COMBINATORICS. EXTRA PROBLEMS

Main topics of the course:

(1) Generating functions
(2) Catalan numbers
(3) Nonintersecting paths
(4) Generating functions and asymptotics
(5) Partitions and their generating functions
(6) Dirichlet generating functions. Möbius inversion and multiplicative arithmetic functions
(7) Enumeration of trees and exponential formula
(8) Counting spanning trees
(9) Enumeration of embedded graphs

Problem 1. Find $[x^n]\dfrac{1}{(a-x)(b-2x)}$ (where $a \neq \frac{b}{2}$).

Problem 2. Using generating functions, solve the recurrence relation

$$a_{n+1} = 3a_n + 4n(n-1) \quad (\text{where } n \geq 0), \qquad a_0 = -1.$$

(Be very careful in computations! Check your answer, it is very nice and *simple*. You may also use exponential generating functions, this will lead to differential equations. I do not know which way is quicker or simpler.)

Problem 3. Consider the (non-usual!) power series $F(u) := \sum_{k=0}^{\infty} u^{-k} = 1+\dfrac{1}{u}+\dfrac{1}{u^2}+\ldots$. (That is, the center of the power series expansion is $u = \infty$ instead of the usual $u = 0$.) Compute similar power series expansions for $F(u-1)$ and $F(u+1)$. (Note that you substitute inside $F(u)$ something with non-zero free term, but here this operation is algebraically *legal*.)

Problem 4. Show that the number of triangulations of $(n+2)$-polygon with $n-1$ diagonals is the Catalan number C_n (this is Problem 3.20). Give *two* proofs of this fact. (Both were discussed in class. You may use the planar trees interpretation for Catalan numbers, the Catalan recurrence relation, etc.)

Problem 5. Consider the sequence $a_n := \dfrac{n+1}{n+2}\dbinom{2n+1}{n+1}$ (where $n \geq 0$). The generating function for a_n is

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = \frac{1-2x-\sqrt{1-4x}}{4x^2\sqrt{1-4x}}.$$

Using this generating function,

(1) Find the singular points of $f(x)$ and thus find R such that a_n grows slower than $(\frac{1}{R}+\epsilon)^n$ for any $\epsilon > 0$.
(2) Compute a_{n+1}/a_n. Ensure that $\{a_n\}$ is a hypergeometric sequence (see problem set 5). Find hypergeometric-type asymptotics for a_n.
(3) Using Stirling's formula, find *exact* asymptotics for a_n.

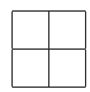
(Of course, in all three cases your answers should agree with each other.)

Problem 6. Find the generation function for partitions $\sum q^{|\lambda|}$, where the sum is taken over all partitions λ in which all even parts must be distinct (and no restrictions on odd parts). Example of a partition we sum over: $\lambda = (7, 7, 6, 3, 3, 3, 2, 1)$.

Problem 7. Let $f_k(n)$ be the arithmetic function $f_k(n) := GCD(k, n)$ (greatest common divisor; $k \geq 1$ is a fixed integer). By a simple number-theoretic argument show that for any k, $f_k(n)$ is a multiplicative arithmetic function in n. Compute the Dirichlet generating function ($k = 6$):

$$F_6(s) := \sum_{n=1}^{\infty} \frac{f_6(n)}{n^s}$$

(you may use Problem 8.6 which is a "key" to our problems about multiplicative arithmetic functions).

Problem 8. Consider the 3×3 grid graph . Write the Laplacian matrix, and compute the number of spanning trees of this graph. (The Laplacian matrix is 9×9, so to compute the number of spanning trees you may want to use a computer.)

Problem 9. Compute the number of labeled rooted forests on n vertices such that every component is a star-shaped tree (i.e., it has one central vertex and every other vertex is attached to it with an edge).

Problem 10. Consider the graph with one vertex and three loops attached to this vertex (i.e., there are 6 half-edges around this vertex). Describe all surfaces in which one can embed this graph. (Different embeddings differ by the cycle order of half-edges around this vertex.)