Height Functions and Extremal Matchings on General Planar Bipartite Graphs

LP

1 Setup

We work with **planar bipartite graphs** — connected bipartite graphs equipped with a fixed embedding in the plane (or the sphere). Height functions are defined on *faces* of the embedding, so planarity is essential: without it there is no well-defined facial structure (e.g., $K_{3,3}$ is not planar, and height functions make no sense there). The general theory goes back to Thurston [Thu90] for domino tilings and extends to arbitrary planar bipartite graphs; see Kenyon [Ken09], Propp [Pro03].

2 Superposition and alternating cycles

Fix a *reference perfect matching* M_{ref}. Given any other perfect matching M, the symmetric difference

$$M \oplus M_{\mathrm{ref}} \;=\; \big(M \setminus M_{\mathrm{ref}}\big) \;\cup\; \big(M_{\mathrm{ref}} \setminus M\big) \tag{2.1}$$

has degree 0 or 2 at every vertex, so it decomposes into disjoint *alternating cycles* whose edges alternate between M and M_{ref}.

To extract a height function from this data, we impose a canonical orientation: direct every M-edge from **black** to **white**, and every M_{ref}-edge from **white** to **black**. Each alternating cycle then becomes a consistently directed closed curve in the plane.

We illustrate with the 2×4 grid graph G (three square faces, five perfect matchings $M_1, \ldots, M_5$). Taking $M_{\mathrm{ref}} = M_5$ (all vertical edges) and $M = M_2$ (all horizontal), the symmetric difference produces two disjoint clockwise cycles, one around each end face (Figure 1).

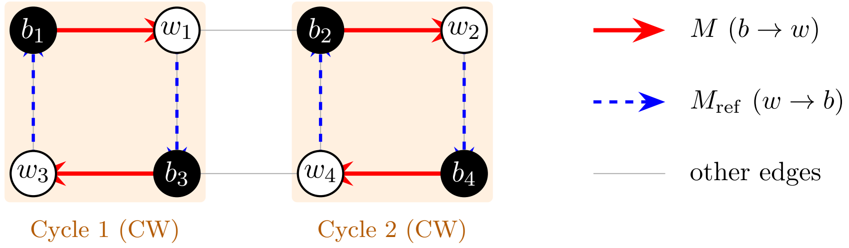

Figure 1: Superposition of $M = M_2$ (horizontal, red) and $M_{\mathrm{ref}} = M_5$ (vertical, blue) on G. The symmetric difference produces two disjoint clockwise alternating cycles (shaded), one around each end face. The middle face's horizontal boundary edges belong to neither matching.

3 The height function

Set $h = 0$ on the outer (unbounded) face. For two adjacent faces f, f' sharing a directed edge e on their boundary, set

$$h(f') = h(f) + 1 \quad \text{if } e \text{ flows to the \textit{left} of the crossing path,} \tag{3.1}$$

and $h(f') = h(f) - 1$ if e flows to the right. Edges in $M \cap M_{\text{ref}}$ carry no directed flow, and the height does not change across them. Well-definedness follows from planarity — the consistent orientation of alternating cycles guarantees zero height gain around any closed walk.

Figure 2 shows the resulting height functions for two matchings of G, with faces colored by height value (green for positive, red for negative, gray for zero).

(a) M_3: heights $(0, -1, 0)$

(b) M_2: heights $(+1, 0, +1)$

Figure 2: Height functions for two matchings of G with $M_{\text{ref}} = M_5$. Matching edges shown in red. The height vector (h_L, h_M, h_R) records the height of the left, middle, and right face.

4 Local flips

A face is *flippable* if its entire boundary forms an alternating cycle of M- and M_{ref}-edges (equivalently, a single directed cycle under the canonical orientation). In lozenge tilings this is a hexagonal face; in domino tilings, a 2×2 square. The cycle direction determines the effect of the flip:

- **Clockwise** boundary — local height maximum; flipping (swapping M-edges for M_{ref}-edges on the face) strictly *decreases* the height.
- **Counter-clockwise** boundary — local height minimum; flipping strictly *increases* the height.

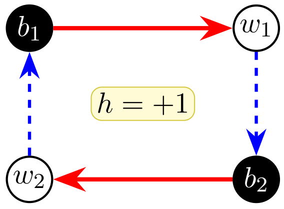

Figure 3: Before flip: clockwise cycle (local height maximum).

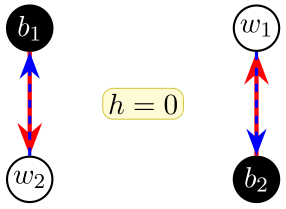

Figure 4: After flip: M and M_{ref} now agree on the vertical edges (both arrows on each edge), so the alternating cycle disappears and h drops from +1 to 0.

After the flip, M and M_{ref} agree on the flipped face, so the alternating cycle there disappears. The net effect is that the height of the flipped face changes by ±1.

5 Extremal matchings and the distributive lattice

Starting from any perfect matching M, we can construct the unique *maximal matching* M_{max} by repeatedly flipping all faces with clockwise alternating boundaries (local height minima), and the unique *minimal matching* M_{min} by flipping all counter-clockwise boundaries (local height maxima).

The flip order does not matter. The set of all perfect matchings, partially ordered by pointwise comparison of height functions, forms a **distributive lattice** — see Thurston [Thu90], Propp [Pro03], and the general theory of $\mathbb{Z}$-flows on planar graphs. In particular:

- The lattice has a unique maximum M_{max} and a unique minimum M_{min}.
- Every increasing chain of flips terminates at M_{max}, regardless of order; every decreasing chain terminates at M_{min}.
- The lattice operations $M_1 \vee M_2$ and $M_1 \wedge M_2$ correspond to pointwise max and min of the height functions.

This confluent structure — no local traps, no dependence on flip sequence — is the key reason height functions are such a powerful tool for the analysis of the space of perfect matchings of planar bipartite graphs.

For the graph G, the five perfect matchings and their height vectors (h_L, h_M, h_R) are shown in Figure 5.

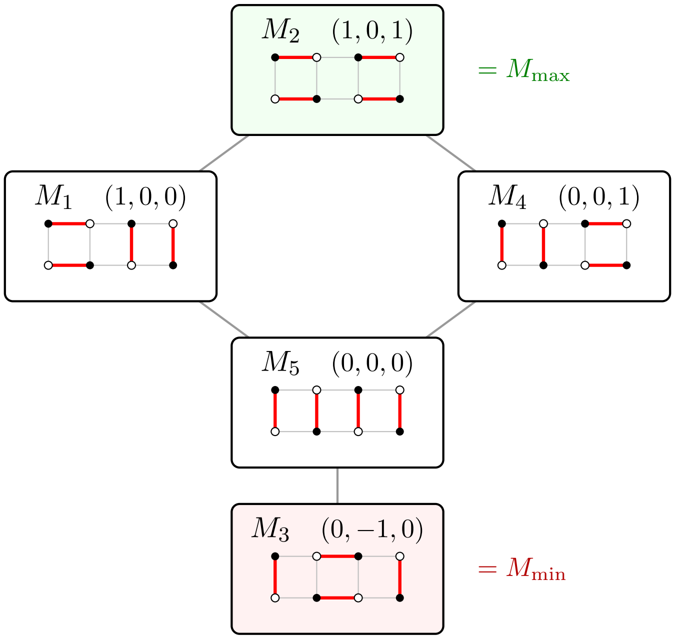

Figure 5: The distributive lattice of all five perfect matchings of G, partially ordered by pointwise comparison of height functions. Each node shows the matching name, height vector (h_L, h_M, h_R), and a thumbnail (red edges = matching). Edges of the Hasse diagram connect matchings that differ by a single face flip.

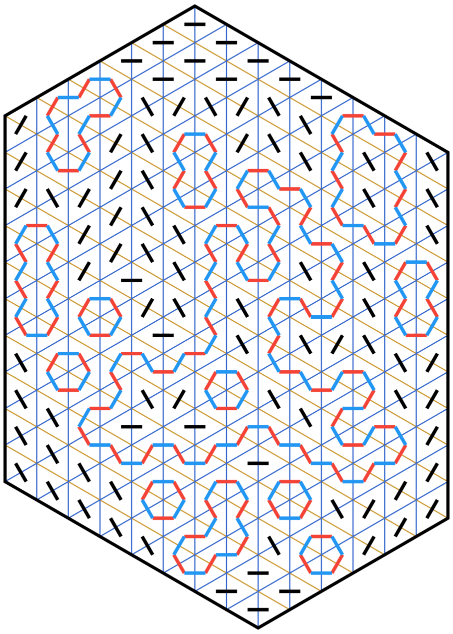

Figure 6: A uniform random double-dimer configuration on a large lozenge graph ($n = 188$), illustrating the macroscopic loop structure that emerges from the superposition of two independent perfect matchings.

References

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[Pro03] J. Propp. Generalized domino-shuffling. *Theoretical Computer Science*, 303(2-3):267–301, 2003. arXiv:math/0111034 [math.CO]. 1, 3

[Thu90] W. P. Thurston. Conway's tiling groups. *Amer. Math. Monthly*, 97:757–773, 1990. 1, 3