COMBINATORICS.
PROBLEM SET 1.
BASIC ENUMERATION

SEMINAR PROBLEMS

Problem 1.1. There are 7 different roads from Moscow to Yekaterinburg, and 3 different roads from Yekaterinburg to Novosibirsk. How many ways are to drive from Moscow to Novosibirsk (through Yekaterinburg)?

Problem 1.2. What is the number of 4-digit numbers (not starting from zero) with all distinct digits?

Problem 1.3. How many ways are there to arrange $n \geq 1$ people to sit around a (round) table, if we do not distinguish arrangements which are different only by a rotation?

Problem 1.4. What is the total number of ways in which the letters in the word "MOSCOW" can be arranged?

Problem 1.5. There are two parallel lines on the plane; one has p marked points, and the other has q marked points. What is the total number of triangles with vertices at the marked points?

Problem 1.6. In the shop there are candies of four different kinds A,B,C, and D. How many ways are there to buy 7 candies?

Problem 1.7. Prove the Binomial Theorem: $(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$.

Problem 1.8. Give a combinatorial proof of the fact that $\binom{n}{k} = \binom{n}{n-k}$.

Problem 1.9. Give a combinatorial proof of the fact that $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$.

Problem 1.10. Prove using a combinatorial argument:
$$\binom{n}{1} + 2\binom{n}{2} = n^2.$$

Problem 1.11. Give a combinatorial proof of the identity
$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n.$$

Problem 1.12. Give a combinatorial proof of the identity
$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots + (-1)^n \binom{n}{n} = \begin{cases} 0, & \text{if } n > 0; \\ 1, & \text{if } n = 0. \end{cases}$$

Problem 1.13. Give a combinatorial proof of the identity
$$\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n}.$$

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Homework

Problem 1.14. There are 7 different roads from Moscow to Yekaterinburg, and 3 different roads from Yekaterinburg to Novosibirsk. (a) (*1*) How many ways are to drive from Moscow to Novosibirsk and back not using *one way* twice? (b) (*1*) How many ways are to drive from Moscow to Novosibirsk and back never using the *same road* twice?

Problem 1.15 (*1*)**.** You have 4 pairs of pants, 7 sweaters, and 5 hats. What is the total number of different outfits you can assemble?

Problem 1.16 (*1*)**.** Some ancient tribe had a language with 6 vowels and 8 consonants. In every word vowels and consonants must be interlacing. How many 9-letter words were in that language?

Problem 1.17 (*2*)**.** A man forgot his 5-digit PIN code (e.g. 03462) but only remembers that the code contains substrings of the form "92" and "89" (e.g. 89392, but not 90289). How many times must he attempt at entering the code to certainly enter a correct code?

Problem 1.18 (*1*)**.** How many 5-letter words containing at least one "a" can be made out of 26 letters?

Problem 1.19 (*2*)**.** What is the total number of ways in which the letters in the word "BOOKKEEPER" can be arranged?

Problem 1.20. Prove using a combinatorial argument:

(a) (*2*) $\binom{n}{1} + 6\binom{n}{2} + 6\binom{n}{3} = n^3$;   (b) (*2*) $\binom{n}{1} + 14\binom{n}{2} + 36\binom{n}{3} + 24\binom{n}{4} = n^4$.

Problem 1.21 (*2*)**.** Deduce the identities

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n$$

and

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots + (-1)^n \binom{n}{n} = 0$$

from the Binomial Theorem.

Problem 1.22 (*2*)**.** How many numbers from 0 to 999 999 do not have two identical digits as neighbors (i.e., we forbid numbers such as 184492)?

Problem 1.23 (*3*)**.** What is the total number of injective maps $f\colon \{1,\ldots,6\} \to \{1,\ldots,6\}$ such that $|f(1) - f(4)| \geq 2$?

Problem 1.24 (*3*)**.** Find the sum: $\binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \cdots + n\binom{n}{n}$.

Problem 1.25 (*2*)**.** There are n telephone subscribers (customers) in a network. What is the number of ways to organize three conversations among these subscribers (a conversation is an event when two subscribers connect to each other)?

Problem 1.26 (*2*)**.** Find the number of ways to choose 4 different 7-subsets out of a subset with 30 elements.

Problem 1.27 (*1*)**.** What is the number of ways to choose a team out of 11 men with some number ≥ 1 of members in the team?

Problem 1.28 (*3*)**.** Compute the sum of all numbers which are obtained by permuting the digits 1,2,3,4,5.

Supplementary problems

Problem 1.29 (*3*)**.** What is the total number of positive divisors of the number $N = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_k^{\alpha_k}$, where $p_1, \ldots, p_k$ are pairwise distinct primes?

Problem 1.30 (*3*)**.** What is the number of ways to choose three numbers from 1 to 100 such that their sum is divisible by 3?

Problem 1.31 (*4*)**.** Find the sum: $\binom{n}{0} + \binom{n}{3} + \binom{n}{6} + \ldots$.