DOMINO TILINGS OF THE AZTEC DIAMOND IN RANDOM ENVIRONMENT AND SCHUR GENERATING FUNCTIONS

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Abstract. We study the asymptotic behavior of random domino tilings of the Aztec diamond of size M in a random environment, where the environment is a one-periodic sequence of i.i.d. random weights attached to domino positions (i.e., to the edges of the underlying portion of the square grid). We consider two cases: either the variance of the weights decreases at a critical scale $1/M$, or the distribution of the weights is fixed. In the former case, the unrescaled fluctuations of the domino height function are governed by the sum of a Gaussian Free Field and an independent Brownian motion. In the latter case, we establish fluctuations on the much larger scale $\sqrt{M}$, given by the Brownian motion alone.

To access asymptotic fluctuations in random environment, we employ the method of Schur generating functions. Moreover, we substantially extend the known Law of Large Numbers and Central Limit Theorems for particle systems via Schur generating functions in order to apply them to our setting. These results might be of independent interest.

1. Introduction

1.1. **Overview.** Domino tilings of Aztec diamond were introduced and enumerated by [EKLP92]. Since then, random domino tilings of Aztec diamond were found to be a very rich and exciting research topic. For a uniformly random choice of a tiling, the existence of the arctic curve (see Figure 1 for an illustration) was first established in [JPS98]. The local behavior of the tiling in the neighborhood of the arctic curve was studied in [Joh05] — it turned out to be related to the KPZ universality class. The fluctuations of the height function of the tiling were established in [CJY15], [BG18] and turn out to be related to the Gaussian Free Field, in accordance with the Kenyon-Okounkov conjecture, see [KOS06], [KO07].

Much attention has also been devoted to non-uniform random tilings. A standard choice, inspired by statistical mechanics, is to assign a positive weight to each potential domino position in the Aztec diamond; and to select a tiling with probability proportional to the product of the weights of all dominos it contains.

A completely arbitrary choice of such weights seems to lead to random domino tilings which currently available tools are not strong enough to analyze in detail. However, several specific choices of weights were thoroughly analyzed in the last twenty years, leading, in particular, to beautiful connections with other fields of mathematics. The simplest out of these choices is a one-periodic weighting. It is well understood that such a choice of weights leads to a Schur measure on Young diagrams, the object that was introduced in and studied in [Oko01], [OR03].

Even richer phenomena emerge in random domino tilings of the Aztec diamond when two-periodic or, more generally, multi-periodic weights are considered; see, for instance, [CJY15], [CY14], [CJ16], [DK20], [Ber21], [BB23], [KP24], [BB24a], [BdT24], [BB24b], [Mas22] for an (incomplete) list of recent results in this direction.

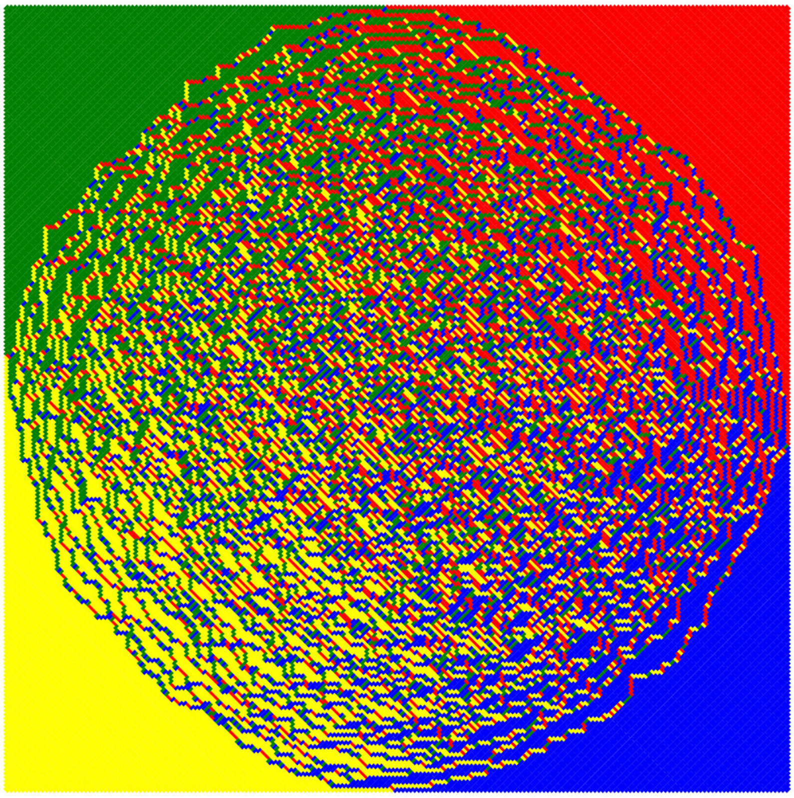

FIGURE 1. A uniformly random domino tiling of the Aztec diamond of size $M = 200$.

In the present work we move in a different direction by allowing the weights themselves to be random. We focus on what is arguably the simplest nontrivial setting: a one-periodic model in which the parameters $W_1, W_2, \ldots, W_M$ are i.i.d.[1] with a common distribution $\boldsymbol{\mathcal{B}}$ (see Section 1.2 below for a precise definition, and Figure 2, left, for an illustration of the weighting). As far as we are aware, there is essentially no mathematical literature on dimer models with random edge weights. Related studies in mathematical physics, such as [PRLD+12], treat the Aztec diamond with all weights chosen i.i.d., but the specific model examined here seems to be new. As we were preparing this manuscript, we learned of an ongoing work [DVP25] on a different model involving domino tilings of the Aztec diamond with random edge weights.

We distinguish two regimes for domino tilings of the Aztec diamond of size $M \to \infty$ with random weights:

- *Critical vanishing variance.* The variance of $\boldsymbol{\mathcal{B}} = \boldsymbol{\mathcal{B}}_M$ decays like M^{-1}, so that the fluctuations produced by the randomness of the weights and those present in the uniform model occur on the same scale. These critically scaled random weights do not change the limit shape of the height function of the domino tiling (Theorem 4.2). We establish the Central Limit Theorem (Theorems 4.4 and 4.5). See Theorem 1.1 for an informal version of the statement.
- *Fixed variance.* The distribution $\boldsymbol{\mathcal{B}}$ is independent of M. Here, we prove both a Law of Large Numbers and a Central Limit Theorem for the height function; see

[1] Independent identically distributed.

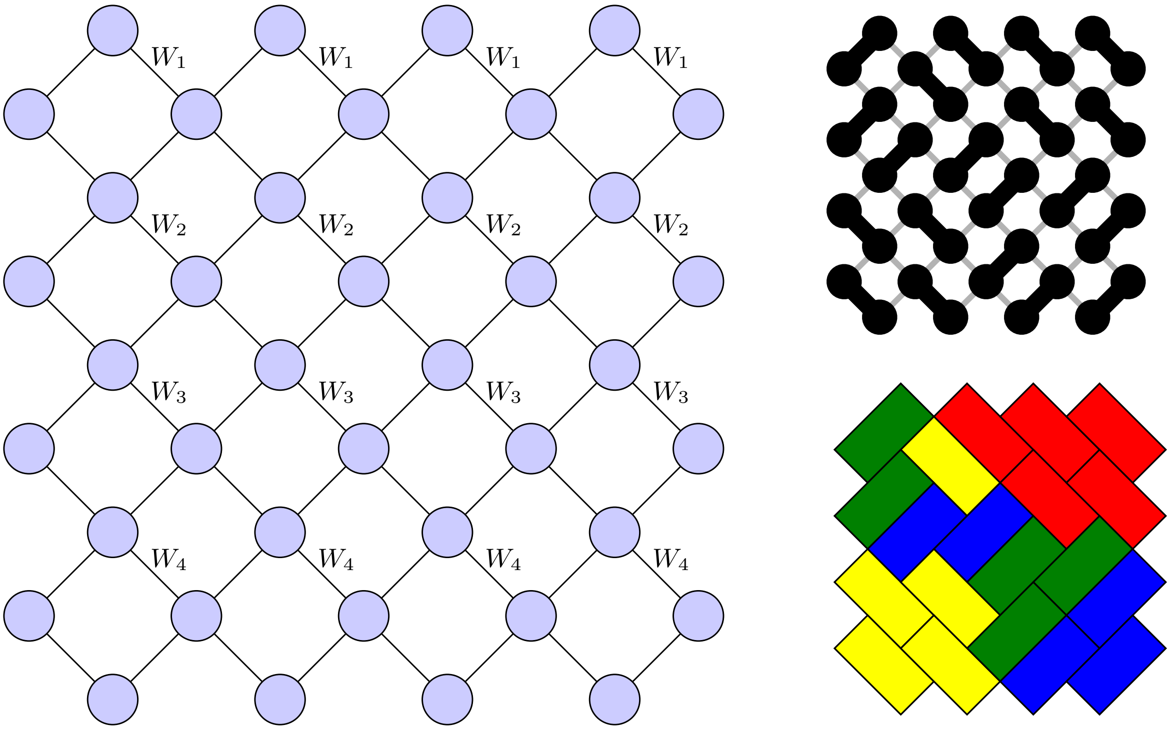

FIGURE 2. Left: One-periodic weights W_i attached to the edges of the Aztec diamond graph of size $M = 4$. The edges which are not labeled have weight 1. We assume that W_i's form an i.i.d. random sequence with common distribution $\boldsymbol{\mathcal{B}}$. Right: An example of a dimer configuration on the Aztec diamond graph, and the corresponding domino tiling. Given the weights W_i, this configuration has conditional probability proportional to $W_1^3 W_2^2$. In terms of domino colors, we assign nontrivial Boltzmann weights to red dominos.

Proposition 5.12 and Theorem 7.1, as well as Theorem 1.2 for an informal version of the statements.

Both families of results are universal in the sense that they do not depend on the specific distribution $\boldsymbol{\mathcal{B}}$ of the random weights (provided that the distribution satisfies suitable mild conditions).

To establish these asymptotic results, we employ the method of Schur generating functions developed in [BG15], [BG18], [BG19]; One of important applications of the method of Schur generating functions is the analysis of global fluctuations of uniformly random domino tilings of the Aztec diamond (performed in [BG18]). Further applications and extensions of the method of Schur generating functions include, for example, [BK18] (domino tilings of more general domains), [BGS20], [GS22] (random matrix models with unitary symmetry), and [Hua21], [CD25] (extension to Jack generating functions, with applications to general Beta random matrix models).

The Schur generating function techniques of [BG15], [BG18] also cover deterministic one-periodic weights. Moreover, these techniques also cover the case of decreasing variance of random weights, as we show in Section 4. The reason for this is that the fluctuations are of the same order as in the case of deterministic weights, even though the distribution of the fluctuations changes. On the other hand, the presence of random weights with a fixed distribution $\boldsymbol{\mathcal{B}}$ (the second case above) involves markedly different scalings and lies beyond the scope of the known works, so we must significantly extend their general results. The Law of Large Numbers (Theorem 5.6) and Central Limit Theorem (Theorem 6.2) for Schur generating functions obtained here constitute the principal contributions of the present paper. Although we apply them only for the Aztec diamond, we expect these theorems to be useful in other settings.

Let us remark that, although the determinantal process approach (either via asymptotics of the kernel, or through connections to orthogonal polynomials) is one of the standard methods for studying dimer models, a dimer model with generic random weights produces a point process that is likely not determinantal.

1.2. **Domino tilings with random edge weights. Model and results.** The Aztec diamond graph is a classical bipartite graph embedded in the square lattice; see the left panel of Figure 2. Let M denote its size (order), that is, the number of lattice vertices along each side. Dimer coverings (which are the same as perfect matchings) of this graph are essentially the same as tilings of the Aztec diamond by 1×2 (or 2×1) dominos. We refer to Figure 2 for the correspondence. We equip the edges of the Aztec diamond graph with one-periodic weights $(W_1, \ldots, W_M)$ shown in Figure 2. Here, "one-periodic" means that the weights change in one direction, but are repeated in the other. The weights $W_1, \ldots, W_M$ are assumed to be independent and identically distributed according to a probability law $\boldsymbol{\mathcal{B}}_M$ on $\mathbb{R}_{>0}$ (possibly depending on M). Our aim is to study the asymptotic behavior, as $M \to \infty$, of a random domino tiling on the graph with these weights. In detail, we consider a probability measure on the set of all dimer coverings (perfect matchings) D of the Aztec diamond graph defined as

$$\mathbf{P}(D) \coloneqq \frac{1}{Z} \prod_{e \in D} \text{weight}(e),$$

where $\text{weight}(e)$ is either equal to 1, or is one of the random weights W_i.

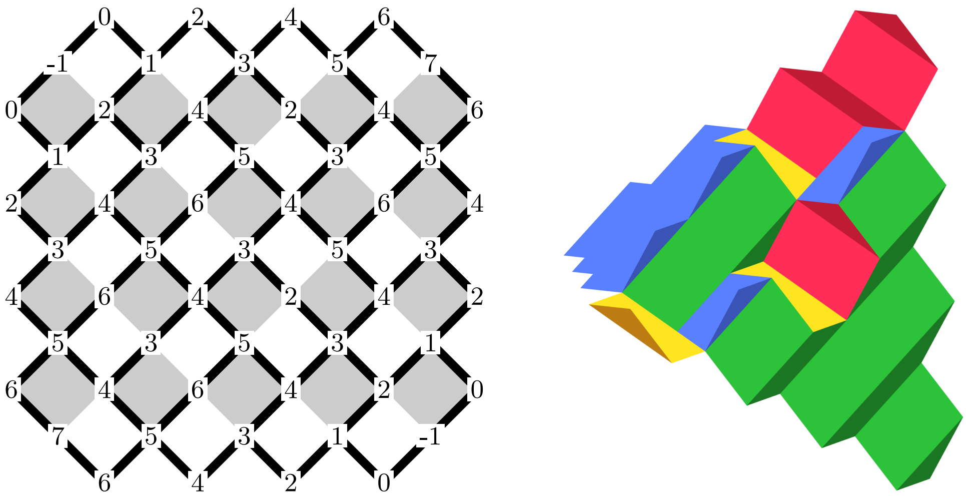

FIGURE 3. A domino tiling of the Aztec diamond of size $M = 4$ together with the corresponding height function and its 3D plot. Interactive 3D plots of domino height functions is available online at [Pet25a]. The height function changes by ± 1 along the domino edges, and by ± 3 across the dominos. The sign of the change depends on the parity with respect to the checkerboard coloring of the cells of $\mathbb{Z}^2$. The height function (up to a global shift) determines the domino tiling uniquely.

Each domino tiling of the Aztec diamond determines a height function [Thu90]. See Figure 3 for an illustration. We describe the asymptotic behavior of our random domino tilings through this height function:

Theorem 1.1 (Informal version of Theorems 4.2, 4.4 and 4.5)**.** *Assume that the variance of the distribution $\boldsymbol{\mathcal{B}}_M$ decays at rate $1/M$, and that the corresponding random weights $W_i = W_i(M)$ converge in probability to a deterministic constant $w > 0$. Then the height function has the same limit shape as in the case of fixed deterministic weights $W_i \equiv w$.*[2] *Furthermore, the fluctuations of the height function in suitable coordinates are described by the sum of two independent random objects: (a deterministic image of) the Gaussian Free Field, and a one-dimensional Brownian motion.*

Kenyon-Okounkov conjecture [Ken01], [Ken08], [KO07] generally predicts Gaussian Free Field fluctuations of random tilings in a variety of models, see [Gor21, Ch. 11] for an exposition. This behavior was established in several tiling (dimer) models, see, e.g., [Ken01], [Ken08], [BF14], [Pet15], [BLR20], [BN25]. For domino tilings of the Aztec diamond with nonrandom unit edge weights the Gaussian Free Field fluctuations were established in [CJY15], [BG18].

We show that introducing random edge weights alters the fluctuations of the height function even when the variance of the weight distribution tends to zero. Under the scaling of the variance by $1/M$, as in Theorem 1.1, the new fluctuations produced by random weights and the "old" fluctuations arising from fixed weights (which give the Gaussian Free Field) occur on the same scale, and contribute independently to the total fluctuations of the height function.

Theorem 1.2 (Informal version of Proposition 5.12 and Theorem 7.1)**.** *Assume that the distribution $\boldsymbol{\mathcal{B}}_M$ is independent of M and possesses moments of all orders. Then the height function satisfies a Law of Large Numbers. Moreover, its fluctuations occur on scale $\sqrt{M}$ and, in suitable coordinates, are described by a one-dimensional Brownian motion.*

Note that in this setting (typical for processes in random environment, when the edge weights are sampled once from a fixed distribution independent of M), the height function behaves very differently from the fixed-weight case. In general, its limit shape does not coincide with that of any fixed-weight domino tiling of the Aztec diamond (see Figures 4 and 5 for illustrations; the first figure shows a special case in which Bernoulli random edge weights yield the same limit shape as a tiling with deterministic periodic weights). Moreover, the fluctuations of the height function are now much larger — of order $\sqrt{M}$, in contrast to the unnormalized fluctuations in Theorem 1.1 — and are governed by the Brownian motion. Intuitively this may be explained by the fact that by the classical Central Limit Theorem, linear statistics of M i.i.d. random variables fluctuate on the scale $\sqrt{M}$. These classical Gaussian fluctuations dominate the much smaller fluctuations produced by the randomness of the domino tiling with fixed weights.

1.3. Asymptotics via Schur generating functions. General results.

Our key technical tool for the asymptotic analysis of domino tilings of the Aztec diamond with random edge weights is the method of Schur generating functions. We refer to

[2] In particular, the arctic curve in this case is an ellipse [JPS98] inscribed into the square bounding the Aztec diamond.

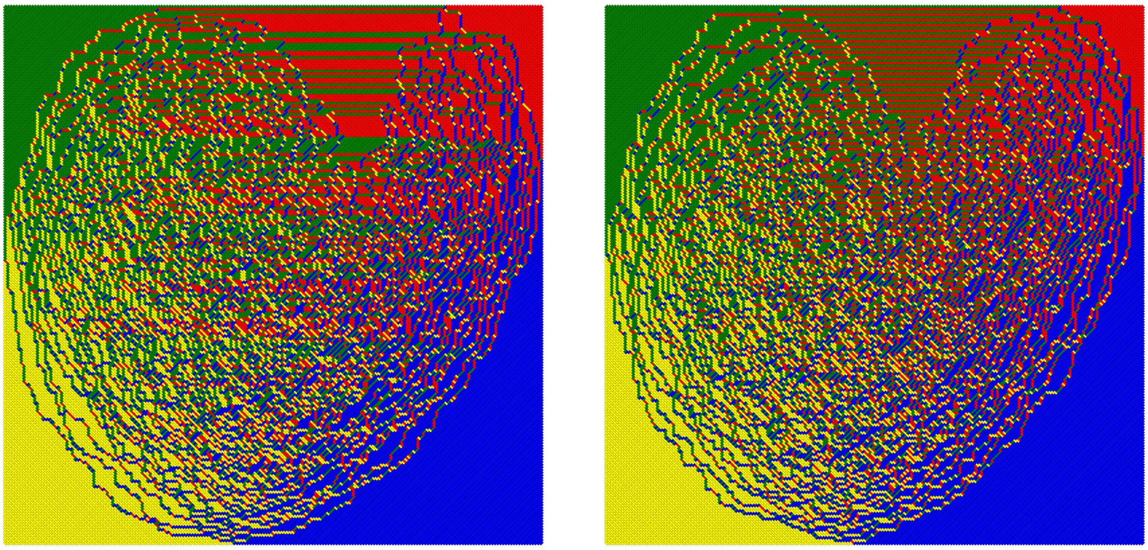

FIGURE 4. Left: A random domino tiling of the Aztec diamond of size $M = 200$ with i.i.d. Bernoulli weights with $\mathbf{P}(W_i = \frac{1}{2}) = \mathbf{P}(W_i = 5) = \frac{1}{2}$. Right: A random domino tiling of the Aztec diamond of size $M = 200$ with fixed nonrandom weights $W_i = \frac{1}{2}$ and $W_i = 5$ for odd and even i, respectively. The limit shapes for these two choices of weights — random and deterministic periodic with the same values of W_i in the same proportions — are the same. See also Figure 7 for limit shapes and arctic curves which we derive in this work. These samples were generated with the domino shuffling algorithm of [Pro03]; the implementation we used is available online at [Pet25b].

Proposition 3.2 for a precise statement connecting the domino tiling model to Schur generating functions (and also to Remark 3.3 for comparison with other approaches).

In this subsection, we present the most technical part of our results, which concern asymptotic behavior of particle systems via Schur generating functions. They extend the results of [BG15], [BG18], and [BG19], which were not sufficient for our model with random edge weights.

An N-tuple of non-increasing integers $\lambda = (\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_N)$ is called a *signature* of length N. We denote by $\mathbb{GT}_N$ the set of all signatures of length N. Let $s_\lambda(x_1, \ldots, x_N)$ be the *Schur function* parameterized by λ More precisely, $s_\lambda(x_1, \ldots, x_N)$ is a homogeneous symmetric Laurent polynomial in the x_i's of degree $|\lambda| = \lambda_1 + \ldots + \lambda_N$ (see (2.1) for the full definition). Let ρ_N be a probability measure on the set $\mathbb{GT}_N$. A *Schur generating function* $S_{\rho_N}(x_1, \ldots, x_N)$ is a symmetric Laurent power series in $x_1, \ldots, x_N$ given by

$$S_{\rho_N}(x_1, \ldots, x_N) \coloneqq \sum_{\lambda \in \mathbb{GT}_N} \rho_N(\lambda) \frac{s_\lambda(x_1, \ldots, x_N)}{s_\lambda(1^N)}.$$

Define the empirical measure

$$m[\rho_N] \coloneqq \frac{1}{N} \sum_{i=1}^{N} \delta\left(\frac{\lambda_i + N - i}{N}\right), \qquad \text{where } \lambda = (\lambda_1, \ldots, \lambda_N) \text{ is } \rho_N\text{-distributed}.$$

If $F_{m[\rho_N]}(x)$ denotes the cumulative distribution function of $m[\rho_N]$, then its rescaling of the form $N F_{m[\rho_N]}(x/N)$ corresponds to the (integer-valued) height function of

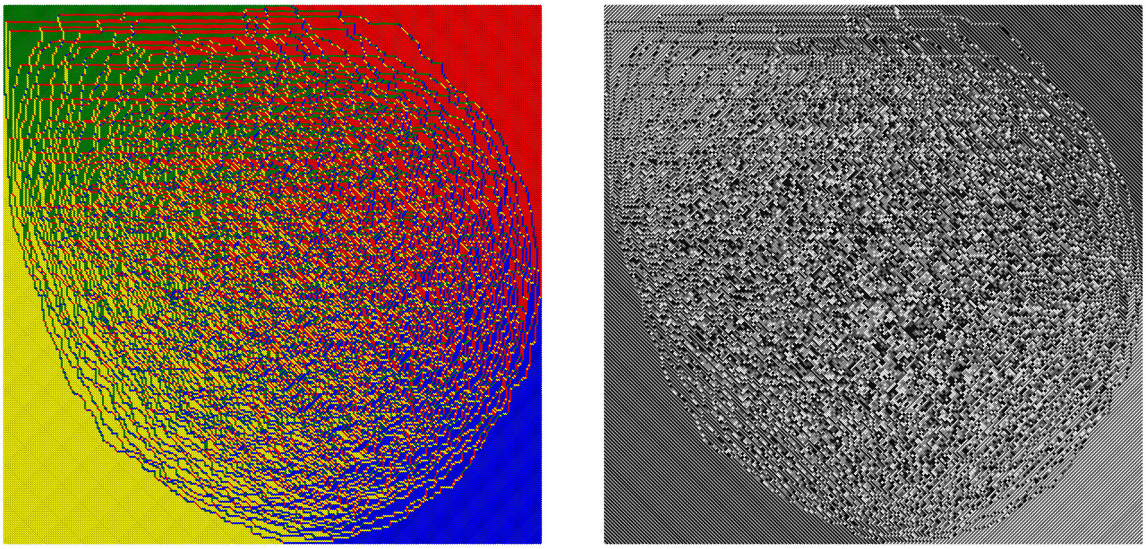

FIGURE 5. A random domino tiling of the Aztec diamond of size $M = 300$ with i.i.d. continuous uniform random weights W_i on $[0,2]$. The right image is a grayscale version with eight shades depending on the parity of the dominos with respect to the lattice. See also Figure 7 for limit shapes and arctic curves which we derive in this work. These samples were generated with the domino shuffling algorithm of [Pro03]; the implementation we used is available online at [Pet25b].

the domino tiling of the Aztec diamond (illustrated in Figure 3) along a given one-dimensional slice. We refer to Section 2.2 for details on the correspondence between particle configurations and domino tilings.

Asymptotic behavior of the empirical measure $m[\rho_N]$ can be understood by looking at the asymptotics of its Schur generating function, as $N \to \infty$. Our first general result is as follows:

Theorem 1.3 (Law of Large Numbers)**.** *Let ρ_N, $N \in \mathbb{Z}_{\geq 1}$, be a sequence of probability measures on the sets $\mathbb{GT}_N$. Assume that there exists a sequence of symmetric functions $F_k(z_1, \dots, z_k)$, $k \geq 1$, analytic in a (complex) neighborhood of $1^k := (1, \dots, 1) \in \mathbb{C}^k$,[3] such that*

$$\lim_{N \to \infty} \sqrt[N]{S_{\rho_N}(u_1, \dots, u_k, 1^{N-k})} = F_k(u_1, \dots, u_k), \tag{1.1}$$

where k is fixed, and the the convergence is uniform in a complex neighborhood of 1^k. Then, as $N \to \infty$, the random measure $m[\rho_N]$ converges in probability, in the sense of moments,[4] to a deterministic probability measure $\boldsymbol{\mu}$ on $\mathbb{R}$. Moreover, the moments

[3]Throughout the paper, we use the notation 1^m to denote the vector $(1, \dots, 1)$ of m ones, where $m \geq 0$ is an arbitrary integer, and similarly for 0^m.

[4]That is, all moments of the empirical measure $m[\rho_N]$, which are random variables, converge in probability to the moments of the limiting measure.

$(\boldsymbol{\mu}_k)_{k\ge1}$ *of the limiting measure* $\boldsymbol{\mu}$ *are given by*

$$\boldsymbol{\mu}_k = \sum_{l=0}^{k} \binom{k}{l} \frac{1}{(l+1)!} \times \partial_u^l \left[(1+u)^k \left(\partial_1 F_{l+1}(1+u, 1+uw_{l+1}, 1+uw_{l+1}^2, \dots, 1+uw_{l+1}^l) \right)^{k-l} \right]\Big|_{u=0}, \quad (1.2)$$

where $w_m \coloneqq \exp\left(\frac{2\pi\sqrt{-1}}{m}\right)$ *is the* m*-th root of unity,* ∂_u *is the partial derivative with respect to* u*, and* ∂_1 *is the partial derivative with respect to the first variable in the function* F_{l+1}*.*

Theorem 1.3 provides a substantial generalization of Theorem 5.1 in [BG15]. We recover it in Corollary 5.7 below.

Our second general result is on fluctuations of the empirical measure $m[\rho_N]$ determined by the Schur generating function. For a probability measure ρ_N on $\mathbb{GT}_N$, define the k-th moment as

$$p_k^{(\rho_N)} \coloneqq \sum_{i=1}^{N} (\lambda_i + N - i)^k, \qquad \text{where } \lambda = (\lambda_1, \dots, \lambda_N) \text{ is } \rho_N\text{-distributed.} \quad (1.3)$$

Theorem 1.4 (Central Limit Theorem)**.** *Under the assumptions of Theorem 1.3, the vector*

$$\left(\frac{p_k^{(\rho_N)} - \mathbf{E}[p_k^{(\rho_N)}]}{N^{k+1/2}} \right)_{k\ge1} \quad (1.4)$$

converges to a mean-zero Gaussian vector with covariance given by formula (6.3) *below.*

Remark 1.5. The principal distinction between our setting and those in [BG18], [BG19] lies in the fluctuation scale. Namely, these earlier works considered unnormalized fluctuations of the (integer-valued) height function $N\left(F_{m[\rho_N]}(y) - \mathbf{E}[F_{m[\rho_N]}(y)]\right)$, where y belongs to a compact interval. Then we have

$$\frac{p_k^{(\rho_N)} - \mathbf{E}[p_k^{(\rho_N)}]}{N^k} = \int_{-\infty}^{\infty} y^k \, d\left[N\left(F_{m[\rho_N]}(y) - \mathbf{E}[F_{m[\rho_N]}(y)]\right)\right], \quad (1.5)$$

and in [BG18], these moments are shown to have jointly Gaussian asymptotics (we recall one of these results in Theorem 2.7 below).

In contrast, for random domino tilings of the Aztec diamond with random edge weights with a fixed distribution $\boldsymbol{\mathcal{B}}$, the denominators in (1.5) must be replaced by $N^{k+1/2}$. This means that the fluctuations of the height function are now growing on the scale $\sqrt{N}$. This new scaling alters the behavior of the Schur generating functions, a feature captured by Theorem 1.4.

Our proofs of Theorems 1.3 and 1.4 largely follow the approach of [BG15], [BG18], [BG19]; for this reason we keep the proofs of our theorems relatively brief, focusing on the modifications necessitated by the new setting. The moment computation for Theorem 1.3 required several new ideas, most notably the use of the complex domain. The argument for Theorem 1.4 is closer to that in [BG18] and, in some respects, even simpler. Nevertheless, the new fluctuation scale and the significantly broader assumptions introduced additional challenges that we had to overcome.

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2. Preliminaries

2.1. **Signatures and Schur functions.** An N-tuple of non-increasing integers $\lambda = (\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_N)$ is called a *signature* of length N. We denote by $\mathbb{GT}_N$ the set of all signatures of length N. Let also $|\lambda| \coloneqq \sum_{i=1}^N \lambda_i$. Signatures $\lambda \in \mathbb{GT}_N$ and $\mu \in \mathbb{GT}_{N-1}$ *interlace* (notation $\mu \prec \lambda$), if $\lambda_i \ge \mu_i \ge \lambda_{i+1}$, for all $i = 1, \ldots, N-1$. Signatures $\nu \in \mathbb{GT}_N$ and $\lambda \in \mathbb{GT}_N$ *interlace vertically* (notation $\lambda \prec_v \nu$), if $\nu_i - \lambda_i \in \{0,1\}$ for all $i = 1, \ldots, N$.

The *Schur function* s_λ, $\lambda \in \mathbb{GT}_N$, is a symmetric Laurent polynomial in $x_1, \ldots, x_N$ of degree $|\lambda|$ defined by

$$s_\lambda(x_1, \ldots, x_N) \coloneqq \frac{\det\left[x_i^{\lambda_j+N-j}\right]}{\det\left[x_i^{N-j}\right]} = \frac{\det\left[x_i^{\lambda_j+N-j}\right]}{\prod_{i<j}(x_i - x_j)}. \tag{2.1}$$

Schur functions s_λ form a linear basis in the space of symmetric Laurent polynomials in $x_1, \ldots, x_N$, where λ runs over $\mathbb{GT}_N$.

2.2. **Domino tilings and sequences of signatures.** For each $t = 1, 2, \ldots, M$, let $\lambda^{(t)}$, $\vartheta^{(t)}$ be signatures of length t.

Definition 2.1. Let the coefficients $\kappa_\beta(\lambda^{(t)} \to \vartheta^{(t)})$ be defined via the following expansion into the linear basis of Schur functions:

$$\frac{s_{\lambda^{(t)}}(x_1, \ldots, x_t)}{s_{\lambda^{(t)}}(1^t)} \prod_{i=1}^t (1 - \beta + \beta x_i) = \sum_{\vartheta^{(t)} \in \mathbb{GT}_t} \kappa_\beta(\lambda^{(t)} \to \vartheta^{(t)}) \frac{s_{\vartheta^{(t)}}(x_1, \ldots, x_t)}{s_{\vartheta^{(t)}}(1^t)}. \tag{2.2}$$

It follows from the Pieri rule (see, e.g., [BK18, Lemma 2.12]) that these coefficients have the explicit form

$$\kappa_\beta(\lambda^{(t)} \to \vartheta^{(t)}) = \begin{cases} \beta^{|\vartheta^{(t)}|-|\lambda^{(t)}|}(1-\beta)^{t-(|\vartheta^{(t)}|-|\lambda^{(t)}|)} \frac{s_{\vartheta^{(t)}}(1^t)}{s_{\lambda^{(t)}}(1^t)}, & \lambda^{(t)} \prec_v \vartheta^{(t)}; \\ 0, & \textit{otherwise}, \end{cases} \tag{2.3}$$

By setting $x_i \equiv 1$ in (2.2), we see that $\kappa_\beta(\lambda^{(t)} \to \vartheta^{(t)})$ sum to one over all $\vartheta^{(t)} \in \mathbb{GT}_t$.

Definition 2.2. Let the coefficients $\mathrm{pr}\left(\vartheta^{(t)} \to \lambda^{(t-1)}\right)$ be defined through the following expansion into the linear basis of Schur functions:

$$\frac{s_{\vartheta^{(t)}}(x_1, \ldots, x_{t-1}, 1)}{s_{\vartheta^{(t)}}(1^t)} = \sum_{\lambda^{(t-1)} \in \mathbb{GT}_{t-1}} \mathrm{pr}\left(\vartheta^{(t)} \to \lambda^{(t-1)}\right) \frac{s_{\lambda^{(t-1)}}(x_1, \ldots, x_{t-1})}{s_{\lambda^{(t-1)}}(1^{t-1})}. \tag{2.4}$$

It follows from the branching rule for Schur functions that these coefficients have the explicit form

$$\mathrm{pr}\left(\vartheta^{(t)} \to \lambda^{(t-1)}\right) = \begin{cases} \dfrac{s_{\lambda^{(t-1)}}(1^{t-1})}{s_{\vartheta^{(t)}}(1^t)}, & \lambda^{(t-1)} \prec \vartheta^{(t)}, \\ 0, & \text{otherwise}. \end{cases} \tag{2.5}$$

By setting $x_i \equiv 1$ in (2.4), we see that the coefficients $\mathrm{pr}\left(\vartheta^{(t)} \to \lambda^{(t-1)}\right)$ sum to one over all $\lambda^{(t-1)} \in \mathbb{GT}_{t-1}$.

Definition 2.3. Given a sequence of parameters $(\beta_1, \ldots, \beta_M)$, $0 < \beta_i < 1$, define the probability measure $\mathbf{P}_{\beta_1,\ldots,\beta_M}$ on the sequence of signatures of the form

$$(\lambda^{(M)}, \vartheta^{(M)}, \lambda^{(M-1)}, \vartheta^{(M-1)}, \ldots, \lambda^{(2)}, \vartheta^{(2)}, \lambda^{(1)}, \vartheta^{(1)}) \tag{2.6}$$

(we also set $\lambda^{(0)} = \varnothing$) by the formula

$$\mathbf{P}_{\beta_1,\ldots,\beta_M}\left(\lambda^{(M)}, \vartheta^{(M)}, \lambda^{(M-1)}, \vartheta^{(M-1)}, \ldots, \lambda^{(2)}, \vartheta^{(2)}, \lambda^{(1)}, \vartheta^{(1)}\right)$$
$$\coloneqq 1_{\lambda^{(M)}=(0^M)} \prod_{i=1}^{M} \kappa_{\beta_i}(\lambda^{(i)} \to \vartheta^{(i)})\, \mathrm{pr}_{i\to(i-1)}(\vartheta^{(i)} \to \lambda^{(i-1)}). \tag{2.7}$$

By induction, these weights sum to one over all sequences of signatures (2.6).

Let $\mathbb{S}_M$ be the set of sequences (2.6) with nonzero probability measure $\mathbf{P}_{\beta_1,\ldots,\beta_M}$. From explicit formulas (2.3) and (2.5) it follows that each configuration from $\mathbb{S}_M$ has probability

$$\mathbf{P}_{\beta_1,\ldots,\beta_M}\left(\lambda^{(M)}, \vartheta^{(M)}, \ldots, \vartheta^{(2)}, \lambda^{(1)}\right) = \prod_{i=1}^{M}(1-\beta_i)^i \prod_{i=1}^{M}\Big(\frac{\beta_i}{1-\beta_i}\Big)^{|\vartheta^{(i)}|-|\lambda^{(i)}|}.$$

The measure from Definition 2.3 corresponds to domino tilings of the Aztec diamond with one-periodic weights:

Proposition 2.4. *There is a bijection between $\mathbb{S}_M$ and the set of domino tilings of the Aztec diamond of size M. Under this bijection, the measure $\mathbf{P}_{\beta_1,\ldots,\beta_M}$ (2.7) turns into the measure on domino tilings with (fixed, deterministic) edge weights W_i as in Figure 2, left, where $W_i = \beta_i/(1-\beta_i)$, $i = 1, \ldots, M$.*

Proof. This correspondence is classical; see, for example, [Joh02], [JN06]. A detailed presentation in our notation is given in [BK18, Section 2], where the same construction is applied to domino tilings of more general domains. Figure 6 illustrates the bijection for the Aztec diamond. □

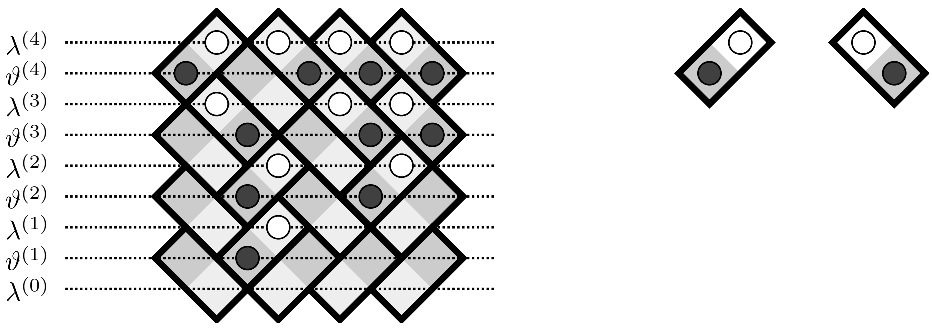

FIGURE 6. Correspondence between domino tilings of the Aztec diamond and sequences of signatures (here, $M = 4$). We single out two types of dominos (red and green in the coloring in Figure 2), and place black and white particles into them. The black (resp., white) particle configurations on each horizontal slice correspond to the signatures $\vartheta^{(i)}$ (resp., $\lambda^{(i)}$). To read a signature, one counts the number of unoccupied positions to the left of each particle. In particular, we have $\lambda^{(4)} = (0,0,0,0)$, $\vartheta^{(4)} = (1,1,1,0)$, $\lambda^{(3)} = (1,1,0)$, $\vartheta^{(3)} = (2,2,1)$, and so on, until $\vartheta^{(1)} = (1)$ and $\lambda^{(0)} = (0)$. Note that $\sum_{i=1}^{M} |\vartheta^{(i)}| - |\lambda^{(i)}|$ is the number of NW-SE dominos in the corresponding row which contain particles. We assign nontrivial Boltzmann weights to these dominos.

Proposition 2.4 allows to translate results about sequences of signatures (2.6) into the geometric language of domino tilings of the Aztec diamond. Throughout the paper, we mostly formulate and prove the results in the language of arrays and signatures, as they are directly related to Schur generating functions described in the next Section 2.3.

2.3. **Schur generating functions.** Let ρ_N be a probability measure on $\mathbb{GT}_N$. A *Schur generating function* $S_{\rho_N}(x_1,\dots,x_N)$ is a symmetric Laurent (formal) power series in $x_1,\dots,x_N$ defined by

$$S_{\rho_N}(x_1,\dots,x_N) \coloneqq \sum_{\lambda\in\mathbb{GT}_N} \rho_N(\lambda)\,\frac{s_\lambda(x_1,\dots,x_N)}{s_\lambda(1^N)}.$$

In what follows, we always assume that the measure ρ_N is such that this formal series is well-defined, i.e., the sum converges in an open complex neighborhood of (1^N).

Proposition 2.5. *Let $\mathbf{P}_{\beta_1,\dots,\beta_M}$ be a probability measure on sequences of signatures* (2.6) *as in Definition 2.3. Let $\mathfrak{r}_N$ be a projection of this measure to the signature $\lambda^{(N)}$, where $1\le N\le M$. Then this measuse admits the following Schur generating function:*

$$S_{\mathfrak{r}_N}(x_1,\dots,x_N) = \prod_{i=1}^{N}\prod_{j=N+1}^{M}\left(1-\beta_j+x_i\beta_j\right).$$

Proof. This is a standard computation with Schur processes, a class of measures on sequences of signatures introduced in [OR03] (see also [Bor11], [BBB+17]). Namely, by summing (2.7) over all signatures except for $\lambda^{(N)}$ and using usual and dual Cauchy identities, we obtain

$$\mathfrak{r}_N\left(\lambda^{(N)}=\lambda\right) = \left(\prod_{i=N+1}^{M}(1-\beta_i)^N\right)\cdot s_\lambda(1^N)\,s_{\lambda'}\left(\frac{\beta_M}{1-\beta_M},\frac{\beta_{M-1}}{1-\beta_{M-1}},\dots,\frac{\beta_{N+1}}{1-\beta_{N+1}}\right). \quad (2.8)$$

In particular, the random signature $\lambda^{(N)}$ is distributed according to the Schur measure [Oko01]. Applying the Cauchy identity once again to (2.8), one obtains the desired Schur generating function. □

2.4. **Asymptotics via Schur generating functions. Known results.** Here, we summarize the main results of [BG15] and [BG18] that describe the asymptotic behavior of random particle configurations on $\mathbb{Z}$ via Schur generating functions.

Let ρ_N be a sequence of probability measures on $\mathbb{GT}_N$, $N=1,2,\dots$. The main object of interest for us is the asymptotic behavior of the following *random measure*:

$$m[\rho_N] \coloneqq \frac{1}{N}\sum_{i=1}^{N}\delta\left(\frac{\lambda_i+N-i}{N}\right), \qquad \text{where } \lambda=(\lambda_1,\dots,\lambda_N) \text{ is } \rho_N\text{-distributed}. \quad (2.9)$$

We study it via its moments:

$$p_k^{(\rho_N)} \coloneqq \sum_{i=1}^{N}(\lambda_i+N-i)^k, \qquad \text{where } \lambda=(\lambda_1,\dots,\lambda_N) \text{ is } \rho_N\text{-distributed}. \quad (2.10)$$

Theorem 2.6 (Law of Large Numbers; Theorem 5.1 of [BG15])**.** *Suppose that the sequence ρ_N is such that for every k one has*

$$\lim_{N\to\infty}\frac{1}{N}\log\left(S_{\rho_N}(x_1,\dots,x_k,1^{N-k})\right) = \mathsf{F}(x_1)+\dots+\mathsf{F}(x_k), \quad (2.11)$$

where F is an analytic function in a complex neighborhood of 1*, and the convergence is uniform in an open complex neighborhood of 1^k. Then the random measures $m[\rho_N]$*

converge as $N \to \infty$ in probability, in the sense of moments to a deterministic *measure* $\mathbf{m}$ *on* $\mathbb{R}$*, whose moments are given by the following integral formula:*

$$\int_{\mathbb{R}} x^k \mathbf{m}(dx) = \frac{1}{2\pi\mathbf{i}(k+1)} \oint_{|z|=\varepsilon} \frac{dz}{1+z} \left(\frac{1}{z} + 1 + (1+z)\mathsf{F}(1+z) \right)^{k+1}. \tag{2.12}$$

Theorem 2.7 (Central Limit Theorem; Theorem 2.8 of [BG18])**.** *Assume that*

$$\lim_{N\to\infty} \frac{\partial_1 \log S_{\rho_N}(x_1,\ldots,x_k,1^{N-k})}{N} = \mathsf{F}(x_1), \qquad \textit{for any} \ \ k \ge 1,$$

$$\lim_{N\to\infty} \partial_1\partial_2 \log S_{\rho_N}(x_1,\ldots,x_k,1^{N-k}) = \mathsf{G}(x_1,x_2), \qquad \textit{for any} \ \ k \ge 1,$$

where $\mathsf{F}(x), \mathsf{G}(x,y)$ are holomorphic functions, and the convergence is uniform in a complex neighborhood of unity. Here, ∂_1 and ∂_2 denote the partial derivatives with respect to the first and second variables, respectively.

Then the collection of random variables

$$\left(\frac{p_k^{(\rho_N)} - \mathbf{E}[p_k^{(\rho_N)}]}{N^k} \right)_{k\ge1}$$

converges, as $N \to \infty$, in the sense of moments, to a Gaussian vector with zero mean and covariance

$$\lim_{N\to\infty} \frac{\operatorname{Cov}(p_{k_1}^{(\rho_N)}, p_{k_2}^{(\rho_N)})}{N^{k_1+k_2}} = \frac{1}{(2\pi\mathbf{i})^2} \oint_{|z|=\varepsilon} \oint_{|w|=2\varepsilon} \left(\frac{1}{z} + 1 + (1+z)\mathsf{F}(1+z) \right)^{k_1}$$
$$\times \left(\frac{1}{w} + 1 + (1+w)\mathsf{F}(1+w) \right)^{k_2} \left(\frac{1}{(z-w)^2} + \mathsf{G}(1+z,1+w) \right) dz\, dw,$$

where the integration contours are counterclockwise, and $\varepsilon \ll 1$.

We also note that [BG18, Section 2.4] provides multilevel generalizations of the above Central Limit Theorem, which concern limits of covariances of the form $N^{-k_1-k_2} \operatorname{Cov}\big(p_{k_1}^{(\rho_{\lfloor \alpha_1 N\rfloor})}, p_{k_2}^{(\rho_{\lfloor \alpha_2 N\rfloor})}\big)$.

Remark 2.8. Observe that after exponentiation, the left-hand side of (2.11) becomes the same as the expression in our Theorem 1.3 from the Introduction. However, this simple correspondence does not extend to the level of the Central Limit Theorem (compare to Theorem 6.2 below). Indeed, the CLT computations require differentiating either the logarithm or the N-root of the Schur generating function, which, in general, yields different results.

3. Domino tilings of the Aztec diamond with random edge weights

Let us now describe the model we study in the present paper. We consider domino tilings of the Aztec diamond of size M with random edge weights (illustrated in Figure 2). Equivalently, via the encoding of Section 2.2, we can (and will) work with sequences of signatures $(\lambda^{(N)}, \vartheta^{(N)}, \lambda^{(N-1)}, \vartheta^{(N-1)}, \ldots, \lambda^{(2)}, \vartheta^{(2)}, \lambda^{(1)}, \vartheta^{(1)})$ (2.6) and the measure $\mathbf{P}_{\beta_1,\ldots,\beta_M}$ on them (Definition 2.3), but now we also choose the parameters $(\beta_1,\ldots,\beta_M)$ to be random. We will use the notation $(\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \ldots, \boldsymbol{\beta}_M)$ for these parameters in order to emphasize that from now on these are **random variables** rather than fixed reals.

Definition 3.1 (Random environment with i.i.d. weights)**.** We consider the case where $\{\boldsymbol{\beta}_i\}_{i=1}^M$ are i.i.d. random variables drawn from a distribution $\boldsymbol{\mathcal{B}}_M$, which may depend on M. We assume that the distribution $\boldsymbol{\mathcal{B}}_M$ has finite moments of all orders.

We will study two regimes: one in which the distribution is fixed, and another in which it varies with M so that the fluctuations due to the random weights and those due to the domino tiling randomness occur on the same scale.

In the context of random environments, it is common to distinguish between two types of expectations: the **quenched** expectation, where we first fix the random environment (in our case, the parameters $\vec{\boldsymbol{\beta}} = (\boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_M)$) and then compute expectations with respect to the domino tiling, and the **annealed** expectation, where we average over both the randomness in the tiling and the randomness in the environment. Let us denote by $\mathbf{E}_{\boldsymbol{\mathcal{B}}}$ the expectation with respect to the randomness coming from the environment (i.e., the distribution of $\vec{\boldsymbol{\beta}}$). This notation is not strictly necessary (we might as well write the usual expectation $\mathbf{E}$), but it helps emphasize the randomness coming from the environment.

For the i.i.d. environment $\vec{\boldsymbol{\beta}} = (\boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_M)$, the annealed Schur generating function has the following explicit form:

Proposition 3.2. *In the sequence of signatures* $(\lambda^{(N)}, \vartheta^{(N)}, \ldots, \lambda^{(1)}, \vartheta^{(1)})$ (2.6) *under the measure* $\mathbf{P}_{\boldsymbol{\beta}_1, \boldsymbol{\beta}_2 \ldots, \boldsymbol{\beta}_M}$ *(with* $\boldsymbol{\beta}_i$ *random as in Definition 3.1), let* ρ_N *be the marginal distribution of* $\lambda^{(N)}$*. Then its annealed Schur generating function is given by*

$$S_{\rho_N}(x_1, \ldots, x_N) = \left(\mathbf{E}_{\boldsymbol{\mathcal{B}}} \prod_{i=1}^{N} (1 - \boldsymbol{\beta} + x_i \boldsymbol{\beta}) \right)^{M-N}. \tag{3.1}$$

Proof. Applying Proposition 2.5, we have

$$S_{\rho_N}(x_1, \ldots, x_N) = \mathbf{E}_{\boldsymbol{\mathcal{B}}} \prod_{i=1}^{N} \prod_{j=N+1}^{M} (1 - \boldsymbol{\beta}_j + x_i \boldsymbol{\beta}_j),$$

which simplifies to the right-hand side of (3.1) thanks to the independence of the environment random variables $\boldsymbol{\beta}_j$. ☐

Remark 3.3. Although the proof of Proposition 3.2 is very short, it plays an important conceptual role. Domino tilings of the Aztec diamond is a well-studied model, which is amenable to asymptotic analysis by a number of approaches. These include analysis of the shuffling algorithm [JPS98] and further application of cluster algebras [DFSG14], variational principle [CKP01], [KO07], determinantal point process methods [Joh02], [JN06], [CJY15], the use of orthogonal polynomials [DK20], (nonrigorous) tangent method [DFL18], Schur generating functions [BG15], [BG18], [BK18], and graph embeddings focused on conformal structure [CR24], [BNR24].

Proposition 3.2 shows that the method of Schur generating functions is particularly suited to analyzing random weights in one-periodic situations (equivalently, random parameters of Schur measures and processes). All other methods listed above seem to encounter immediate technical obstacles in the random environment setting.

We also note that Schur generating functions, while suitable for studying the global behavior of domino tilings (even in one-periodic random environment), are not well-suited for their local behavior. Thus, understanding the local behavior of domino tilings in a random environment requires overcoming these technical obstacles.

4. Decreasing variance: Gaussian Free Field plus Brownian motion

4.1. **Law of Large Numbers and Central Limit Theorem.**

Here we study domino tilings with random edge weights (Section 3) under the assumption that the distribution $\boldsymbol{\mathcal{B}} = \boldsymbol{\mathcal{B}}_M$ depends on the size M of the Aztec diamond, has compact support on

$\mathbb{R}_{>0}$ uniformly in M, and its first two moments satisfy

$$\lim_{M\to\infty} \mathbf{E}[\boldsymbol{\beta}] = \beta, \qquad \lim_{M\to\infty} M \cdot \mathrm{Var}[\boldsymbol{\beta}] = \sigma^2, \qquad \boldsymbol{\beta} \sim \boldsymbol{\mathcal{B}}_M, \tag{4.1}$$

where $0 < \beta < 1$ and $\sigma > 0$. Note that (4.1) implies that as $M \to \infty$, $\boldsymbol{\beta}$ convreges in probability to the deterministic value β. Let us also pick $0 < \alpha < 1$, and assume that the index N of the random signature $\lambda^{(N)}$ in the sequence of signatures (2.6) (corresponding to a domino tiling of the Aztec diamond of size M) satisfies

$$\lim_{M\to\infty} \frac{N}{M} = \alpha. \tag{4.2}$$

Remark 4.1. The results of this section do not require the enhanced asymptotics via Schur generating functions formulated in Section 1.3, and are obtained as a combination of Proposition 3.2 and the known results of [BG15], [BG18].

Theorem 4.2. *Under assumptions* (4.1)–(4.2)*, let h_M be the (unrescaled) domino height function of an Aztec diamond of size M with edge weights $\boldsymbol{\beta}_i$. Then the rescaled height profile $M^{-1}h_M(\lfloor Mu \rfloor, \lfloor Mv \rfloor)$ converges (in probability, in the sense of moments of the measures $m[\rho_N]$* (2.9) *for any $N \sim \alpha M$, $0 < \alpha < 1$) to a deterministic limit shape which is the same as in the case of nonrandom parameters $\boldsymbol{\beta}_i \equiv \beta$ for all i.*

Proof. By Proposition 3.2 , the normalized logarithm of the annealed Schur generating function equals

$$\frac{1}{N} \log S_{\rho_N}(x_1, \ldots, x_k, 1^{N-k}) = \frac{M-N}{N} \log \mathbf{E}_{\boldsymbol{\mathcal{B}}} \prod_{i=1}^{k} (1 - \boldsymbol{\beta} + x_i \boldsymbol{\beta}). \tag{4.3}$$

Since $\boldsymbol{\beta}$ converges in probability to β, the expectation factorizes in the limit. Thus, the right-hand side of (4.3) converges to the same expression $(\alpha^{-1}-1)\sum_{i=1}^{k} \log(1-\beta+x_i\beta)$ as for the deterministic parameters $\boldsymbol{\beta}_i \equiv \beta$. By Theorem 2.6, we get the desired result. ☐

Next, let us compute the asymptotic quantities entering Theorem 2.7:

Proposition 4.3. *Under assumptions* (4.1)–(4.2)*, for any $k \in \mathbb{Z}_{\geq 1}$, we have*

$$\lim_{M\to\infty} \frac{1}{N} \partial_1 \log\Big(\mathbf{E}_{\boldsymbol{\mathcal{B}}} \prod_{i=1}^{k} \big(1 - \boldsymbol{\beta} + x_i \boldsymbol{\beta}\big)\Big)^{M-N} = \Big(\frac{1}{\alpha} - 1\Big) \frac{\beta}{1 - \beta + \beta x_1},$$

and

$$\lim_{M\to\infty} \partial_1 \partial_2 \log\Big(\mathbf{E}_{\boldsymbol{\mathcal{B}}} \prod_{i=1}^{k} \big(1 - \boldsymbol{\beta} + x_i \boldsymbol{\beta}\big)\Big)^{M-N} = \frac{(1-\alpha)\sigma^2}{(1 - \beta + \beta x_1)^2 (1 - \beta + \beta x_2)^2}.$$

Proof. Differentiating with respect to x_1, we have

$$\partial_1 \log\left(\mathbf{E}_{\boldsymbol{\mathcal{B}}} \prod_{i=1}^{k} (1 - \boldsymbol{\beta} + x_i \boldsymbol{\beta})\right) = \frac{\mathbf{E}_{\boldsymbol{\mathcal{B}}}\left[\boldsymbol{\beta} \prod_{i=2}^{k} (1 - \boldsymbol{\beta} + x_i \boldsymbol{\beta})\right]}{\mathbf{E}_{\boldsymbol{\mathcal{B}}}\left[\prod_{i=1}^{k} (1 - \boldsymbol{\beta} + x_i \boldsymbol{\beta})\right]}.$$

Since $\boldsymbol{\beta}$ converges to a deterministic value β as $M \to \infty$, we immediately get the first statement of the proposition.

Further differentiating with respect to x_2, we have

$$\partial_2\partial_1 \log\left(\mathbf{E}_{\mathcal{B}} \prod_{i=1}^{k} (1-\boldsymbol{\beta}+x_i\boldsymbol{\beta})\right) = \frac{\mathbf{E}_{\mathcal{B}}\left[\boldsymbol{\beta}^2 \prod_{i=3}^{k}(1-\boldsymbol{\beta}+x_i\boldsymbol{\beta})\right]\mathbf{E}_{\mathcal{B}}\left[\prod_{i=1}^{k}(1-\boldsymbol{\beta}+x_i\boldsymbol{\beta})\right]}{\mathbf{E}_{\mathcal{B}}\left[\prod_{i=1}^{k}(1-\boldsymbol{\beta}+x_i\boldsymbol{\beta})\right]^2} - \frac{\mathbf{E}_{\mathcal{B}}\left[\boldsymbol{\beta}\prod_{i=2}^{k}(1-\boldsymbol{\beta}+x_i\boldsymbol{\beta})\right]\mathbf{E}_{\mathcal{B}}\left[\boldsymbol{\beta}\prod_{i=1;i\neq 2}^{k}(1-\boldsymbol{\beta}+x_i\boldsymbol{\beta})\right]}{\mathbf{E}_{\mathcal{B}}\left[\prod_{i=1}^{k}(1-\boldsymbol{\beta}+x_i\boldsymbol{\beta})\right]^2}.$$

Set

$$\xi := \boldsymbol{\beta} - \beta, \qquad \psi := \prod_{i=3}^{k}(1-\boldsymbol{\beta}+\boldsymbol{\beta}x_i).$$

Next, the numerator of the expression above,

$$\mathbf{E}_{\mathcal{B}}\left[\boldsymbol{\beta}^2\psi\right]\mathbf{E}_{\mathcal{B}}\left[(1-\boldsymbol{\beta}+x_1\boldsymbol{\beta})(1-\boldsymbol{\beta}+x_2\boldsymbol{\beta})\psi\right] - \mathbf{E}_{\mathcal{B}}\left[(1-\boldsymbol{\beta}+x_1\boldsymbol{\beta})\boldsymbol{\beta}\psi\right]\mathbf{E}_{\mathcal{B}}\left[(1-\boldsymbol{\beta}+x_2\boldsymbol{\beta})\boldsymbol{\beta}\psi\right],$$

can be expanded into a Taylor series with terms of order 1, ξ, ξ^2, and $O(\xi^3)$. After a direct calculation, one sees that the leading term as $M\to\infty$ is

$$\left(\mathbf{E}_{\mathcal{B}}(\xi^2) - \mathbf{E}_{\mathcal{B}}(\xi)^2\right)\mathbf{E}_{\mathcal{B}}[\psi^2] \approx \frac{\sigma^2}{M}\,\mathbf{E}_{\mathcal{B}}[\psi^2].$$

Taking into account the denominator and using the convergence of $\boldsymbol{\beta}$ to β, we obtain the second statement of the proposition. □

Recall that ρ_N denotes the distribution of $\lambda^{(N)}$, and its moments $p_k^{(\rho_N)}$ are defined in (2.10). The next theorem is the first main result of this paper.

Theorem 4.4. *Under assumptions* (4.1)–(4.2)*, the normalized moments* $M^{-k}p_k^{(\rho_N)}$*,* $k\in\mathbb{Z}_{\geq 1}$*, are jointly asymptotically Gaussian, with the limiting covariance given by*

$$\lim_{N\to\infty}\frac{\mathrm{Cov}\left(p_{k_1}^{(\rho_N)}, p_{k_2}^{(\rho_N)}\right)}{M^{k_1+k_2}} = \frac{\alpha^{k_1+k_2}}{(2\pi\mathbf{i})^2}\oint_{|z|=\varepsilon}\oint_{|w|=2\varepsilon}\left(\frac{1}{z}+1+\frac{(1+z)(1-\alpha)\beta}{\alpha(1-\beta+\beta(z+1))}\right)^{k_1} \times\left(\frac{1}{w}+1+\frac{(1+w)(1-\alpha)\beta}{\alpha(1-\beta+\beta(w+1))}\right)^{k_2}\left(\frac{(1-\alpha)\sigma^2}{(1+\beta z)^2(1+\beta w)^2}+\frac{1}{(z-w)^2}\right)dz\,dw. \quad (4.4)$$

The integration contours are counterclockwise and $\varepsilon \ll 1$*.*

Proof. This is a straightforward combination of Theorem 2.7 and Proposition 4.3. □

One also has a multilevel version of Theorem 4.4:

Theorem 4.5. *Let* k_1, k_2 *be two integers, and let* $N_1 \leq N_2$ *be two sequences such that* $N_1/M \to \alpha_1$*,* $N_2/M\to\alpha_2$*, for* $\alpha_1<\alpha_2$*. Then, the family of random variables* $\{M^{-k}p_k^{(\rho_{N_1})}, M^{-l}p_l^{(\rho_{N_2})}\}_{k,l\in\mathbb{Z}_{\geq 1}}$ *is asymptotically Gaussian, with the limiting covariance* *given by*

$$\lim_{N\to\infty}\frac{\operatorname{Cov}\left(p_{k_1}^{(\rho_{N_1})},p_{k_2}^{(\rho_{N_2})}\right)}{M^{k_1+k_2}}=\frac{\alpha_1^{k_1}\alpha_2^{k_2}}{(2\pi\mathbf{i})^2}\oint_{|z|=\varepsilon}\oint_{|w|=2\varepsilon}\left(\frac{1}{z}+1+\frac{(1+z)(1-\alpha_1)\beta}{\alpha_1(1-\beta+\beta(z+1))}\right)^{k_1}$$
$$\times\left(\frac{1}{w}+1+\frac{(1+w)(1-\alpha_2)\beta}{\alpha_2(1-\beta+\beta(w+1))}\right)^{k_2}\left(\frac{(1-\alpha_2)\sigma^2}{(1+\beta z)^2(1+\beta w)^2}+\frac{1}{(z-w)^2}\right)dz\,dw. \quad (4.5)$$

Proof. The extension of Theorem 4.4 to the multilevel setting follows the same steps as the passage from one-level to multilevel results in [BG18]. That paper introduces multilevel Schur generating functions; for the Schur process these functions can be evaluated explicitly, and with random parameters the computation is analogous to Proposition 3.2. Finally, instead of Theorem 2.7 we invoke its multilevel analogue, [BG18, Theorem 2.11], which applies to multilevel distributions on signatures. We omit further details of the proof. □

4.2. **Limiting covariance: Gaussian Free Field plus Brownian motion.** Let us interpret the covariance structure (4.5) obtained in Theorem 4.5. We can represent the integral as a sum of two terms, corresponding to the two summands

$$\frac{(1-\alpha_2)\sigma^2}{(1+\beta z)^2(1+\beta w)^2}+\frac{1}{(z-w)^2} \quad (4.6)$$

in one of the brackets. The second term already appears in the case of deterministic edge weights. As shown in [BG18, Section 9], this term gives rise to the *Gaussian Free Field* (GFF). That is, in an appropriate complex structure determined by the limit shape, the contour integral expression can be transformed into an expression coming from the GFF.

In more detail, if we map the liquid region inside the Aztec diamond (i.e., the region bounded by the arctic ellipse inside of which one sees a random mixture of dominos of all types, cf. Figure 1) to the complex upper half-plane $\mathbb{H} := \{z \in \mathbb{C} : \operatorname{Im}(z) > 0\}$ via a suitable diffeomorphism,[5] then the covariance of fluctuations at points $z, w \in \mathbb{H}$ will be given by

$$\frac{-1}{2\pi^2}\log\left(\frac{z-w}{z-\overline{w}}\right),$$

which is a scalar multiple of the Green function of the Laplace operator in $\mathbb{H}$ with Dirichlet boundary conditions.

The term with the first summand in (4.6) is new. It appears because of the randomness coming from the edge weights. Under the same diffeomorphism of the liquid region onto $\mathbb{H}$, we obtain the following covariance term in the GFF complex structure:

$$\frac{4\operatorname{Im}(z)\operatorname{Im}(w)\min(1-\alpha_1,1-\alpha_2)}{|1+\beta z|^2|1+\beta w|^2}, \qquad z,w\in\mathbb{H}.$$

One can verify that this fluctuation term is interpreted as coming from (a deterministic pushforward of) the one-dimensional Brownian motion. Note that in Theorem 4.5 we assumed that $\alpha_1 < \alpha_2$, but since the left-hand side of (4.5) is symmetric in α_1 and α_2, we can write $1-\alpha_2 = \min(1-\alpha_1, 1-\alpha_2)$ in order to emphasize the Brownian motion-like covariance structure.

To summarize, it is natural to interpret the additional Brownian motion appearing in the covariance as arising from the limiting fluctuations of the i.i.d. parameters $(\boldsymbol{\beta}_1,\ldots,\boldsymbol{\beta}_M)$. At the same time, tuning the variance of the edge weights as $\sim 1/M$,

[5] Which is deterministic, that is, does not depend on our random edge weights.

the fluctuation structure still retains the Gaussian Free Field component, which is the same as in the case of deterministic edge weights.

5. Law of Large Numbers

In this section we prove the law of large numbers for discrete particle systems determined by their Schur generating functions. Our result vastly generalizes the one from [BG15] (formulated as Theorem 2.6 above). We require this generalization in order to analyze domino tilings of the Aztec diamond with random edge weights of a fixed distribution (as opposed to the decreasing variance case in Section 4). However, this law of large numbers is certainly of an independent interest.

For the proof in this and the next section, we use the method of Schur generating functions that was developed in [BG15], [BG18]. While we recall all the necessary steps, we focus primarily on new ideas and computations. We refer to [BG18] and [BZ24] for a more detailed discussion of the basics of the method. Preliminary definitions and results on Schur generating functions from [BG15], [BG18] are given in Sections 2.3 and 2.4 above.

5.1. **Symmetrization lemmas.** First, we recall a symmetrization result from [BG15]. Define $P \coloneqq \{(a,b) \in \mathbb{Z}^2_{\geq 1} : 1 \leq a < b \leq n\}$.

Lemma 5.1 ([BG15, Lemma 5.4]). *Let n be a positive integer. Let $f(z_1,\dots,z_n)$ be a function. We consider its symmetrization with respect to P:*

$$f_P(z_1,\dots,z_n) \coloneqq \frac{1}{n!}\sum_{\pi\in S_n}\frac{f(z_{\pi(1)},\dots,z_{\pi(n)})}{\prod_{(a,b)\in P}(z_{\pi_{(a)}}-z_{\pi_{(b)}})}$$

(where S_n is the symmetric group). Then:

- *If f is analytic in a complex neighborhood of 1^n, then f_P is also analytic in a complex neighborhood of 1^n.*
- *If $(f^{(m)}(z_1,\dots,z_n))_{m\geq 1}$ is a sequence of analytic functions converging to zero uniformly in a complex neighborhood of 1^n, then so is the sequence $(f_P^{(m)})_{m\geq1}$.*

We need a new symmetrization identity which we were not able to locate in the literature. Throughout the rest of the paper, we will use the notation

$$w_m \coloneqq \exp\left(\frac{2\pi\sqrt{-1}}{m}\right)$$

for the m-th root of unity.

Lemma 5.2. *Let $G(u_1,u_2,\dots,u_{m+1})$ be a complex-analytic function in a neighborhood of (0^{m+1}), which is symmetric in $u_2,\dots,u_{m+1}$ (but not necessarily in u_1). We have*

$$\sum_{i=1}^{m+1}\frac{G(u_i,u_1,u_2,\dots,\widehat{u_i},\dots,u_{m+1})}{\prod_{j=1;j\neq i}^{m+1}(u_i-u_j)}\Bigg|_{u_1=\dots=u_{m+1}=0} = \frac{1}{m!}\partial_u^m\big[G(u,uw_{m+1},\dots,uw_{m+1}^m)\big]\Big|_{u=0}. \tag{5.1}$$

Proof. By Lemma 5.1, the limit in the left-hand side of (5.1) exists. In order to compute it, we chose $u_i = uw_{m+1}^{i-1}$, $i=1,2,\dots,m+1$, and let u tend to zero. Define also

$$\widetilde{F}(u) \coloneqq G(u,uw_{m+1},\dots,uw_{m+1}^m).$$

For our choice of u_i's, one has

$$\widetilde{F}(u_i) = G(u_i,u_1,u_2,\dots,\widehat{u_i},\dots,u_{m+1}),$$

due to the symmetry of G with respect to all variables except of the first one, and the fact that w_{m+1} is a root of unity. Therefore,

$$\sum_{i=1}^{m+1} \frac{G(u_i, u_1, u_2, \dots, \widehat{u_i}, \dots, u_{m+1})}{\prod_{j=1; j \neq i}^{m+1} (u_i - u_j)} \bigg|_{u_1 = \dots = u_{m+1} = 0} = \sum_{i=1}^{m+1} \frac{\widetilde{F}(u_i)}{\prod_{j=1; j \neq i}^{m+1} (u_i - u_j)} \bigg|_{u_1 = \dots = u_{m+1} = 0}. \quad (5.2)$$

The symmetrization in the right-hand side of (5.2) is now standard. It is well-known (see, e.g., [BG15, Lemma 5.5]) that this right-hand side is equal to $\frac{1}{m!} \partial_u^m \widetilde{F}(u)\big|_{u=0}$. This completes the proof. □

The following corollary is obtained immediately from Lemma 5.2 via a shift of variables.

Corollary 5.3. *Let $G(u_1, u_2, \dots, u_{m+1})$ be a complex-analytic function in a neighborhood of (1^{m+1}), which is symmetric in $u_2, \dots, u_{m+1}$ (but not necessarily in u_1). Then*

$$\sum_{i=1}^{m+1} \frac{G(u_i, u_1, u_2, \dots, \hat{u}_i, \dots, u_{m+1})}{\prod_{j=1; j \neq i}^{m+1} (u_i - u_j)} \bigg|_{u_1 = \dots = u_{m+1} = 1}$$
$$= \frac{1}{m!} \partial_u^m \left[G(1 + u, 1 + u w_{m+1}, \dots, 1 + u w_{m+1}^m) \right] \Big|_{u=0}. \quad (5.3)$$

The next observation will be important for simplifying formulas in applications:

Lemma 5.4. *Let $P(u_1, u_2, \dots, u_m)$ be a constant-free polynomial of degree strictly less than m. Then $P(1, w_m, w_m^2, \dots, w_m^{m-1}) = 0$.*

Proof. Power sums of degree less than m evaluated at all m-th roots of unity vanish, that is, $\sum_{i=0}^{m-1} w_m^{i \cdot k} = 0$ for all $1 \leq k < m$. Any polynomial P as in the hypothesis can be expressed, as a linear combination of products of such power sums. This completes the proof. □

5.2. **Law of Large Numbers.** Let ρ_N be a probability measure on $\mathbb{GT}_N$. Recall that our main object of interest is the asymptotic behavior of the empirical measure $m[\rho_N]$ defined by (2.9). The main technical role in our proofs is played (as in [BG15], [BG18]) by differential operators acting on analytic functions of N variables,

$$\big(\mathcal{D}_k f\big)(u_1, \dots, u_N) \coloneqq \frac{1}{V_N(\vec{u})} \sum_{i=1}^{N} (u_i \partial_i)^k [V_N(\vec{u}) f(u_1, \dots, u_N)],$$

where $V_N(\vec{u}) \coloneqq \prod_{1 \leq i < j \leq N} (u_i - u_j)$ is the Vandermonde determinant, and ∂_i denotes the derivative with respect to u_i. They are useful because they act diagonally on the Schur functions:

Proposition 5.5. *For $\lambda \in \mathbb{GT}_N$, we have*

$$\mathcal{D}_k s_\lambda = s_\lambda \cdot \sum_{i=1}^{N} (\lambda_i + N - i)^k.$$

Proof. Immediate from the definition of Schur functions (2.1). □

We now state and prove the main result of this section:

Theorem 5.6. *Let ρ_N, $N \geq 1$, be a sequence of probability measures on $\mathbb{GT}_N$. Assume that there exists a sequence of symmetric functions $F_k(z_1, \dots, z_k)$, $k \geq 1$, analytic in a complex neighborhood of 1^k, such that for every fixed k we have*

$$\lim_{N \to \infty} \sqrt[N]{S_{\rho_N}(u_1, \dots, u_k, 1^{N-k})} = F_k(u_1, \dots, u_k), \quad (5.4)$$

where the convergence is uniform in a complex neighborhood of 1^k. Then, as $N \to \infty$, the random measures $m[\rho_N]$ converge in probability, in the sense of moments, to a deterministic probability measure $\boldsymbol{\mu}$ on $\mathbb{R}$. The moments $(\boldsymbol{\mu}_k)_{k\geq 1}$ of this measure are given by

$$\boldsymbol{\mu}_k = \sum_{l=0}^{k} \binom{k}{l} \frac{1}{(l+1)!} \times \partial_u^l \Big[(1+u)^k \big(\partial_1 F_{l+1}(1+u, 1+uw_{l+1}, 1+uw_{l+1}^2, \dots, 1+uw_{l+1}^l) \big)^{k-l} \Big] \Big|_{u=0}. \quad (5.5)$$

Proof. **Step 1.** First, we show convergence in expectation. By Proposition 5.5, we have

$$\mathcal{D}_k S_{\rho_N}(u_1, u_2, \dots, u_N)|_{u_1=\dots=u_N=1} = \mathbf{E}\Bigg[\sum_{i=1}^{N} (\lambda_i + N - i)^k \Bigg],$$

where λ is distributed according to ρ_N. The expression

$$\frac{1}{V_N(\vec{u})} \sum_{i=1}^{N} u_i^n \partial_i^n [V_N(\vec{u}) S_{\rho_N}(u_1, \dots, u_N)] \quad (5.6)$$

can be written as

$$\sum_{m=0}^{k} \sum_{\substack{l_0,\dots,l_m=1 \\ l_i \neq l_j \text{ for } i\neq j}}^{N} \binom{k}{m} \frac{u_{l_0}^k \partial_{l_0}^{k-m} S_{\rho_N}(u_1, \dots, u_N)}{(u_{l_0} - u_{l_1}) \dots (u_{l_0} - u_{l_m})}. \quad (5.7)$$

Further on, using the form $S_{\rho_N} = (\sqrt[N]{S_{\rho_N}})^N$ when applying the differential operator, one can write (5.7) as a linear combination of terms of the form

$$\frac{u_{b_0}^k \left(\partial_{b_0} \sqrt[N]{S_{\rho_N}}\right)^{l_1} \dots \left(\partial_{b_0}^{k-m} \sqrt[N]{S_{\rho_N}}\right)^{l_{k-m}}}{(u_{b_0} - u_{b_1}) \dots (u_{b_0} - u_{b_m})} + \dots + \frac{u_{b_m}^k \left(\partial_{b_m} \sqrt[N]{S_{\rho_N}}\right)^{l_1} \dots \left(\partial_{b_m}^{k-m} \sqrt[N]{S_{\rho_N}}\right)^{l_{k-m}}}{(u_{b_m} - u_{b_0}) \dots (u_{b_m} - u_{b_{m-1}})}, \quad (5.8)$$

where $l_1 + 2l_2 + \dots + (k-m)l_{k-m} = k - m$, $b_0, \dots, b_m \in \{1, \dots, N\}$ are distinct, and $m = 0, \dots, k$.

By Lemma 5.1, expression (5.8) has a limit as $u_1, \dots, u_N \to 1$. Furthermore, the symmetry of S_{ρ_N} guarantees that the limit does not depend on $b_0, \dots, b_m \in \{1, \dots, N\}$. Consequently, this limit will appear $m!\binom{N}{m+1}$ times in $\mathcal{D}_k S_{\rho_N}|_{u_1=\dots=u_N=1}$. Therefore, the leading order of asymptotics in N of $\mathcal{D}_k S_{\rho_N}|_{u_1=\dots=u_N=1}$ is N^{k+1}, and it is obtained from the terms of the form (5.8) with $l_1 = k - m$ and $l_2 = \dots = l_{k-m} = 0$. Note that the leading power N^{k+1} precisely corresponds to taking the moments of the empirical measure $m[\rho_N]$, see (2.9).

As a conclusion, the leading asymptotics in N of $\mathcal{D}_k S_{\rho_N}|_{u_1=\dots=u_N=1}$ is given by

$$N^{k+1} \sum_{m=0}^{k} \binom{k}{m} \frac{1}{(m+1)!} \left(\sum_{i=1}^{m+1} \frac{u_i^k (\partial_1 F_k(u_i, u_1, \dots, \hat{u}_i, \dots, u_k))^{k-m}}{\prod_{j=1; j\neq i}^{m+1} (u_i - u_j)} \Bigg|_{u_1, u_2, \dots, u_{m+1}=1} \right).$$

Applying Lemma 5.2, we arrive at the fact that the expected moments of the empirical measure $m[\rho_N]$ converge to the moments $\boldsymbol{\mu}_k$ given by (5.5).

Step 2. It remains to show that the random moments of the empirical measure $m[\rho_N]$ concentrate around their expected values. This would follow from

$$\lim_{N\to\infty} \mathbf{E}\left[\frac{1}{N^{k+1}}\sum_{i=1}^{N}(\lambda_i+N-i)^k\right]^2 = (\boldsymbol{\mu}_k)^2, \qquad \text{for every } k\geq 1, \tag{5.9}$$

and will guarantee the desired convergence of moments in probability.

Up to a power of N, the left-hand side of (5.9) can be obtained by applying $\mathcal{D}_k$ to S_{ρ_N} twice, and then setting $u_1=\cdots=u_N=1$. We have

$$(\mathcal{D}_k)^2 S_{\rho_N} = \mathcal{D}_{2k}S_{\rho_N} + \frac{1}{V_N(\vec{u})}\sum_{\substack{m,n=1\\ m\neq n}}^{N}(u_n\partial_n)^k(u_m\partial_m)^k[V_N(\vec{u})S_{\rho_N}]. \tag{5.10}$$

The leading power of N in $\mathcal{D}_{2k}S_{\rho_N}$ is N^{2k+1}, and so it will not contribute to the limit in (5.9) which will be of the order N^{2k+2}. Next, in the second summand in (5.10), only terms of the form

$$\frac{1}{V_N(\vec{u})}\sum_{\substack{m,n=1\\ m\neq n}}^{N} u_n^k u_m^k \partial_n^k\partial_m^k[V_N(\vec{u})S_{\rho_N}]\Big|_{u_1=\cdots=u_N=1}, \tag{5.11}$$

contribute to the degree N^{2k+2}. Expression (5.11) is a linear combination of terms

$$\frac{u_{a_0}^k u_{b_0}^k \partial_{a_0}^{k-\nu}\partial_{b_0}^{k-\mu}S_{\rho_N}}{(u_{a_0}-u_{a_1})\ldots(u_{a_0}-u_{a_\nu})(u_{b_0}-u_{b_1})\ldots(u_{b_0}-u_{b_\mu})}, \tag{5.12}$$

where $a_0\neq b_0$, $a_i\neq a_j$, $b_i\neq b_j$ and $|\{b_0\}\cap\{a_1,\ldots,a_\nu\}|+|\{a_0\}\cap\{b_1,\ldots,b_\mu\}|\leq 1$. It is convenient to symmetrize these terms in $\{u_{a_i}\}$ and $\{u_{b_j}\}$. By Lemma 5.1, such symmetrizazions have a limit as $u_i\to 1$, $1\leq i\leq N$.

Analogously to, e.g., [BG15, Theorem 5.1], the leading contribution in the large-N expansion comes from the terms where the indices $a_0,\ldots,a_\nu,b_0,\ldots,b_\mu\in\{1,\ldots,N\}$ are all distinct. This contribution can be written in the factorized form

$$\left(\frac{u_{a_0}^k\left(\partial_{a_0}\sqrt[N]{S_{\rho_N}}\right)^{k-\nu}}{(u_{a_0}-u_{a_1})\ldots(u_{a_0}-u_{a_\nu})}+\ldots+\frac{u_{a_\nu}^k\left(\partial_{a_\nu}\sqrt[N]{S_{\rho_N}}\right)^{k-\nu}}{(u_{a_\nu}-u_{a_0})\ldots(u_{a_\nu}-u_{a_{\nu-1}})}\right)$$
$$\times\left(\frac{u_{b_0}^k\left(\partial_{b_0}\sqrt[N]{S_{\rho_N}}\right)^{k-\mu}}{(u_{b_0}-u_{b_1})\ldots(u_{b_0}-u_{b_\mu})}+\ldots+\frac{u_{b_\mu}^k\left(\partial_{b_\mu}\sqrt[N]{S_{\rho_N}}\right)^{k-\mu}}{(u_{b_\mu}-u_{b_0})\ldots(u_{b_\mu}-u_{b_{\mu-1}})}\right),$$

where each of the two factors is as in (5.8). This implies (5.9), which completes the proof of the theorem. □

A particular case of Theorem 5.6 recovers [BG15, Theorem 5.1] (which we fomulated as Theorem 2.6 above):

Corollary 5.7. *In the notation of Theorem 5.6, assume additionally that there exists a function* $\mathsf{F}(u)$ *such that*

$$F_k(u_1,\ldots,u_k)=\exp\left(\mathsf{F}(u_1)+\mathsf{F}(u_2)+\cdots+\mathsf{F}(u_k)\right), \qquad \text{for any } k\in\mathbb{Z}_{\geq 1}.$$

Then the moments of the limiting measure can be written as

$$\boldsymbol{\mu}_k=\sum_{l=0}^{k}\binom{k}{l}\frac{1}{(l+1)!}\,\partial_u^l\left[(1+u)^k[\mathsf{F}'(1+u)]^{k-l}\right]\Big|_{u=0}. \tag{5.13}$$

Proof. We have

$$\partial_1 F_k(u_1,\dots,u_k) = \mathsf{F}'(u_1)F_k(u_1,\dots,u_k),$$

and

$$\partial_1 F_{l+1}(1+u, 1+uw_{l+1},\dots,1+uw_{l+1}^l) = \mathsf{F}'(1+u)\left(1+O\left(u^{l+1}\right)\right),$$

due to the fact that the function $F_{l+1}(u_1,u_2,\dots,u_{l+1})$ is symmetric in all variables, and thanks to Lemma 5.4. Therefore, the expression under ∂_u^l in (5.5) can be simplified for this special F_k, and we arrive at (5.13). □

Remark 5.8. Compared to [BG15] and [BG18], we work with the N-th root of the Schur generating function, rather than with its logarithm, as N tends to infinity. It would be possible to state our results in terms of the limit of the logarithm of the Schur generating function; moreover, the double-summation formula for the covariance in Theorem 6.2 below would then be slightly simpler. However, our main application in this paper — the Schur measure with random parameters — is more straightforward to analyze with the current formulation.

Remark 5.9. The use of complex numbers (roots of unity) in (5.5) is a key technical tool, even if it originally might look a bit artificial. We are able to write several equivalent formulas for this expression, which involve real coefficients only. In particular, we have the following representation for the moments:

$$\begin{aligned}\boldsymbol{\mu}_k = &\sum_{m=0}^{k}\binom{k}{m}\frac{1}{(m+1)!}\sum_{n=1}^{m+1}\binom{m+1}{n}\\ &\times(-1)^{n+1}\sum_{l_1+\dots+l_n=m}\binom{m}{l_1,\dots,l_n}\partial_n^{l_n}\dots\partial_1^{l_1}\left[u_1^k(\partial_1 F(u_1,\dots,u_k))^{k-m}\right]\Big|_{u_1=\dots=u_k=1}.\end{aligned} \tag{5.14}$$

We omit the proof, because expression (5.5) for the moments $\boldsymbol{\mu}_k$ is far more suitable for our purposes: simplifying (5.14) for our main application — the domino tilings of the Aztec diamond — would require substantial additional work. We also believe that (5.5) will be more convenient for other potential applications.

5.3. **Free cumulants.** Theorem 5.1 in [BG15] is closely related to free probability; see, e.g., [BG15, Sections 1.4 and 1.5], [MN18], and [CNŚ18]. In this section, we derive an expression for the free cumulants of the limiting measure whose moments are given by formula (5.5). For basics on free cumulants, see [Spe11].

Let $\{G_k(u_1,\dots,u_k)\}_{k\ge1}$ be a sequence of functions that are symmetric in all variables except possibly for the first one, and such that

$$G_k(u_1,\dots,u_k) = G_{k+1}(u_1,\dots,u_k,1), \qquad k\ge 1.$$

They should be thought of as functions appearing in the right-hand side of (5.4), differentiated once with respect to u_1 (thus, no longer symmetric in u_1).

Lemma 5.10. *Let $\{G_k(u_1,\dots,u_k)\}_{k\ge1}$ be a sequence of functions as above. Then, for any $k,l\ge1$, and any $1\le s\le\min(k,l)$, we have*

$$\partial_u^s G_{l+1}(1+u,1+w_{l+1}u,\dots,1+w_{l+1}^l u)\Big|_{u=0} = \partial_u^s G_{k+1}(1+u,1+w_{k+1}u,\dots,1+w_{k+1}^k u)\Big|_{u=0}.$$

Proof. Without loss of generality, assume that $k>l$. We need to prove that the Taylor coefficients up to degree l of the functions

$$G_{k+1}(1+u,1+w_{l+1}u,\dots,1+w_{l+1}^l u,1,1,\dots,1)$$

and

$$G_{k+1}(1+u,1+w_{k+1}u,\dots,1+w_{k+1}^k u)$$

coincide. To see this, note that the Newton power sums of the k-element sets $(w_{k+1},\dots,w_{k+1}^k)$ and $(w_{l+1},\dots,w_{l+1}^l,0,\dots,0)$ coincide up to degree l. Therefore, any symmetric polynomial in these variables also has the same value on the two sets. Owing to the symmetry with respect to all variables except the first one, the Taylor coefficients involve only such polynomials, which completes the proof of the lemma. □

Therefore, the moments $\boldsymbol{\mu}_k$ (5.5) can be written as

$$\boldsymbol{\mu}_k = \sum_{l=0}^{k} \binom{k}{l} \frac{1}{(l+1)!} \partial_u^l \big[(1+u)^k (\partial_1 F_{k+1}(1+u, 1+w_{k+1}u, \dots, 1+w_{k+1}^k u))^{k-l}\big]\Big|_{u=0},$$

the difference with (5.5) is that now only the function F_{k+1} appears in the right-hand side.

Arguing as in [BG15, Equation 6.2], we can write this summation as a contour integral:

$$\frac{1}{2\pi i(k+1)} \oint_1 \frac{1}{u} \left(u \partial_1 F_{k+1}(u, 1-w_{k+1}+w_{k+1}u, \dots, 1-w_{k+1}^k + w_{k+1}^k u) + \frac{u}{u-1} \right)^{k+1} du,$$

where we integrate over a small contour around 1, oriented counterclockwise. Under the change of variables $u = e^v$, this integral becomes

$$\frac{1}{2\pi i(k+1)} \oint_0 \left(e^v \partial_1 F_{k+1}(e^v, 1-w_{k+1}+w_{k+1}e^v, \dots, 1-w_{k+1}^k + w_{k+1}^k e^v) + \frac{e^v}{e^v-1} \right)^{k+1} dv, \tag{5.15}$$

where the contour $\mathcal{C}_1$ is now a small positively oriented circle around $v = 0$. This computation leads to the following:

Proposition 5.11. *The moments $(\boldsymbol{\mu}_k)_{k\ge1}$ of the limiting measure from Theorem 5.6 are given by*

$$\boldsymbol{\mu}_k = \sum_{l=0}^{k} \binom{k}{l} \frac{1}{(l+1)!} \partial_v^l \bigg(\frac{e^v}{e^v-1} - \frac{1}{v} + e^v \partial_1 F_{k+1}\big(e^v,\, 1-w_{k+1}+w_{k+1}e^v,\, \dots,\, 1-w_{k+1}^k + w_{k+1}^k e^v\big) \bigg)^{k-l} \bigg|_{v=0}, \tag{5.16}$$

and its free cumulants $(\boldsymbol{c}_k)_{k\ge1}$ are given by

$$\boldsymbol{c}_k = \frac{1}{(k-1)!} \partial_v^{k-1} \bigg(\frac{e^v}{e^v-1} - \frac{1}{v} + e^v \partial_1 F_{k+1}\big(e^v,\, 1-w_{k+1}+w_{k+1}e^v,\, \dots,\, 1-w_{k+1}^k + w_{k+1}^k e^v\big) \bigg) \bigg|_{v=0}.$$

Proof. Starting from the right-hand side of (5.16) and applying Cauchy's integral formula together with the binomial theorem, we arrive at (5.15). This establishes the claim for the moments. Having done that, we can appeal to [BZ24, Lemma 4] to obtain the desired expression for the free cumulants. □

5.4. **Schur measures with random parameters.** In this section we address the specific form of Schur generating functions that appears in domino tilings of Aztec diamond with random edge weights.

Proposition 5.12. *Let ρ_N, $N \geq 1$, be a sequence of probability measures on $\mathbb{GT}_N$, such that their Schur generating function has a form*

$$S_{\rho_N}(x_1, \dots, x_N) = \left(\mathbf{E}_{\boldsymbol{\mathcal{B}}} \prod_{i=1}^{N} (1 - \boldsymbol{\beta} + x_i \boldsymbol{\beta}) \right)^{M_N},$$

where the distribution $\boldsymbol{\mathcal{B}}$ of the random variable $\boldsymbol{\beta}$ is independent of M, and has moments of all orders. Assume that $\lim_{N\to\infty} M_N/N = a \in \mathbb{R}_{>0}$. Then the random measure $m[\rho_N]$ converges, as $N \to \infty$, in probability, in the sense of moments to a deterministic probability measure $\boldsymbol{\mu}$ on $\mathbb{R}$, with moments $(\boldsymbol{\mu}_k)_{k\geq 1}$ given by

$$\boldsymbol{\mu}_k = \frac{1}{2\pi\sqrt{-1}(k+1)} \oint_{\mathcal{C}_1} \frac{dz}{z} \mathcal{F}(z)^{k+1}, \tag{5.17}$$

where

$$\mathcal{F}(z) := \frac{z}{z-1} + a\, z\, \mathbf{E}_{\boldsymbol{\mathcal{B}}} \frac{\boldsymbol{\beta}}{1 - \boldsymbol{\beta} + \boldsymbol{\beta} z}, \tag{5.18}$$

and $\mathcal{C}_1$ is a contour oriented counterclockwise around $z = 1$ that encloses no other poles of the integrand.

Proof. We have

$$\lim_{N\to\infty} \sqrt[N]{S_{\rho_N}(u_1, \dots, u_k, 1^{N-k})} = \left(\mathbf{E}_{\boldsymbol{\mathcal{B}}} \prod_{i=1}^{k} (1 - \boldsymbol{\beta} + u_i \boldsymbol{\beta}) \right)^{a},$$

and this is our function $F_k(u_1, \dots, u_k)$. Therefore, in the notation of Theorem 5.6, we have

$$\partial_1 F_k(u_1, \dots, u_k) = a\, \mathbf{E}_{\boldsymbol{\mathcal{B}}} \left(\prod_{i=1}^{k} (1 - \boldsymbol{\beta} + u_i \boldsymbol{\beta}) \right)^{a-1} \mathbf{E}_{\boldsymbol{\mathcal{B}}} \left(\frac{\boldsymbol{\beta}}{1 - \boldsymbol{\beta} + u_1 \boldsymbol{\beta}} \left(\prod_{i=1}^{k} (1 - \boldsymbol{\beta} + u_i \boldsymbol{\beta}) \right) \right).$$

Noting that $\prod_{i=1}^{k} (1 - \boldsymbol{\beta} + u_i \boldsymbol{\beta})$ is a symmetric function in all variables and applying Lemma 5.4, we see that its nonzero degree terms in the u_i's will not contribute after the symmetrization. This, only the expression

$$a \left(\mathbf{E}_{\boldsymbol{\mathcal{B}}} \left(\frac{\boldsymbol{\beta}}{1 - \boldsymbol{\beta} + u_1 \boldsymbol{\beta}} \right) \right)$$

contributes to the limit. Applying Theorem 5.6 and the argument from [BG15, Equation 6.2], we arrive at the result. □

A key step in recovering the limiting density of the empirical measure from its moments is the analysis of the equation $\mathcal{F}(z) = y$. For the moment formula (5.17), it was shown, see e.g. [BK18, Lemma 4.1 and Theorem 4.3], that if, for each $y \in \mathbb{R}$, this equation has a unique complex root $z(y)$ lying in the upper half-plane, then the density of the limiting measure is given by $\frac{1}{\pi} \operatorname{Arg}(z(y))$.

To apply this to the Aztec diamond, one needs a change of variables in the function $\mathcal{F}(z) = \mathcal{F}(z; a)$ (5.18) and the corresponding equation. Namely, one needs to find the solution $z = z(\alpha, y)$ in the complex upper half-plane to the equation

$$\mathcal{F}(z; \alpha^{-1} - 1) = \alpha^{-1} y, \tag{5.19}$$

where $\alpha, y \in [0, 1]$ serve as coordinates in the Aztec diamond. A few examples of the limiting density

$$(\alpha, y) \mapsto \frac{1}{\pi} \operatorname{Arg}(z(\alpha, y)) \tag{5.20}$$

(which is a suitable linear transformation of the domino height function given in Figure 3) are presented in Figure 7 (recall that the edge weights are related to the parameters as $W_i = \boldsymbol{\beta}_i/(1-\boldsymbol{\beta}_i)$). Note that for non-discrete distributions $\boldsymbol{\mathcal{B}}$, equation (5.19) can be solved only numerically. On the other hand, the arctic curve has an exact parametrization, expressing (α, y) as functions of the double root z of the equation (5.19). We have used this parametrization to plot the arctic curves in Figure 7.

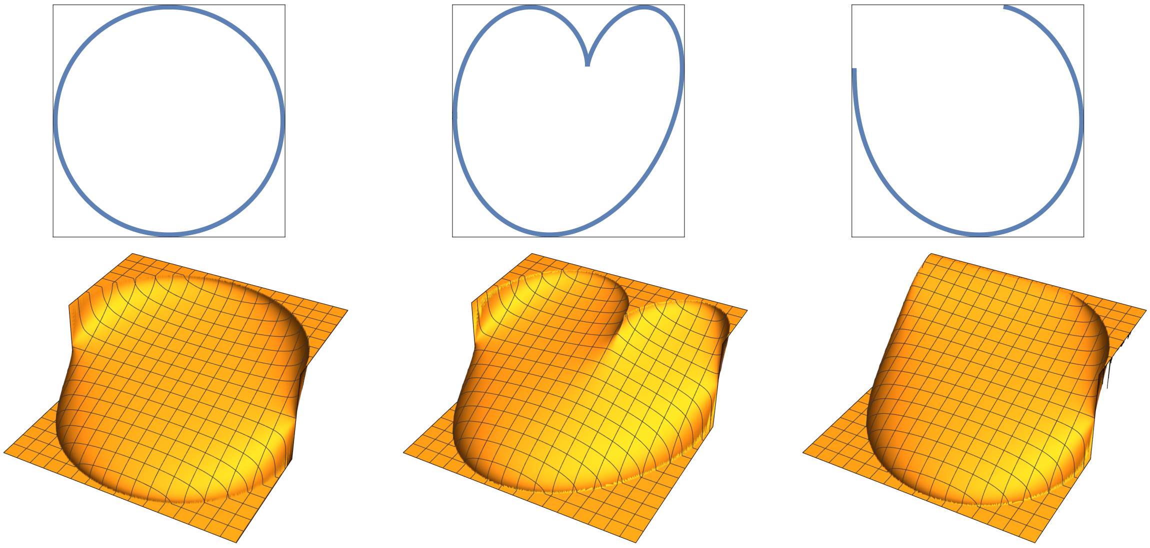

FIGURE 7. Arctic curves (top row) and plots of the limiting density (5.20) (bottom row) for domino tilings of the Aztec diamond with constant edge weights $W_i \equiv 1$ (left), i.i.d. Bernoulli edge weights with $\mathbf{P}(W_i = \frac{1}{2}) = \mathbf{P}(W_i = 5) = \frac{1}{2}$ (middle), and i.i.d. uniform edge weights on $[0, 2]$ (right). These plots are obtained from the equation (5.19): arctic curves are the sets (α, y) for which (5.19) has a double root in z, and the limiting density comes from the argument of the complex root as in (5.20). We see agreement with the samples in Figures 4 and 5.

5.5. **Random Matrix degeneration.** It is well known (see e.g. [GM05], [BG15]) that the framework of Schur generating functions admits a degeneration to results on random matrices. We now present the degeneration of Theorem 5.6 explicitly.

Let A be a random Hermitian $N \times N$ matrix with (possibly random) eigenvalues $\{\lambda_1(A) \le \ldots \le \lambda_N(A)\}$. Its *Harish-Chandra transform* (also known as a multivariate Bessel generating function) is defined by

$$\mathbb{E}[HC(x_1, \ldots, x_N; \lambda_1(A), \ldots, \lambda_N(A))] := \mathbb{E} \int_{U(N)} \exp(\mathrm{Tr}(AUBU^*)) \boldsymbol{m}_N(dU), \quad (5.21)$$

where B is a deterministic diagonal matrix with eigenvalues $x_1, x_2, \ldots, x_N \in \mathbb{C}$, and the integration is with respect to a Haar-distributed $N \times N$ unitary matrix $U \in U(N)$. We refer to [BG15] for the definition and a detailed discussion, see also [BZ24] for recent applications of the transform.

Theorem 5.13. *Let $A = A_N$ be a random Hermitian matrix of size N. Assume that there exists a sequence of symmetric functions $\Phi_k(x_1, x_2, \ldots, x_k)$, $k \ge 1$, analytic in a complex neighborhood of (0^k), such that for every fixed $k \ge 1$ one has*

$$\lim_{N\to\infty} \sqrt[N]{\mathbf{E}[HC(x_1, \ldots, x_r, 0^{N-k}; \lambda_1(A), \ldots, \lambda_N(A))]} = \Phi_k(x_1, x_2, \ldots, x_k), \quad (5.22)$$

uniformly in a complex neighborhood of 0^k*. Then the random atomic measure* $\frac{1}{N}\sum_{i=1}^{N}\delta\left(\frac{\lambda_i(A)}{N}\right)$ *converges, as* $N\to\infty$*, in probability, in the sense of moments to a deterministic probability measure* $\boldsymbol{\nu}$ *on* $\mathbb{R}$*, with moments* $(\boldsymbol{\nu}_k)_{k\geq 1}$ *given by*

$$\boldsymbol{\nu}_k = \sum_{l=0}^{k}\binom{k}{l}\frac{1}{(l+1)!}\partial_u^l\left[\left(\partial_1\Phi_{l+1}(u, uw_{l+1}, uw_{l+1}^2, \dots, uw_{l+1}^l)\right)^{k-l}\right]\Bigg|_{u=0}.$$

Proof. The simplest way to prove the result is to repeat all the arguments in the proof of Theorem 5.6, adjusting them to accommodate the slightly different differential operators; see, e.g., [BG15] and [BZ24, Section 3]. No substantial modifications are required, and the resulting formula differs only in that the factor $(1+u)^k$ is absent, owing to a minor distinction in the differential operators used. □

6. Cental Limit Theorem

6.1. **Statement.** In this section, we prove a general Central Limit Theorem for probability measures on signatures defined by their Schur generating functions. We continue to work in the limit regime introduced in the previous section (which in an application will correspond to domino tilings of the Aztec diamond with i.i.d. edge weights coming from a fixed distribution). This regime leads to particle systems with fluctuations on the scale $1/\sqrt{N}$.

Definition 6.1. The sequence of probability measures $\rho=\rho_N$ on $\mathbb{GT}_N$, $N\geq 1$, is called *asymptotically appropriate*, if there exists a sequence of symmetric functions $F_k^{(\rho)}(z_1,\dots,z_k)$, $k\geq 1$, which are analytic in a complex neighborhood of 1^k, such that for every fixed k, one has

$$\lim_{N\to\infty}\sqrt[N]{S_{\rho_N}(u_1,\dots,u_k,1^{N-k})} = F_k^{(\rho)}(u_1,\dots,u_k), \tag{6.1}$$

uniformly in a complex neighborhood of 1^k.

Theorem 6.2. *Let* $\rho=(\rho_N)_{N\geq1}$ *be an asymptotically appropriate sequence of probability measures on* $\mathbb{GT}_N$*. Then the vector of normalized moments* (2.10)*,*

$$\left(\frac{p_k^{(\rho_N)}-\mathbf{E}(p_k^{(\rho_N)})}{N^{k+\frac12}}\right)_{k\geq1} \tag{6.2}$$

converges to a mean zero Gaussian vector, with covariance given by

$$\lim_{N\to\infty}\frac{1}{N^{k+l+1}}\left(\mathbf{E}\big(p_k^{(\rho_N)}p_l^{(\rho_N)}\big)-\mathbf{E}\big(p_k^{(\rho_N)}\big)\,\mathbf{E}\big(p_l^{(\rho_N)}\big)\right)$$
$$=\sum_{q=0}^{k-1}\sum_{r=0}^{l-1}\frac{k\,l}{(q+1)!\,(r+1)!}\binom{l-1}{r}\binom{k-1}{q}$$
$$\times\,\partial_1^q\partial_2^r\Big[\Big(\partial_1\partial_2F_{q+r+2}^{(\rho)}\big(1+x_1,1+x_2,1+x_1w_{q+1},\dots,1+x_1w_{q+1}^q,1+x_2w_{r+1},\dots,1+x_2w_{r+1}^r\big)$$
$$-\,\partial_1F_{q+1}^{(\rho)}\big(1+x_1,1+x_1w_{q+1},\dots,1+x_1w_{q+1}^q\big)\,\partial_2F_{r+1}^{(\rho)}\big(1+x_2,1+x_2w_{r+1},\dots,1+x_2w_{r+1}^r\big)\Big)$$
$$\times\,(1+x_1)^k\big(\partial_1F_{q+1}^{(\rho)}(1+x_1,1+x_1w_{q+1},\dots,1+x_1w_{q+1}^q)\big)^{k-1-q}$$
$$\times\,(1+x_2)^l\big(\partial_2F_{r+1}^{(\rho)}(1+x_2,1+x_2w_{r+1},\dots,1+x_2w_{r+1}^r)\big)^{l-1-r}\Big]\Bigg|_{x_1=x_2=0}. \tag{6.3}$$

Remark 6.3. In Theorem 6.2 we permit the limiting Gaussian vector to be identically zero. This situation arises, for instance, when the theorem is applied to measures falling within the limit regime of [BG18] (and in particular, to domino tilings of the Aztec diamond with deterministic weights or random weights with variance decaying as $1/M$). The theorem becomes nontrivial once one shows that the covariance at the scale (6.2) remains non-zero in the limit.

Remark 6.4. Theorem 6.2 can also be degenerated to a random matrix setting, similarly to Section 5.5, but we do not state the corresponding result explicitly.

6.2. **First and second moment.** Throughout the rest of this section, we prove Theorem 6.2. Its proof is close to that in [BG18]. In some aspects, it is even easier since there are fewer terms that contribute to the limiting covariance in our limit regime. Therefore, we do not repeat all the details of [BG18, Sections 5 and 6], but rather provide a sketch of them and focus on those parts that differ from the argument there. We also keep the same notation as in [BG18].

Definition 6.5. For any $N \geq 1$, let $F_N(\vec{x})$, $\vec{x} = (x_1, x_2, \ldots, x_N)$, be a function of N variables. For $D \in \mathbb{Z}$, we will say that the sequence $(F_N)_{N\geq 1}$ *has N-degree at most D*, if for any integer $s \geq 0$ (not depending on N) and any indices $i_1, \ldots, i_s$, we have

$$\lim_{N\to\infty} \frac{1}{N^D} \partial_{i_1} \ldots \partial_{i_s} F_N(\vec{x})\Big|_{\vec{x}=1} = c_{i_1,\ldots,i_s}, \tag{6.4}$$

for some constants $c_{i_1,\ldots,i_s}$. In particular, the limit

$$\lim_{N\to\infty} \frac{1}{N^D} \left. F_N(\vec{x})\right|_{\vec{x}=1}$$

should exist (this corresponds to $s = 0$).

Similarly, we will say that the sequence $(F_N)_{N\geq 1}$ *has N-degree less than D*, if for any $s \geq 0$ (not depending on N) and any indices $i_1, \ldots, i_s$, we have

$$\lim_{N\to\infty} \frac{1}{N^D} \partial_{i_1} \ldots \partial_{i_s} F_N(\vec{x})\Big|_{\vec{x}=1} = 0.$$

Let $\rho = \{\rho_N\}$ be an asymptotically appropriate sequence of measures on $\mathbb{GT}_N$ with the Schur generating function $S_N(\vec{x}) = S_{\rho_N}(x_1, \ldots, x_N)$. For an integer $l > 0$, let us introduce the function

$$\mathcal{F}_{(l)}(\vec{x}) := \frac{1}{S_N(\vec{x}) V_N(\vec{x})} \sum_{i=1}^{N} (x_i \partial_i)^l \, V_N(\vec{x}) \, S_N(\vec{x}). \tag{6.5}$$

Lemma 6.6. *The functions $\mathcal{F}_{(l)}(\vec{x})$ have N-degree at most $l + 1$.*

Proof. The result follows from the argument in Section 5. We now give a brief, informal outline.

The power of N can arise either from derivatives applied to S_N — we use the identity $\partial_i S_N = N(\sqrt[N]{S_N})^{N-1} \partial_i \sqrt[N]{S_N}$, and any additional derivatives acting on $\partial_i \sqrt[N]{S_N}$ do not raise the power of N — or from the summations over indices, since extra indices appear when we differentiate the Vandermonde determinant. Because the operator involves one summation and l derivatives, the maximal power of N is $l + 1$. After we divide by $S_N(\vec{x}) \, V_N(\vec{x})$, further derivatives cannot increase this power, although, unlike in [BG18], they do not necessarily decrease it either. ☐

For positive integers l_1, l_2, let us define one more function via

$$\mathcal{G}_{(l_1,l_2)}(\vec{x}) := l_1 \sum_{r=0}^{l_1-1} \binom{l_1-1}{r} \sum_{\{a_1,\dots,a_{r+1}\}\subset[N]} (r+1)! \times Sym_{a_1,\dots,a_{r+1}} \frac{x_{a_1}^{l_1}\, \partial_{a_1}\big[\mathcal{F}_{(l_2)}\big] \big(\partial_{a_1} \sqrt[N]{S_N}\big)^{l_1-1-r}}{(x_{a_1}-x_{a_2})\cdots(x_{a_1}-x_{a_{r+1}})}. \tag{6.6}$$

Here, $Sym_{a_1,\dots,a_{r+1}}$ denotes the symmetrization over the indices $a_1,\dots,a_{r+1}$. The following lemma clarifies the meaning of this function: it describes the covariance in our probability models.

Lemma 6.7. *For any positive integers l_1, l_2, we have*

$$\frac{1}{V_N S_N} \sum_{i_1=1}^{N} (x_{i_1}\partial_{i_1})^{l_1} \sum_{i_2=1}^{N} (x_{i_2}\partial_{i_2})^{l_2} [V_N S_N] = \mathcal{F}_{(l_1)}(\vec{x})\mathcal{F}_{(l_2)}(\vec{x}) + \mathcal{G}_{(l_1,l_2)}(\vec{x}) + \tilde{T}(\vec{x}), \tag{6.7}$$

where $\mathcal{G}_{(l_1,l_2)}(\vec{x})$ has N-degree at most l_1+l_2+1, and $\tilde{T}(\vec{x})$ has N-degree at most l_1+l_2.

Proof. The proof is analogous to [BG18, Lemma 5.7]. Let us present it in a sketched form. The left-hand side of (6.7) can be written as

$$\frac{1}{V_N S_N} \sum_{i_1=1}^{N} (x_{i_1}\partial_{i_1})^{l_1} \big[V_N S_N \mathcal{F}_{(l_2)}(\vec{x})\big]. \tag{6.8}$$

The left-hand side of (6.8) has l_1 differentiations. If all of them are applied to V_N or S_N, we obtain the first term in the right-hand side. If exactly one differentiation is applied to $\mathcal{F}_{(l_2)}$, this yields a contribution of degree $l_1 + l_2 + 1$; the choice of which differentiation to apply gives rise to the factor l_1 in (6.6), while the other factors arise from choosing which differentiations are applied to the Vandermonde. If more than one differentiation is applied to $\mathcal{F}_{(l_2)}$, then the maximal possible degree is $l_1 + l_2$, since these two (or more) differentiations do not increase the degree. ☐

6.3. **Several moments.** For $s \in \mathbb{Z}_{\geq 1}$ and a subset $\{j_1,\dots,j_p\} \subset \{1,2,\dots,s\}$, let $\mathcal{P}^s_{j_1,\dots,j_p}$ be the set of *all pairings* of $\{1,2,\dots,s\}\setminus\{j_1,\dots,j_p\}$. The set $\mathcal{P}^s_{j_1,\dots,j_p}$ is empty whenever the cardinality of $\{1,2,\dots,s\}\setminus\{j_1,\dots,j_p\}$ is odd. Analogously, define $\mathcal{P}^{2;s}_{j_1,\dots,j_p}$ as the set of *all pairings* of $\{2,\dots,s\}\setminus\{j_1,\dots,j_p\}$. For a pairing P we write $\prod_{(a,b)\in P}$ for the product over all pairs (a,b) contained in P.

Proposition 6.8. *For any $s, l_1, \dots, l_s \in \mathbb{Z}_{\geq 1}$, we have*

$$\frac{1}{V_N S_N} \sum_{i_1=1}^{N} (x_{i_1}\partial_{i_1})^{l_1} \cdots \sum_{i_s=1}^{N} (x_{i_s}\partial_{i_s})^{l_s} [V_N S_N] = \sum_{p=0}^{s} \sum_{\{j_1,\dots,j_p\}\in[s]} \mathcal{F}_{(l_{j_1})}(\vec{x}) \dots \mathcal{F}_{(l_{j_p})}(\vec{x}) \left(\sum_{P\in\mathcal{P}^s_{j_1,\dots,j_p}} \prod_{(a,b)\in P} \mathcal{G}_{(l_a,l_b)}(\vec{x}) + \tilde{T}^{1;s}_{j_1,\dots,j_p}(\vec{x}) \right),$$

where $\tilde{T}^{1;s}_{j_1,\dots,j_p}(\vec{x})$ has N-degree less than $\sum_{i=1}^{s} l_i - \sum_{i=1}^{p} l_{j_i} + \frac{s-p}{2}$.

Proof. The proof parallels that of [BG18, Proposition 5.10]. We now sketch the main ideas. We argue by induction in s. For the induction step, one needs to analyze the expression

$$\frac{1}{V_N S_N}\left(\sum_{i_1=1}^{N}(x_{i_1}\partial_{i_1})^{l_1}\right)\Bigg[V_N S_N \sum_{p=0}^{s-1}\sum_{\{j_1,\dots,j_p\}\in[2;s]}\mathcal{F}_{(j_1)}(\vec{x})\dots\mathcal{F}_{(j_p)}(\vec{x})\\ \times\left(\sum_{P\in\mathcal{P}^{2;s}_{j_1,\dots,j_p}}\prod_{(a,b)\in P}\mathcal{G}_{(k_a,k_b)}(\vec{x})+\tilde{T}^{2;s}_{j_1,\dots,j_p}(\vec{x})\right)\Bigg],$$

for any choice of the set of indices $J_{\text{old}} \coloneqq \{j_1,\dots,j_p\}\subset[2;s]$. If all new differentiations act on $V_N S_N$, they generate a new function $\mathcal{F}$. If exactly one differentiation acts on an existing $\mathcal{F}$, it produces a new $\mathcal{G}$ in place of that $\mathcal{F}$. Any other allocation of differentiations does not yield a sufficiently high N-degree, and thus does not contribute to the leading term in the limit. ☐

For a positive integer l, define

$$E_l \coloneqq \mathcal{F}_{(l)}(1^N) = \frac{1}{V_N S_N}\sum_{i=1}^{N}(x_i\partial_i)^l V_N S_N\Big|_{\vec{x}=1}. \tag{6.9}$$

This is the expectation of the l-th moment of the probability measure with the Schur generating function S_N.

Lemma 6.9. *For any $s, l_1,\dots,l_s\in\mathbb{Z}_{\ge1}$, we have*

$$\frac{1}{V_N S_N}\left(\sum_{i_1=1}^{N}(x_{i_1}\partial_{i_1})^{l_1}-E_{l_1}\right)\left(\sum_{i_2=1}^{N}(x_{i_2}\partial_{i_2})^{l_2}-E_{l_2}\right)\times\dots\\ \times\left(\sum_{i_s=1}^{N}(x_{i_s}\partial_{i_s})^{l_s}-E_{l_s}\right)V_N S_N\Big|_{\vec{x}=1}=\sum_{P\in\mathcal{P}^s_{\varnothing}}\prod_{(a,b)\in P}\mathcal{G}_{(l_a,l_b)}(\vec{x})\Big|_{\vec{x}=1}+\tilde{T}_{\varnothing}(\vec{x})\Big|_{\vec{x}=1},$$

where $\tilde{T}_{\varnothing}(\vec{x})$ has N-degree less than $\sum_{i=1}^{s} l_i+\frac{s}{2}$.

Proof. The derivation of this lemma from Proposition 6.8 follows the standard combinatorial argument that relates moments and cumulants of random variables; it is identical to the proof of [BG18, Lemma 5.11]. ☐

6.4. **Computation of covariance.** Here we finalize the proof of Theorem 6.2 by computing the limiting covariance. We need to establish formula (6.3).

We the indices $k,l\in\mathbb{Z}_{\ge1}$ in (6.3). By Lemma 6.7, the left-hand side of (6.3) before the limit and without the factor N^{-k-l-1} is equivalent (in the sense of the top degree in N) to the expression

$$\mathcal{G}_{(k,l)}(1^N)=k\sum_{r=0}^{k-1}\binom{k-1}{r}\sum_{\{a_1,\dots,a_{r+1}\}\subset[N]}(r+1)!\\ \times Sym_{a_1,\dots,a_{r+1}}\frac{x_{a_1}^k\partial_{a_1}\left[\mathcal{F}_{(l)}\right]\left(N\partial_{a_1}\sqrt[N]{S_N}\right)^{k-1-r}}{(x_{a_1}-x_{a_2})\dots(x_{a_1}-x_{a_{r+1}})}\Bigg|_{x_1=\dots=x_N=1}.$$

Recall that $\mathcal{F}_{(l)}(\vec{x})$ can also be written as a sum of symmetrizations over indices. Using this, we obtain

$$\mathcal{G}_{(k,l)}(1^N) \approx k \sum_{q=0}^{k-1} \sum_{\{a_1,\dots,a_{q+1}\}\subset[N]} \binom{k-1}{q} (q+1)!\, Sym_{a_1,\dots,a_{q+1}} \frac{x_{a_1}^k \left(N\partial_{a_1} \sqrt[N]{S_N}\right)^{k-1-q}}{(x_{a_1}-x_{a_2})\cdots(x_{a_1}-x_{a_{q+1}})}$$

$$\times\, \partial_{a_1}\left[\sum_{r=0}^{l} \sum_{\{b_1,\dots,b_{r+1}\}\subset[N]} \binom{l}{r} (r+1)!\, Sym_{b_1,\dots,b_{r+1}} \frac{x_{b_1}^l \left(N \frac{\partial_{b_1} \sqrt[N]{S_N}}{\sqrt[N]{S_N}}\right)^{l-r}}{(x_{b_1}-x_{b_2})\cdots(x_{b_1}-x_{b_{r+1}})}\right]\Bigg|_{x_1=\ldots=x_N=1}.$$

Observe that the terms of maximal N-degree in this expression (namely, of degree N^{k+l+1}), arise precisely when $\{a_1,\dots,a_{q+1}\}\cap\{b_1,\dots,b_{r+1}\}=\varnothing$. In this case the outer differentiation ∂_{a_1} acts only on the factor $(\partial_{b_1}\sqrt[N]{S_N})/\sqrt[N]{S_N}$ as

$$\partial_{a_1}\left(\frac{\partial_{b_1}\sqrt[N]{S_N}}{\sqrt[N]{S_N}}\right)^{l-r} = (l-r)\left(\frac{\partial_{b_1}\sqrt[N]{S_N}}{\sqrt[N]{S_N}}\right)^{l-r-1} \frac{(\partial_{a_1}\partial_{b_1}\sqrt[N]{S_N})\cdot\sqrt[N]{S_N} - (\partial_{b_1}\sqrt[N]{S_N})\cdot(\partial_{a_1}\sqrt[N]{S_N})}{\sqrt[N]{S_N}^2}.$$

Applying Lemma 5.2 twice, passing to the limit, and performing minor transformations (which coincide with those performed in [BG18, Proposition 6.4, case 1]), we arrive at the right-hand side of the desired expression (6.3). This completes the proof of Theorem 7.1.

Remark 6.10. While the proof of Gaussianity is almost identical to that in [BG18], the computation of the covariance was more involved in the setting of that previous work. In our scaling regime, some of the terms considered in [BG18] dominate the others, so we needed to evaluate only the contribution of these dominant terms, which reduced the workload.

7. CLT for domino tilings in iid environment

We now apply the results of the previous Sections 5 and 6 to the case of domino tilings of Aztec diamond with i.i.d. one-periodic weights as in Figure 2.

Theorem 7.1. *Let $\rho=(\rho_N)_{N\ge1}$ be measures on signatures as in Proposition 5.12. Then the vector*

$$\left(\frac{p_k^{(\rho_N)} - \mathbf{E}[p_k^{(\rho_N)}]}{N^{k+\frac12}}\right)_{k\ge1} \tag{7.1}$$

converges, as $N\to\infty$, to a Gaussian vector with zero mean, and covariance given by

$$\lim_{N\to\infty}\frac{\mathrm{Cov}\left(p_k^{(\rho_N)},p_l^{(\rho_N)}\right)}{N^{k+l+1}} = \frac{1}{(2\pi\mathbf{i})^2}\oint_{|z|=\varepsilon}\oint_{|w|=2\varepsilon}\left(\frac1z+1+(1+z)\mathsf{F}(1+z)\right)^k \times\left(\frac1w+1+(1+w)\mathsf{F}(1+w)\right)^l \mathsf{G}(1+z,1+w)\,dz\,dw, \tag{7.2}$$

where the integration contours are counterclockwise, $\varepsilon\ll1$, and the functions $\mathsf{F}(z)$ and $\mathsf{G}(z,w)$ are given by

$$\mathsf{F}(z) := \mathbf{E}_{\boldsymbol{\mathcal{B}}}\left(\frac{\boldsymbol\beta}{1-\boldsymbol\beta+\boldsymbol\beta z}\right), \qquad \mathsf{G}(z,w) := \mathrm{Cov}_{\boldsymbol{\mathcal{B}}}\left(\frac{\boldsymbol\beta}{1-\boldsymbol\beta+\boldsymbol\beta z},\frac{\boldsymbol\beta}{1-\boldsymbol\beta+\boldsymbol\beta w}\right). \tag{7.3}$$

Recall that the subscript $\boldsymbol{\mathcal{B}}$ in the expectation and covariance in (7.3) indicates that these averages are not taken with respect to a random signature, but rather with respect to the distribution $\boldsymbol{\mathcal{B}}$ defining the random environment.

Proof of Theorem 7.1. Note that the sequence ρ_N as in the hypothesis of the theorem is appropriate in the sense of Definition 6.1. We have

$$\lim_{N\to\infty} \sqrt[N]{S_{\rho_N}(u_1,\dots,u_k,1^{N-k})} = \left(\mathbf{E}_{\boldsymbol{\mathcal{B}}}\prod_{i=1}^{k}(1-\boldsymbol{\beta}+u_i\boldsymbol{\beta})\right)^a = F_k^{(\rho)}(u_1,\dots,u_k),$$

and

$$\partial_1 F_k^{(\rho)}(u_1,\dots,u_k) = a\left(\mathbf{E}_{\boldsymbol{\mathcal{B}}}\prod_{i=1}^{k}(1-\boldsymbol{\beta}+u_i\boldsymbol{\beta})\right)^{a-1}\mathbf{E}_{\boldsymbol{\mathcal{B}}}\left(\frac{\boldsymbol{\beta}}{1-\boldsymbol{\beta}+\boldsymbol{\beta}u_1}\prod_{i=1}^{k}(1-\boldsymbol{\beta}+u_i\boldsymbol{\beta})\right).$$

Differentiating further, we obtain

$$\begin{aligned}\partial_1\partial_2 F_k^{(\rho)}(u_1,\dots,u_k) &= a(a-1)\left(\mathbf{E}_{\boldsymbol{\mathcal{B}}}\prod_{i=1}^{k}(1-\boldsymbol{\beta}+u_i\boldsymbol{\beta})\right)^{a-2}\\ &\quad\times \mathbf{E}_{\boldsymbol{\mathcal{B}}}\left(\frac{\boldsymbol{\beta}}{1-\boldsymbol{\beta}+\boldsymbol{\beta}u_2}\prod_{i=1}^{k}(1-\boldsymbol{\beta}+u_i\boldsymbol{\beta})\right)\mathbf{E}_{\boldsymbol{\mathcal{B}}}\left(\frac{\boldsymbol{\beta}}{1-\boldsymbol{\beta}+\boldsymbol{\beta}u_1}\prod_{i=1}^{k}(1-\boldsymbol{\beta}+u_i\boldsymbol{\beta})\right)\\ &\quad+ a\left(\mathbf{E}_{\boldsymbol{\mathcal{B}}}\prod_{i=1}^{k}(1-\boldsymbol{\beta}+u_i\boldsymbol{\beta})\right)^{a-1}\mathbf{E}_{\boldsymbol{\mathcal{B}}}\left[\frac{\boldsymbol{\beta}^2}{(1-\boldsymbol{\beta}+u_1\boldsymbol{\beta})(1-\boldsymbol{\beta}+u_2\boldsymbol{\beta})}\prod_{i=1}^{k}(1-\boldsymbol{\beta}+u_i\boldsymbol{\beta})\right].\end{aligned}$$

We apply Theorem 6.2. It establishes the convergence of moments to a Gaussian vector, and it remains to show that the general formula (6.3) for covariance vastly simplifies in our case of Schur measures with random parameters.

Indeed, by Lemma 5.4, the function $\prod_{i=1}^{k}(1-\boldsymbol{\beta}+\boldsymbol{\beta}u_i)$, which is symmetric in all variables, does not contribute to the limit beyond its free term (which is equal to 1). One can check that the resulting covariance is given by

$$\begin{aligned}\sum_{q=0}^{k-1}\sum_{r=0}^{l-1}\frac{kl}{(q+1)!\,(r+1)!}\binom{l-1}{r}\binom{k-1}{q}\partial_1^q\partial_2^r\Big[\mathsf{G}(1+x_1,1+x_2)\\ \times(1+x_1)^k\big(\mathsf{F}(1+x_1)\big)^{k-1-q}(1+x_2)^l\big(\mathsf{F}(1+x_2)\big)^{l-1-r}\Big]\Big|_{x_1=0,x_2=0}.\end{aligned}$$

Using the Cauchy integral formula and the binomial theorem, this expression turns into the desired right-hand side of (7.2). □

Remark 7.2. Similarly to Section 4.2, one can straightforwardly establish a multilevel analogue of Theorem 7.1 for domino tilings of the Aztec diamond with random edge weights, in the spirit of Theorem 4.5 and [BG18, Theorem 2.11]. This would produce a Brownian motion interpretation of the multilevel version of the covariance in (7.2).

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