# COMBINATORICS. PROBLEM SET 7. PARTITIONS II

## Seminar problems

**Problem 7.1.** Show that the generating functions of the number of partitions of $n$ with following constraints are:

(1) partitions into odd parts: $\dfrac{1}{(1-q)(1-q^3)(1-q^5)(1-q^7)\dots}$;
(2) partitions with distinct parts (so-called *strict* partitions): $(1+q)(1+q^2)(1+q^3)(1+q^4)\dots$;
(3) partitions with distinct odd parts: $(1+q)(1+q^3)(1+q^5)(1+q^7)\dots$;

**Problem 7.2.** Show that the number of partitions of $n$ with exactly $k$ parts is the same as the number of *strict* partitions of $n+\frac{k(k-1)}{2}$ with exactly $k$ parts.

**Problem 7.3** (Euler pentagonal theorem)**.** Show that $(1-q)(1-q^2)(1-q^3)\cdots=\sum_{k=-\infty}^{+\infty}(-1)^k q^{\frac{3k^2+k}{2}}$.

**Problem 7.4.** For a Young diagram $\lambda$, let $f_\lambda$ be the number of ways to grow the diagram $\lambda$ by adding one box at a time (= the *number of standard tableaux* of shape $\lambda$). Show that $\sum_{\lambda:\,|\lambda|=n} f_\lambda^2=n!$

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**Problem 7.5.** Find the generating function for the number of partitions of $n$ with all parts not exceeding $N$ ($N$ is a given fixed number).

**Problem 7.6.** Let $Sym(N)$ be the space of all symmetric polynomials in $N$ variables $z_1,\dots,z_N$. Let $Sym_n(N)$ denote the subspace of *homogeneous* symmetric polynomials of degree $n$. Find the generating function for $\dim Sym_n(N)$. (Here $N$ is fixed.)

**Problem 7.7.** Similarly to Problem 7.6, consider the space $Sym$ of all symmetric polynomials in infinitely many variables $z_1,z_2,\dots$ (examples: $z_1+z_2+z_3+\dots$, or $z_1z_2^2+z_1^2z_2+z_1z_3^2+z_1^2z_3+\dots$). Find the generating function for the sequence $\dim Sym_n$, where $Sym_n$ is the space of homogeneous symmetric polynomials of degree $n$.

**Problem 7.8.** Let $c_n$ be the number of all Young diagrams with *perimeter* $n$. Find the generating function $\sum_{n=0}^{\infty} c_n q^n$.

## Homework

**Problem 7.9.** Let $psym_n$ be the number of symmetric partitions $\lambda$ of $n$ (= the number of symmetric wrt transposition Young diagrams with $n$ boxes). Find the generating function $PSYM(q):=\sum_{n=0}^{\infty} psym_n q^n$.

**Problem 7.10.** Show that the number of all partitions of $n$ is the same as the number of partitions of $2n$ with exactly $n$ parts.

**Problem 7.11.** Show that the number of partitions of $n$ with $\le k$ parts is the same as the number of *strict* partitions of $n+\frac{k(k+1)}{2}$ with exactly $k$ parts.

**Problem 7.12.** Using Problem 7.5, find the generating function for the number of partitions of $n$ with $\le N$ parts (here $N$ is also a given fixed number). (Hint: use Young diagram representation of partitions.)

**Problem 7.13.** Let $f_\lambda$ be as defined in Problem 7.4. Find an explicit formula for $f_{(n,n)}$, where $(n,n)$ is the Young diagram with two rows of length $n$, $n=0,1,2,\dots$.