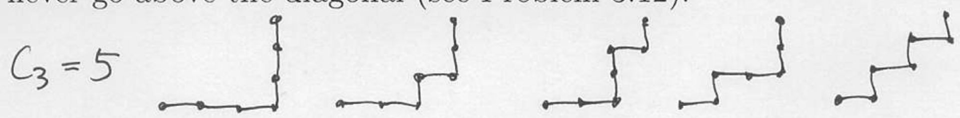
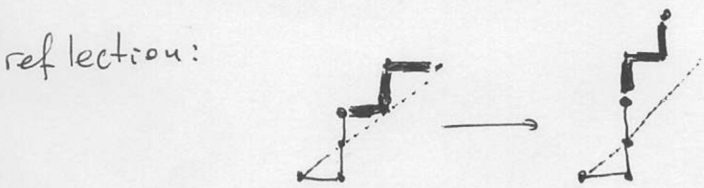
# COMBINATORICS
# PROBLEM SET 4
# PATHS, NONINTERSECTING PATHS AND MORE

## Seminar problems

**Problem 4.1.** Consider the square lattice. *Up-right* paths on this lattice are paths which go only up or right. *Diagonal* is the line that goes through the points $(0,0),(1,1),(2,2),\dots$.

We know that the Catalan number $C_n$ is the number of up-right paths from $(0,0)$ to $(n,n)$ which never go above the diagonal (see Problem 3.12):




(a) Use the *paths reflection principle* to show that the number of up-right paths from $(0,0)$ to $(n,n)$ which *cross* (= go above) the diagonal is the same as the number of *any* up-right paths on the lattice from $(0,0)$ to $(n-1,n+1)$. (This is a bijection argument.)

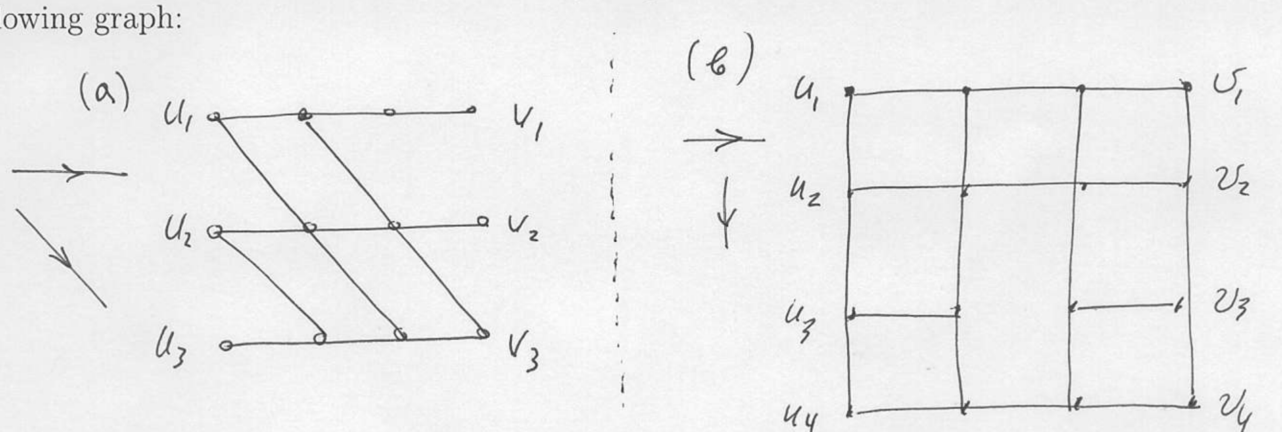
(b) Using (a) and the definition of the binomial coefficients as numbers of certain up-right paths, deduce $C_n = \binom{2n}{n} - \binom{2n}{n-1}$. Simplify this to get the known explicit expression for $C_n$.

**Problem 4.2.** Compute the number of up-right paths on the square lattice from $(0,0)$ to $(n+1,n)$ which never go above the diagonal. Argue as follows:

(a) Use the paths reflection principle to show that the number of up-right paths from $(0,0)$ to $(n+1,n)$ which *cross* (= go above) the diagonal is the same as the number of *all* up-right paths on the lattice from $(0,0)$ to $(n-1,n+2)$.

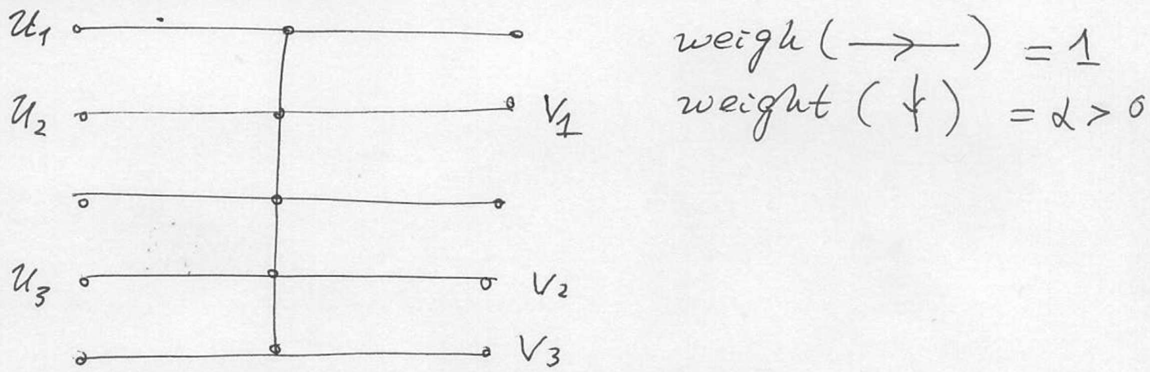
(b) The number of paths we want is $\binom{2n+1}{n} - \binom{2n+1}{n-1}$. Simplify this to get the answer.

**Problem 4.3.** Write KMGLGV matrix $X_{ij}$ ($X_{ij}$ is the number of paths from $u_i$ to $v_j$) for the following graph:




Compute the determinant of the matrix. What does it represent?

**Problem 4.4.** Write KMGLGV matrix $X_{ij}$ ($X_{ij}$ is the sum of weights of all paths from $u_i$ to $v_j$) for the following graph *with weights*:




Compute the determinant of the matrix. What does it represent?

**Problem 4.5.** Compute the determinant by hand: $\det \begin{bmatrix} C_0 & C_1 & C_2 \\ C_1 & C_2 & C_3 \\ C_2 & C_3 & C_4 \end{bmatrix}$.

**Problem 4.6.** A matrix in which each $ij$th element depends only on $i+j$, is called a *Hankel matrix*. Use KMGLGV lemma (= nonintersecting paths lemma) to compute the following determinants of Hankel matrices ($C_n$ is the $n$th Catalan number):

$$\text{(a) } \det[C_{i+j-2}]_{i,j=1}^n = 1 \qquad \text{(b) } \det[C_{i+j-1}]_{i,j=1}^n = 1$$

(for all $n = 1, 2, \dots$).

**Problem 4.7.** Let $X_{ij}$ be a KMGLGV matrix of some graph; take its square submatrix (formed by some rows and columns). What does the determinant of this submatrix (called a *minor* of the initial matrix) represent?

## Homework

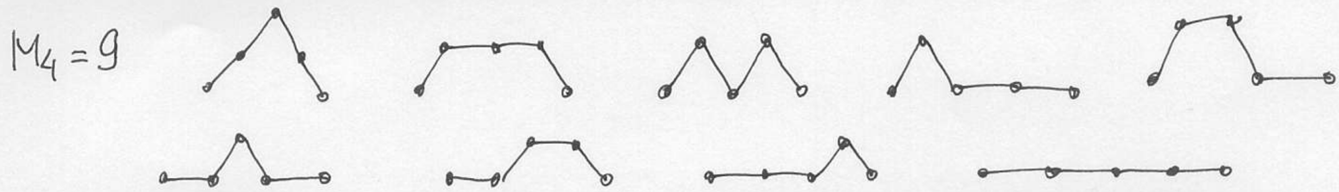
**Problem 4.8** (*2*). Let $f(x) = a_0 + a_1x + a_2x^2 + \dots$ be the generating function of a sequence $\{a_n \dots\}_{n=0}^\infty$. Consider the sequence $\left\{\sum_{j=0}^n a_j\right\}_{n=0}^\infty$, i.e., the sequence starts as $a_0, a_0 + a_1, a_0 + a_1 + a_2, \dots$. Show that the generating function of this new sequence is $\frac{f(x)}{1-x}$.

**Problem 4.9.** Let $\{F_n\}_{n=0}^\infty = \{0, 1, 1, 2, 3, 5, 8, 13, \dots\}$ be the Fibonacci numbers, their generating function is

$$\sum_{n=0}^\infty F_n x^n = \frac{x}{1-x-x^2}.$$

Show (using generating functions) that
(a) (*2*) $F_0 + F_1 + \dots + F_n = F_{n+2} - 1$;
(b) (*2*) $F_0 + F_2 + F_4 + \dots + F_{2n} = F_{2n+1} - 1$;
(c) (*2*) $F_1 + F_3 + F_5 + \dots + F_{2n-1} = F_{2n}$;

**Problem 4.10** (Motzkin numbers (*3*)). Motzkin paths are the same as Dyck paths (see Problem 3.7) but can contain steps $(0, 1)$:

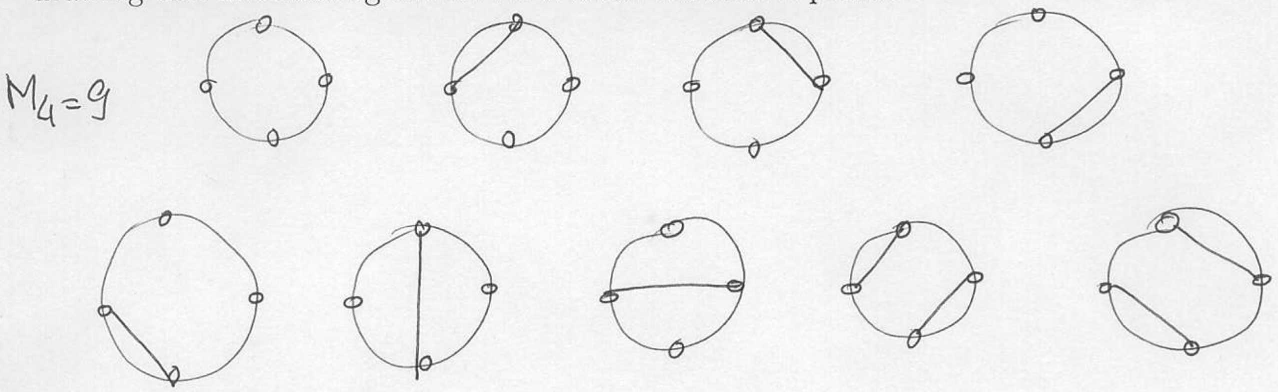



Let $M_n$ be the $n$th Motzkin number — the number of Motzkin paths (by definition, paths with steps $(1, 1), (1, 0)$ and $(1, -1)$ and never going below the $x$-axis) from $(0, 0)$ to $(n, n)$. Show that the Motzkin numbers satisfy the recurrence

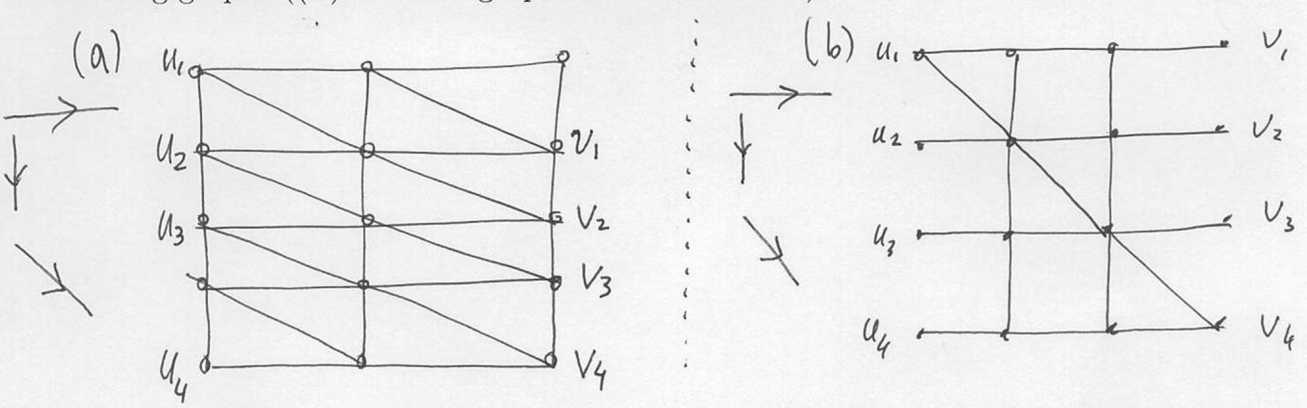
$$M_n = M_{n-1} + \sum_{k=0}^{n-2} M_k M_{n-2-k}, \qquad n \geq 2$$

(the argument is very similar to the Catalan recurrence). Find the equation on the generating function $1 + \sum_{n=1}^{\infty} M_n x^n$ for $M_n$, and then find the generating function itself.

**Problem 4.11** (*2*). Show that the $n$th Motzkin number is the number of different ways of drawing non-intersecting chords on a circle between $n$ points:

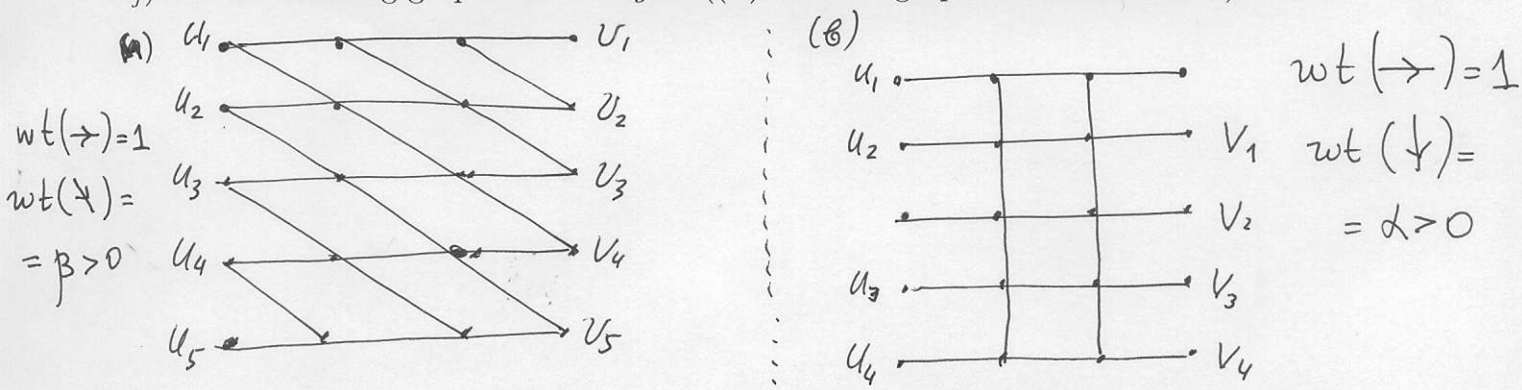



**Problem 4.12.** Write KMGLGV matrix $X_{ij}$ ($X_{ij}$ is the number of paths from $u_i$ to $v_j$) for the following graphs ((*1*) for each graph and determinant):




Compute the determinants of these matrices. What do these determinants represent?

**Problem 4.13.** Write KMGLGV matrix $X_{ij}$ ($X_{ij}$ is the sum of weights of all paths from $u_i$ to $v_j$) for the following graphs *with weights* ((*1*) for each graph and determinant):




Compute the determinants of these matrices. What do these determinants represent?

**Problem 4.14.** Compute the following determinants by hand ($C_n$ is the $n$th Catalan number, $C_0 = C_1 = 1$, $C_2 = 2$, ...):

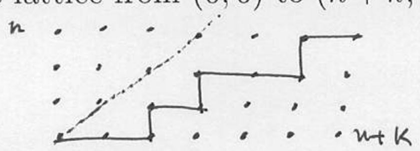
(a) (*1*) $\det \begin{bmatrix} C_1 & C_2 & C_3 \\ C_2 & C_3 & C_4 \\ C_3 & C_4 & C_5 \end{bmatrix}$ (b) (*1*) $\det \begin{bmatrix} C_2 & C_3 & C_4 \\ C_3 & C_4 & C_5 \\ C_4 & C_5 & C_6 \end{bmatrix}$

**Problem 4.15.** Compute the following determinants by hand ($M_n$ is the $n$th Motzkin number, see Problem 4.10, $M_0 = M_1 = 1$, $M_2 = 2$, ...):

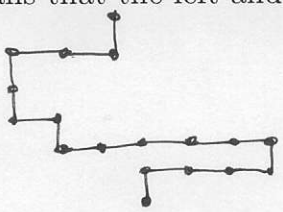
(a) (*1*) $\det \begin{bmatrix} M_0 & M_1 & M_2 & M_3 \\ M_1 & M_2 & M_3 & M_4 \\ M_2 & M_3 & M_4 & M_5 \\ M_3 & M_4 & M_5 & M_6 \end{bmatrix}$ (b) (*1*) $\det \begin{bmatrix} M_1 & M_2 & M_3 \\ M_2 & M_3 & M_4 \\ M_3 & M_4 & M_5 \end{bmatrix}$

**Problem 4.16** (*4*). Use KMGLGV lemma to show that the Hankel determinants $\det[M_{i+j-2}]_{i,j=1}^{n}$ ($M_n$ is the $n$th Motzkin number) are equal to one for all $n = 1, 2, \ldots$.

**Problem 4.17** (*3*). Let $k \geq 0$. Arguing as in Problems 4.1 and 4.2, compute the number of up-right paths on the lattice from $(0, 0)$ to $(n + k, n)$ which never go above the diagonal:




**Problem 4.18** (*4*). Consider paths on the square lattice with steps $(0, 1)$ (up), $(1, 0)$ (right), and $(-1, 0)$ (left). Let $P_n$ be the number of such paths of length $n$ ($n = 1, 2, \ldots$) with no self-intersections (hint: this means that the left and right steps cannot be neighbors):



Find a recurrence relation for the $P_n$'s and show that their generating function is

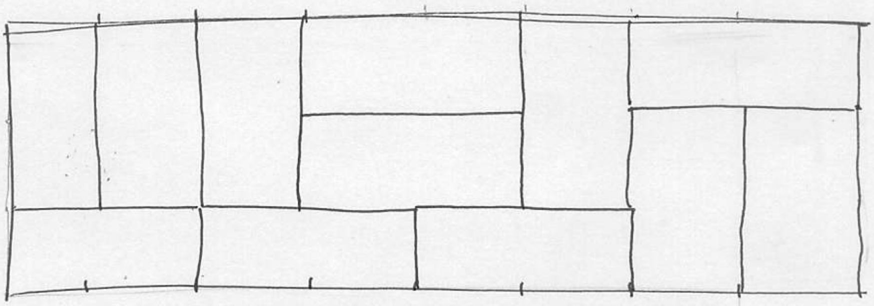
$$1 + \sum_{n=1}^{\infty} P_n x^n = \frac{1 + x}{1 - 2x - x^2}.$$

(Such paths serve as a simple model for *polymers*.)

## Supplementary problems

**Problem 4.19** (*3*). Show that (in notation of Problem 4.9) $F_0^2 + F_1^2 + \cdots + F_n^2 = F_n F_{n+1}$.

**Problem 4.20** (*5*). Let $a_n$ be the number of ways in which the rectangle $3 \times 2n$ can be tiled by $1 \times 2$ dominoes:



Write a recurrence for $a_n$, and then find the generating function $1 + \sum_{n=1}^{\infty} a_n x^n$.

**Problem 4.21** (*3*). Use KMGLGV lemma (= nonintersecting paths lemma) to compute the following determinant: $\det[C_{i+j}]_{i,j=1}^n = n + 1$ (for all $n = 1, 2, \ldots$).

**Problem 4.22.** Let $T = 1, 2, \ldots$, let $0 \le a_1 < \cdots < a_n$ and $0 \le b_1 < \cdots < b_n$. Show that the determinant

$$\det \left[ \binom{T + b_i - a_j}{b_i - a_j} \right]_{i,j=1}^n$$

is nonnegative.

Here (as usual) we use the agreement $\binom{N}{k} = 0$ for $k < 0$ or $k > N$. For example, the above determinant of order 3 has the form

$$\det \begin{bmatrix} \binom{T+b_1-a_1}{b_1-a_1} & \binom{T+b_1-a_2}{b_1-a_2} & \binom{T+b_1-a_3}{b_1-a_3} \\ \binom{T+b_2-a_1}{b_2-a_1} & \binom{T+b_2-a_2}{b_2-a_2} & \binom{T+b_2-a_3}{b_2-a_3} \\ \binom{T+b_3-a_1}{b_3-a_1} & \binom{T+b_3-a_2}{b_3-a_2} & \binom{T+b_3-a_3}{b_3-a_3} \end{bmatrix}.$$

Hint. Use the KMGLGV lemma for the square lattice: