COMBINATORICS. PROBLEM SET 9.
MÖBIUS FUNCTION AND OTHER ARITHMETIC FUNCTIONS

Seminar/homework problems

Problem 9.1 (*1*)**.** Show that if the arithmetic function $f(n)$ is multiplicative, then so is the function $g(n) := \sum_{d\colon\, d|n} f(d)$.

Problem 9.2 (*2*)**.** Show that if the arithmetic function $f(n)$ is multiplicative, then so is the function $h(n) := \sum_{\delta\colon\, \delta|n} f(\delta)d(\frac{n}{\delta})$, where $d(m)$ is the number of all divisors of m.

Problem 9.3 (*1*)**.** Let $\lambda(n) = (-1)^k$, where k is the number of all prime factors in n counted with multiplicities (e.g., $\lambda(18) = \lambda(2 \cdot 3^2) = 3$). Show that $\lambda(n)$ is a multiplicative arithmetic function.

Problem 9.4 (*2*)**.** Let $\lambda(s) = \sum_{n=1}^{\infty} \dfrac{\lambda(n)}{n^s}$ be the Dirichlet GF of the sequence $\lambda(n)$. Find the Dirichlet GF $\lambda(s)\zeta(s)$, where $\zeta(s)$ is the Riemann zeta function.

Problem 9.5 (*1*)**.** Show that the Dirichlet GF of the sequence $\{n\}_{n=1}^{\infty}$ is $\zeta(s-1)$.

Problem 9.6 (*1*)**.** Show that the Dirichlet GF of the sequence $\{n^\alpha\}_{n=1}^{\infty}$ is $\zeta(s-\alpha)$.

Problem 9.7 (*2*)**.** Show that the Dirichlet GF of the sequence $\{\log n\}_{n=1}^{\infty}$ is $-\zeta'(s)$.

Problem 9.8 (*2*)**.** Show that the Dirichlet GF of the sequence $\left\{ \sum_{d\colon\, d|n} d^q \right\}_{n=1}^{\infty}$ is $\zeta(s)\zeta(s-q)$.

Problem 9.9 (*2*)**.** By using the fact that $\zeta(s)\dfrac{\zeta(s-1)}{\zeta(s)} = \zeta(s-1)$, show that for $n \geq 2$ one has $\sum_{d\colon\, d|n} \varphi(d) = n$, where $\varphi(m)$ be the number of numbers among $\{1, \ldots, m-1\}$ which are relatively prime to m.

Problem 9.10 (*2*)**.** What is the number of necklaces with 14 gems made of emeralds, rubys and diamonds? (one can take any number of gems of any type; we do not distinguish necklaces by rotation, but do distinguish necklaces by mirroring: any necklace has a “face” and “back” sides).