COMBINATORICS. PROBLEM SET 12
GRAPHS ON SURFACES

By graphs here we always mean connected non-oriented graphs with loops allowed.

Definition 1. *Surface* = closed orientable two-dimensional manifold. It is homeomorphic to a sphere with a number of handles attached. This number g is called the *genus* of the surface ($g = 0$ — the sphere, $g = 1$ — the torus, etc).

Definition 2. Let Γ be an abstract connected graph. An *embedding* of this graph into a surface M is a drawing of the graph on the surface such that

- Each vertex of the graph is represented by a point in M, and distinct vertices are represented by distinct points;
- Each edge of the graph is represented by a non-selfintersecting curvilinear segment in M, with the ends of the segment coinciding with the vertices connected by the edge. No two segments intersect each other;
- **The complement of the image of Γ in M is a disjoint union of *cells* (two-dimensional domains homeomorphic to the disc).**

A *planar graph* is the same as the graph embedded into the sphere (because plane = sphere without one point).

Seminar problems

Problem 12.1. Recall the definition of a (rooted or unrooted) plane tree. Compare it with the definition of an embedded graph.

Problem 12.2 (Planar Euler formula)**.** Let Γ be a planar graph with V vertices, E edges and F faces (= cells). Show that $V - E + F = 2$. See also Problem 12.5.

Problem 12.3. From Problem 12.5 deduce the following estimate: one cannot embed a graph into a surface of genus $\geq (E - V + 1)/2$.

Definition 3. A *graph with rotations* is a graph endowed with a cyclic order on each set of half-edges issuing from each of its vertices.

Problem 12.4. Show that for any graph with rotations there is a unique surface into which it can be embedded. (Of course, the embedding should agree with the cyclic order!)

Homework/seminar problems

Problem 12.5 (Euler formula)**.** (*3*) Show that for any graph embedded into a surface of genus g, one has (see Problem 12.2) $V-E+F=2-2g$. The number $\chi_g := 2-2g$ is called the *Euler characteristic* of the surface.

Problem 12.6 (*1*)**.** Invent an abstract graph which can be embedded into some surface in two different (= non-homeomorphic) ways.

Problem 12.7 (*2*)**.** Consider a connected graph with 3 vertices having 4 half-edges in two vertices and 2 half-edges in the third vertex, i.e., think of a picture like this:

$$- \cdot - \qquad > \cdot < \qquad > \cdot <$$

Draw all possible graphs which have this described "local structure" (note that loops are allowed).

Problem 12.8 (*4*)**.** Let K_5 be the complete graph with 5 vertices (i.e., every vertex is connected to every other one with exactly one edge). Let $K_{3,3}$ be the complete bipartite graph (i.e., vertices are indexed by $1,2,3$ and $1',2',3'$, and every vertex i is connected to every vertex j'). Using Euler formula, show that K_5 and $K_{3,3}$ cannot be embedded into the sphere. (Hint: use the fact that every face must have ≥ 3 edges. Count semi-edges attached to each face. Example: the triangle embedded into the sphere has 6 semi-edges and two triangular faces.)

Problem 12.9 (*4*)**.** Using Problem 12.3 and the previous problem, show that if the complete graph K_n is embedded into a surface of genus g, then

$$\tfrac{1}{2}\left(2-n+\tfrac{1}{3}\tbinom{n}{2}\right) \le g \le \tfrac{1}{2}\left(\tbinom{n}{2}-n+1\right). \tag{12.1}$$

Problem 12.10 (*2*)**.** Let Γ be an abstract connected graph. An *Eulerian trail* is a path in this graph which visits every edge only once. An *odd* vertex in the graph is a vertex of odd degree (i.e., an odd number of half-edges enter this vertex).

Show that there exists an Eulerian trail iff the number of odd vertices in the graph is ≤ 2.

Problem 12.11 (*1*)**.** Can there be exactly one odd vertex in a graph?

Supplementary problem

Problem 12.12 (*6*)**.** Look at formula (12.1) for $n=7$. Deduce that in principle the graph K_7 can be embedded into the torus. Construct such an embedding. (Hint: use the concept of a *dual graph* with vertices being the faces of the original graph, and new vertices are connected by a new edge iff the original faces are adjacent.)