker}(R)}\big(I_{\mathrm{Ker}(R)} - \Pi_V K|_{\mathrm{Ker}(R)}\big).$$

When R, C depend on parameters, this notation means that R, C, K and Π_V are evaluated at the same parameter value. Equivalently, by Theorem 3.1,

$$\mathcal{T}_{R,C}(\varphi) = Z^{-1}\ \det_{\mathbb{C}^N}\big(R\,A^{-1}\,C\big) = \det_H(I-K)\ \det_{\mathbb{C}^N}\big(R\,A^{-1}\,C\big), \tag{5.6}$$

so identity (4.7) reads $D_N^{\xi,\theta}(\varphi) = G(\varphi)^N\, Z\, \mathcal{T}_{R,C}(\varphi)$.

5.2 Symmetric finite Laurent symbols

Fix $M \geq 1$ and $\mathbf{t} = (t_1, \ldots, t_M) \in \mathbb{R}^M$; the identities below extend holomorphically to $\mathbf{t} \in \mathbb{C}^M$. The symmetric finite Laurent exponential symbol $\varphi(z; \mathbf{t})$ is defined by (5.1). Its Wiener–Hopf factors are $\varphi_\pm(z) = \exp\big(\sum_r t_r z^{\pm r}\big)$, thus, $G(\varphi) = 1$, and the strong Szegő constant is $Z_\mathbf{t} = \exp\big(\sum_{r=1}^M r\, t_r^2\big)$. This implies that

$$\det_H(I - K_\mathbf{t}) = e^{-\sum_r r\, t_r^2}, \qquad \partial_{t_r} \log\det_H(I-K_\mathbf{t}) = -2r\, t_r.$$

Let S, S^* denote the right and left shifts on $H = \ell^2(\mathbb{Z}_{\geq 0})$, $Se_k = e_{k+1}$ and $S^* e_0 = 0$, $S^* e_k = e_{k-1}$ for $k \geq 1$. The ratio

$$b(z;\mathbf{t}) := \frac{\varphi_-(z)}{\varphi_+(z)} = \exp\Big(\sum_r t_r(z^{-r} - z^r)\Big)$$

has Fourier coefficients $b_n(\mathbf{t})$. Since $\widetilde{c} = b$ in this symmetric case, the kernel has the form $K_\mathbf{t} = \mathcal{H}(b)\mathcal{H}(b)^\top$.

Remark 5.5. By [BO00, Remark 2], for $i \neq j$, we can write the BOGC kernel as follows:

$$K_\mathbf{t}(i,j) = -\frac{1}{i-j} \sum_{r=1}^M r t_r \sum_{a=0}^{r-1} \left(b_{i-a}(\mathbf{t}) b_{j+r-a}(\mathbf{t}) - b_{j-a}(\mathbf{t}) b_{i+r-a}(\mathbf{t})\right).$$

By Lemma 3.2, the tilted kernel corresponds to a rank N correction of $K_\mathbf{t}$, that is,

$$(\Pi_V K_\mathbf{t})(i,j) = -\frac{1}{i-j} \sum_{r=1}^M r t_r \sum_{a=0}^{r-1} \left(b_{i-a}(\mathbf{t}) b_{j+r-a}(\mathbf{t}) - b_{j-a}(\mathbf{t}) b_{i+r-a}(\mathbf{t})\right) - \sum_{\alpha=1}^N c_\alpha(i)\, \psi_\alpha(j),$$

where $c_\alpha = C f_\alpha \in H$ and $(\psi_\alpha(j))_{j\geq 0} = f_\alpha^\top \Gamma^{-1} R K_\mathbf{t} e_j$.

For $1 \leq r \leq M$ and $0 \leq a \leq r-1$, define

$$\rho_a^{(r)}[\ell] := b_{\ell+a+1-r}(\mathbf{t}), \qquad h_a^{(r)} := \mathcal{H}(b)\, \rho_a^{(r)}.$$

Lemma 5.6. *In the above setting, the coefficients $b_n(\mathbf{t})$ decay faster than exponentially as $n \to \pm\infty$, locally uniformly for $\mathbf{t}$ in compact subsets of $\mathbb{C}^M$. Consequently $\mathcal{H}(b)$ and $\partial_{t_r}\mathcal{H}(b)$ are Hilbert–Schmidt, $K_\mathbf{t} = \mathcal{H}(b)\mathcal{H}(b)^\top$ and $\partial_{t_r} K_\mathbf{t}$ are trace class, and $\rho_a^{(r)}, h_a^{(r)} \in H$.*

Proof. Cauchy's formula on $|z| = R > 1$ gives, for $n \geq 0$,

$$|b_n(\mathbf{t})| \leq \exp\bigl(C_{\mathbf{t}}(R^M + R^{-M})\bigr) R^{-n},$$

and optimizing in R gives superexponential decay. The coefficients with $n < 0$ are handled by applying the same estimate on $|z| = R^{-1}$. The same bounds apply to $\partial_{t_r} b = (z^{-r} - z^r)b$. Its Fourier coefficients are finite linear combinations of b_{n-r} and b_{n+r}, so they also decay faster than exponentially. Therefore $\partial_{t_r}\mathcal{H}(b) = \mathcal{H}(\partial_{t_r} b)$ is Hilbert–Schmidt by the same criterion; indeed,

$$\sum_{i,\ell \geq 0} |b_{i+\ell+1}|^2 = \sum_{k \geq 1} k|b_k|^2 < \infty,$$

and the same calculation for $\partial_{t_r} b$ applies. Products of two Hilbert–Schmidt operators are trace class, giving the assertion for $K_{\mathbf{t}}$ and $\partial_{t_r} K_{\mathbf{t}}$ (the latter by the Leibniz rule). Finally, $\rho_a^{(r)}$ is a shift of the coefficient sequence of b, and $h_a^{(r)} = \mathcal{H}(b)\rho_a^{(r)}$. □

Proposition 5.7. *For each $1 \leq r \leq M$, we have*

$$\partial_{t_r}\mathcal{H}(b) = (S^*)^r \mathcal{H}(b) - S^r\,\mathcal{H}(b) - \sum_{a=0}^{r-1} e_a\,(\rho_a^{(r)})^\top, \tag{5.7}$$

$$\partial_{t_r} K_{\mathbf{t}} = (S^*)^r K_{\mathbf{t}} - S^r\,K_{\mathbf{t}} + K_{\mathbf{t}}\,S^r - K_{\mathbf{t}}\,(S^*)^r - \sum_{a=0}^{r-1}\bigl[e_a\,(h_a^{(r)})^\top + h_a^{(r)}\,e_a^\top\bigr]. \tag{5.8}$$

The derivatives above are taken in the Hilbert–Schmidt class for the Hankel operators and in the trace-class sense for $K_{\mathbf{t}}$.

Proof. For (5.7), $\partial_{t_r} b(z) = (z^{-r} - z^r)\,b(z)$, so $\partial_{t_r} b_n = b_{n+r} - b_{n-r}$. Hence $\partial_{t_r}\mathcal{H}(b)_{i,\ell} = b_{i+\ell+1+r} - b_{i+\ell+1-r}$. The identities $((S^*)^r\mathcal{H}(b))_{i,\ell} = b_{i+\ell+1+r}$ and $(S^r\mathcal{H}(b))_{i,\ell} = b_{i+\ell+1-r}$ hold for $i \geq r$. For $0 \leq i < r$, the second shifted term is zero, and the boundary sum subtracts $b_{\ell+a+1-r}$ in row $i = a$.

For (5.8), differentiate $K_{\mathbf{t}} = \mathcal{H}(b)\,\mathcal{H}(b)^\top$ by the Leibniz rule:

$$\partial_{t_r} K_{\mathbf{t}} = (\partial_{t_r}\mathcal{H}(b))\,\mathcal{H}(b)^\top + \mathcal{H}(b)\,(\partial_{t_r}\mathcal{H}(b))^\top.$$

Since $((S^*)^r)^\top = S^r$ and $(S^r)^\top = (S^*)^r$, the transpose of the first two terms in (5.7) contributes $K_{\mathbf{t}} S^r - K_{\mathbf{t}}(S^*)^r$. The transpose of the boundary term contributes $-\sum_a h_a^{(r)} e_a^\top$, while the non-transposed boundary term contributes $-\sum_a e_a (h_a^{(r)})^\top$. □

Proposition 5.8. *Fix the tilts ξ_j, θ_i as in Section 4, and let Ξ, Θ be the corresponding tilt operators (Definition 4.1). Set $R(\mathbf{t}) = \Theta T(\varphi_+(\mathbf{t}))$ and $C(\mathbf{t}) = T(\varphi_-(\mathbf{t}))\Xi P_N$. For each $1 \leq r \leq M$, set $Q(\mathbf{t}) \coloneqq (I - K_{\mathbf{t}})^{-1}$ and $Y(\mathbf{t}) \coloneqq R(\mathbf{t})Q(\mathbf{t})C(\mathbf{t})$. Suppressing $\mathbf{t}$ from the notation, we have*

$$\partial_{t_r} Y = R\,(S^*)^r\,Q\,C + R\,Q\,S^r\,C - \sum_{a=0}^{r-1}\bigl[(R\,Q\,e_a)((h_a^{(r)})^\top\,Q\,C) + (R\,Q\,h_a^{(r)})(e_a^\top\,Q\,C)\bigr]. \tag{5.9}$$

Proof. Since $T(\varphi_+) = \exp(\sum_s t_s\, S^s)$ and $T(\varphi_-) = \exp(\sum_s t_s\, (S^*)^s)$, we have $\partial_{t_r} R = R\, S^r$ and $\partial_{t_r} C = (S^*)^r\, C$. Differentiating $Y = RQC$ and using $\partial_{t_r} Q = Q\, (\partial_{t_r} K)\, Q$ gives

$$\partial_{t_r} Y = R\, S^r QC + RQ(S^*)^r C + RQ(\partial_{t_r} K)QC.$$

Substitute (5.8). The four pure-shift products telescope as follows, using $KQ = QK = Q - I$:

$$\begin{aligned} RQ(S^*)^r KQC &= RQ(S^*)^r QC - RQ(S^*)^r C, \\ RQS^r KQC &= RQS^r QC - RQS^r C, \\ RQKS^r QC &= RQS^r QC - RS^r QC, \\ RQK(S^*)^r QC &= RQ(S^*)^r QC - R(S^*)^r QC. \end{aligned}$$

With the signs $+, -, +, -$ from (5.8), the $RQ(S^*)^r QC$ and $RQS^r QC$ terms cancel pairwise. The remaining pure-shift terms are cancelled by $RS^r QC + RQ(S^*)^r C$, leaving $R(S^*)^r QC + RQS^r C$. The boundary part of (5.8) gives the finite rank sum in (5.9). □

Corollary 5.9. *For the time-dependent universal maps $R_m(\mathbf{t}), C_n(\mathbf{t})$, the same identity holds for the rectangular universal block $Y_\varphi^{m,n} = R_m Q C_n$:*

$$\partial_{t_r} Y_\varphi^{m,n} = R_m(S^*)^r QC_n + R_m QS^r C_n - \sum_{a=0}^{r-1} \big[(R_m Q e_a)((h_a^{(r)})^\top QC_n) + (R_m Q h_a^{(r)})(e_a^\top QC_n)\big]. \quad (5.10)$$

Consequently, for polynomial tilts and on the open set where $A_\theta Y_\varphi^{m,n} A_\xi$ is invertible, we have

$$\partial_{t_r} \log \det_{\mathbb{C}^N} \big(A_\theta Y_\varphi^{m,n} A_\xi\big) = \operatorname{tr}_{\mathbb{C}^N} \Big[\big(A_\theta Y_\varphi^{m,n} A_\xi\big)^{-1} A_\theta (\partial_{t_r} Y_\varphi^{m,n}) A_\xi\Big].$$

Proof. The proof of Proposition 5.8 applies verbatim to $R_m(\mathbf{t}), C_n(\mathbf{t})$, since $\partial_{t_r} R_m = R_m S^r$ and $\partial_{t_r} C_n = (S^*)^r C_n$. The logarithmic derivative of a determinant is a standard fact for finite-dimensional matrices. □

Corollary 5.10. *Fix the tilts and set*

$$\Gamma(\mathbf{t}) = R(\mathbf{t})C(\mathbf{t}), \qquad Y(\mathbf{t}) = R(\mathbf{t})Q(\mathbf{t})C(\mathbf{t}), \qquad \mathcal{T}(\mathbf{t}) := \mathcal{T}_{R(\mathbf{t}),C(\mathbf{t})}(\varphi(\cdot;\mathbf{t})).$$

On any open set where $\Gamma(\mathbf{t})$ and $Y(\mathbf{t})$ are invertible, for $1 \le r \le M$ we have

$$\partial_{t_r} \log \mathcal{T}(\mathbf{t}) = -2r\, t_r + \operatorname{tr}_{\mathbb{C}^N} \big(Y(\mathbf{t})^{-1}\, \partial_{t_r} Y(\mathbf{t})\big).$$

Proof. By (5.6), $\log \mathcal{T}(\mathbf{t}) = \log \det_H(I - K_{\mathbf{t}}) + \log \det_{\mathbb{C}^N}(Y(\mathbf{t}))$ on this open set. The first summand is $-\sum_s s\, t_s^2$, whose t_r-derivative is $-2r\, t_r$. The second yields $\operatorname{tr}(Y(\mathbf{t})^{-1}\, \partial_{t_r} Y(\mathbf{t}))$ in the same way as in Corollary 5.9. □

Remark 5.11. For $M = 1$, (5.1) is the Bessel symbol $\varphi_t(z) = e^{t(z+z^{-1})}$. The pure-shift tilts produce determinants of the form

$$\det\big[I_{\nu+i-j}(2t)\big]_{i,j=0}^{n-1}.$$

Up to elementary normalizations and indexing conventions, such determinants are known to form Painlevé III′ tau-function sequences; see Forrester–Witte [FW04] and, for broader tau-function background, [FW02]. For recent asymptotic work, see [CXZ24]. The associated OPUC reflection coefficients and related finite Laurent Toeplitz determinants satisfy discrete Painlevé-type recurrences; see also [CT23].

5.3 Finite-dimensional closure problem

For finite Laurent exponential symbols and polynomial tilts of bandwidth at most d, Sections 5.1 and 5.2 give the first t_r-derivatives of the finite blocks in terms of the following finite list of resolvent matrix elements:

$$\begin{array}{c} Y_\varphi^{m,n}, \quad R_m(S^*)^r QC_n, \quad R_m QS^r C_n, \\ R_m\, Q\, e_a, \quad R_m\, Q\, h_a^{(r)}, \quad e_a^\top\, Q\, C_n, \quad (h_a^{(r)})^\top\, Q\, C_n, \end{array} \qquad (1 \le r \le M,\ 0 \le a \le r-1). \tag{5.11}$$

The shifted blocks $R_m(S^*)^r QC_n$ and $R_m QS^r C_n$ are part of the data in (5.10); they are not, in general, entries of the original block $Y_\varphi^{m,n}$. Thus, (5.11) is not yet a closed finite-dimensional system.

Conjecture 5.12. Assume that φ admits a nonvanishing analytic extension to an annulus containing $\mathbb{T}$, has zero winding number on $\mathbb{T}$, and, for one branch of $\log \varphi$ on this annulus, the logarithmic derivative extends to a rational function,

$$z\, \partial_z \log \varphi(z) \in \mathbb{C}(z).$$

Assume also that the tilts (ξ_j, θ_i) are finite polynomial or rational functions, with the corresponding chart nondegenerate on the parameter domain under consideration. Then, after adjoining finitely many auxiliary shifted-boundary or residue variables to the variables in (5.11), the associated deformation equations for $\mathcal{T}_{R,C}$ should admit a finite-dimensional IIKS/Riemann–Hilbert realization. In such a realization $\mathcal{T}_{R,C}$, up to explicit scalar factors, should coincide with the corresponding isomonodromic tau function.

Remark 5.13. A finite closure, if present, cannot simply come from minors of the original block alone. At a minimum, it should also use Plücker relations among minors of enlarged blocks, as well as rational functions of the deformation parameters.

Numerical experiments point in the same direction. In the test case of polynomial tilts with matrix size $N = 3$ and degree bound $d = 2$, differentiating the minors of the original finite block produced quantities which could not be written as constant linear combinations of the same minors. After adding the nearest boundary-shifted minors, the number of numerically independent sampled quantities increased from 12 to 15, that is, by three. A separate check on the corresponding resolvent quantities showed the same numerical increase, again from 12 to 15. Thus the expected finite closure, if it exists, should allow coefficients depending on the times, at least rationally, rather than only a fixed finite-dimensional span with constant coefficients.

A On spiked soft edge asymptotics of tilted Fredholm formulas

Let us illustrate how the tilted Fredholm determinant on the canonical tail space $Q_N H = \overline{\operatorname{span}}\{e_N, e_{N+1}, \ldots\}$ (Proposition 4.6) can be used in soft edge asymptotic computations. We work out a rank one "spiked" example in the sense of Baik–Ben Arous–Péché [BBP05], presenting only asymptotic computations without rigorous convergence estimates.

A.1

Consider the finite Laurent symbol

$$\varphi_L(z) = \exp L\big(a(z+z^{-1}) + b(z^2+z^{-2})\big), \qquad a>0, \quad 0<b<a/8.$$

Then $G(\varphi_L)=1$ and the strong Szegő constant is

$$Z_L = \exp L^2(a^2+2b^2).$$

For the ordinary Toeplitz determinant, the BOGC identity gives

$$\frac{D_N(\varphi_L)}{Z_L} = \det_{Q_N H}\big(I_{Q_N H} - Q_N K_L Q_N\big).$$

Set

$$\chi = 2a-4b, \qquad c=(a-8b)^{1/3}, \qquad N_L(s) = \lfloor \chi L + cL^{1/3}s \rfloor.$$

To see the soft edge scale, look at the Fourier coefficients of

$$b_L(z) = \frac{\varphi_{L,-}(z)}{\varphi_{L,+}(z)} = \exp L\big(a(z^{-1}-z) + b(z^{-2}-z^2)\big). \tag{A.1}$$

They are given by

$$b_{L,n} = \oint b_L(z) z^{-n} \frac{dz}{2\pi i z}.$$

The saddle point corresponding to $n \approx \chi L$ is $z=-1$. Writing $z=-e^{\lambda}$, and ignoring the harmless sign $(-1)^n$, the exponent in (A.1) with the Fourier factor z^{-n} becomes

$$L\big(2a\sinh\lambda - 2b\sinh(2\lambda)\big) - n\lambda.$$

At $n=\chi L$ the linear term cancels, and

$$2a\sinh\lambda - 2b\sinh(2\lambda) - \chi\lambda = \frac{a-8b}{3}\lambda^3 + O(\lambda^5).$$

Thus the Airy scale is $N = \chi L + O(L^{1/3})$. At the level of Fourier coefficients, the standard steepest descent at $z=-1$ gives

$$(-1)^n cL^{1/3} b_{L,n} \longrightarrow \mathrm{Ai}(x), \qquad n = \chi L + cL^{1/3}x + O(1).$$

Consequently, after conjugation by $(-1)^i$ and the usual Riemann-sum identification of $Q_{N_L(s)}H$ with $L^2(s,\infty)$, one expects $Q_{N_L(s)}K_L Q_{N_L(s)}$ to converge to the Airy kernel K_{Ai} on $L^2(s,\infty)$.

A.2

Consider now the tilted situation, with the tilts depending on L. Assume that the hypotheses of Proposition 4.6 hold for $N=N_L(s)$. Let $\mathsf{K}_L^{\xi,\theta}$ be the fixed-tail kernel on $Q_N H$ from Proposition 4.6. By (4.11), it is equal to the BOGC kernel $Q_N K_L Q_N$ plus a finite-rank correction.

After the same sign conjugation and Riemann-sum identification as above, suppose that

$$Q_N K_L Q_N \longrightarrow K_{\mathrm{Ai}}, \qquad \mathsf{K}_L^{\xi,\theta} - Q_N K_L Q_N \longrightarrow F_\infty$$

in trace norm on $L^2(s,\infty)$, where F_∞ is finite rank and Γ_L is the corresponding Gram matrix. Then continuity of the Fredholm determinant gives

$$\frac{D_N^{\xi,\theta}(\varphi_L)}{Z_L \det_{\mathbb{C}^N}(\Gamma_L)} \longrightarrow \det_{L^2(s,\infty)}\big(I - K_{\mathrm{Ai}} - F_\infty\big).$$

The right-hand side is the Fredholm determinant (on (s,∞)) of the Airy kernel modified by a finite-rank perturbation, which is a typical form for soft edge limits of spiked models [BBP05], [BP08].

A.3

We now specialize to a single column tilt.[1] Fix $w > 0$ and set

$$\alpha_L = -\exp\{-w/(cL^{1/3})\}, \qquad |\alpha_L| < 1.$$

Take

$$\xi_j(z) = 1 \quad (1 \le j < N), \qquad \xi_N(z) = (1 - \alpha_L z)^{-1}, \qquad \theta_i \equiv 1.$$

Set $C_L = T(\varphi_{L,-})\Xi_L P_N$. In this column-only case $R = P_N T(\varphi_{L,+})$, so $RQ_N = 0$ and $\mathrm{Ker}(R) = Q_N H$. The triangular factor $P_N T(\varphi_{L,+}) P_N$ cancels from the Gram matrix, and the fixed-tail kernel becomes

$$\mathsf{K}_L^{\xi,\mathbf{1}} = Q_N K_L Q_N - Q_N C_L (P_N C_L)^{-1} P_N K_L Q_N,$$

provided $P_N C_L$ is invertible. We now compute this correction explicitly.

Write

$$\varphi_{L,-}(z) = \sum_{m\ge 0} h_m z^{-m}, \qquad h_0 = 1.$$

For $0 \le j \le N-2$,

$$C_L e_j = T(\varphi_{L,-}) e_j = \sum_{m=0}^{j} h_m e_{j-m},$$

so these columns have no Q_N-tail. For the last column,

$$\Xi_L e_{N-1} = \sum_{n\ge 0} \alpha_L^n e_{N-1+n},$$

and hence

$$(C_L e_{N-1})_i = \sum_{\substack{m\ge 0 \\ i+m\ge N-1}} h_m \alpha_L^{i+m-N+1}. \tag{A.2}$$

[1] The case of an arbitrary finite number k of columns can be handled similarly, which would lead to the general k-spiked Baik–Ben Arous–Péché (BBP) distribution defined in [BBP05].

For $i = N + r$, $r \geq 0$, the lower constraint on m is automatic, so

$$Q_N C_L e_{N-1} = \left(\sum_{m \geq 0} h_m \alpha_L^m \right) \sum_{r \geq 0} \alpha_L^{r+1} e_{N+r}.$$

At the boundary row $i = N - 1$,

$$(C_L e_{N-1})_{N-1} = \sum_{m \geq 0} h_m \alpha_L^m.$$

The first $N - 1$ columns have no e_{N-1} component. Therefore $P_N C_L$ is block upper triangular with a unit upper-triangular upper-left block and lower-right entry $\sum_{m \geq 0} h_m \alpha_L^m$. In particular, $P_N C_L$ is invertible, and

$$Q_N C_L (P_N C_L)^{-1} = \left(\sum_{r \geq 0} \alpha_L^{r+1} e_{N+r} \right) e_{N-1}^*.$$

Consequently,

$$\mathsf{K}_L^{\xi, \mathbf{1}} = Q_N K_L Q_N - \left(\sum_{r \geq 0} \alpha_L^{r+1} e_{N+r} \right) \otimes (e_{N-1}^* K_L Q_N). \tag{A.3}$$

Equivalently, for $i, j \geq N$,

$$\mathsf{K}_L^{\xi, \mathbf{1}}(i, j) = K_L(i, j) - \alpha_L^{i-N+1} K_L(N - 1, j). \tag{A.4}$$

Thus the one-column tilt gives an exact rank-one correction of the ordinary BOGC tail kernel.

A.4

Put $N = N_L(s)$ and write

$$x = \frac{i - \chi L}{c L^{1/3}}, \qquad y = \frac{j - \chi L}{c L^{1/3}}.$$

For $i = N + r$,

$$\alpha_L^{i-N+1} = (-1)^{r+1} \exp\{-w(r + 1)/(c L^{1/3})\}.$$

After the same sign conjugation used for the ordinary Airy limit, this vector factor tends to $\exp\{-w(x - s)\}$.

For the present symmetric symbol, $\widetilde{c}_L = b_L$, and therefore

$$K_L(i, j) = \sum_{\ell \geq 0} b_{L, i+\ell+1} b_{L, j+\ell+1}, \qquad K_L(N - 1, j) = \sum_{\ell \geq 0} b_{L, N+\ell} b_{L, j+\ell+1}.$$

Inserting the coefficient Airy asymptotics gives the Riemann-sum limit

$$c L^{1/3} (-1)^{N-1+j} K_L(N - 1, j) \longrightarrow \int_0^\infty \mathrm{Ai}(s + t)\, \mathrm{Ai}(y + t)\, dt = K_{\mathrm{Ai}}(s, y).$$

Hence the rank-one term in (A.4) has candidate Airy-scale limit

$$cL^{1/3}(-1)^{i+j}\alpha_L^{i-N+1}K_L(N-1,j) \longrightarrow e^{-w(x-s)}K_{\mathrm{Ai}}(s,y).$$

The limiting fixed-tail kernel is therefore expected to be the boundary form

$$K_{\mathrm{Ai}}(x,y) - E_w(x)K_{\mathrm{Ai}}(s,y), \qquad E_w(x) = e^{-w(x-s)}, \qquad x,y > s. \tag{A.5}$$

The boundary form is not the standard one-spike contour kernel at the operator level. The next Proposition A.1 shows that, nevertheless, its Fredholm determinant on (s,∞) coincides with the one-spike BBP distribution.

First, we need some notation. Fix $s \in \mathbb{R}$ and $w > 0$. Set

$$\Phi_w(x) = \int_0^\infty e^{-wt}\,\mathrm{Ai}(x+t)\,dt = \frac{1}{2\pi i}\int_\Gamma \frac{\exp\{Z^3/3 - xZ\}}{Z-a}\,dZ, \qquad a = -w, \tag{A.6}$$

where Γ is an Airy contour passing to the right of a. Let Σ be the dual Airy contour chosen so that

$$\mathrm{Ai}(y) = \frac{1}{2\pi i}\int_\Sigma \exp\{-W^3/3 + yW\}\,dW$$

and

$$K_{\mathrm{Ai}}(x,y) = \frac{1}{(2\pi i)^2}\int_\Gamma dZ \int_\Sigma dW\, \frac{\exp\{Z^3/3 - xZ\}}{\exp\{W^3/3 - yW\}}\,\frac{1}{Z-W}.$$

Finally, let

$$\mathcal{A}\colon L^2(0,\infty) \to L^2(s,\infty), \qquad (\mathcal{A}f)(x) = \int_0^\infty \mathrm{Ai}(x+t)f(t)\,dt.$$

Thus $K_{\mathrm{Ai}} = \mathcal{A}\mathcal{A}^*$ and $K_{\mathrm{Ai}}(s,\cdot) = \mathcal{A}(\mathrm{Ai}(s+\cdot))$. We write $f \otimes g$ for the rank-one kernel $f(x)g(y)$.

Proposition A.1. *With the notation above,*

$$\det\nolimits_{L^2(s,\infty)}\big(I - K_{\mathrm{Ai}} + E_w \otimes K_{\mathrm{Ai}}(s,\cdot)\big) = \det\nolimits_{L^2(s,\infty)}\big(I - K_{\mathrm{Ai}} + \Phi_w \otimes \mathrm{Ai}\big), \tag{A.7}$$

and

$$\begin{aligned} &K_{\mathrm{Ai}}(x,y) - \Phi_w(x)\,\mathrm{Ai}(y) \\ &\qquad = \frac{1}{(2\pi i)^2}\int_\Gamma dZ \int_\Sigma dW\, \frac{\exp\{Z^3/3 - xZ\}}{\exp\{W^3/3 - yW\}}\,\frac{1}{Z-W}\,\frac{W-a}{Z-a}, \end{aligned} \tag{A.8}$$

which is the standard one-spike BBP kernel from [BBP05].

Proof. The factorization identities above give

$$K_{\mathrm{Ai}} - E_w \otimes K_{\mathrm{Ai}}(s,\cdot) = \big(\mathcal{A} - E_w \otimes \mathrm{Ai}(s+\cdot)\big)\mathcal{A}^*.$$

By Sylvester's identity $\det(I - BC) = \det(I - CB)$,

$$\begin{aligned} &\det\nolimits_{L^2(s,\infty)}\big(I - K_{\mathrm{Ai}} + E_w \otimes K_{\mathrm{Ai}}(s,\cdot)\big) \\ &\qquad = \det\nolimits_{L^2(0,\infty)}\Big(I - \mathcal{A}^*\mathcal{A} + \mathcal{A}^*E_w \otimes \mathrm{Ai}(s+\cdot)\Big). \end{aligned}$$

Moreover,

$$\mathcal{A}^* E_w(t) = \int_s^\infty \mathrm{Ai}(x+t) e^{-w(x-s)}\, dx = \int_0^\infty e^{-wr} \mathrm{Ai}(s+t+r)\, dr = \Phi_w(s+t).$$

After the translation $L^2(0,\infty) \simeq L^2(s,\infty)$, this proves (A.7).

For (A.8), use the elementary identity

$$\frac{1}{Z-W} \frac{W-a}{Z-a} = \frac{1}{Z-W} - \frac{1}{Z-a}.$$

Substituting it into the double contour integral gives the Airy kernel minus $\Phi_w(x)\,\mathrm{Ai}(y)$, using (A.6) and the dual Airy representation above. □

Kernels similar to our initial expression (A.5) appear in asymptotic analysis of the polynuclear growth model with external sources [BR00], [IS04].

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