Solving TASEP via Schur polynomials ①

- TASEP reminder
- Step IC (only)
- Goals: to connect TASEP to Schur
  - to get probabilistic information from Schur
  - : asymptotics of Schur s (Axel – next time)

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2 d dynamics on tilings [Borodin – Ferrari 2008]

(earlier: Johansson 00 for a different dynamics related to RSK; — we do not discuss RSK.)

Step IC; blocking pushing

→ obvious connection to TASEP (w. Step IC)

Discussed: after time t, the distrib. at level k satisfies

$$\boxed{\prod_{i=1}^{k} e^{t(z_i-1)} = \sum_{\mu_1\ge\dots\ge\mu_k} P_t(\mu)\,\frac{s_\mu(z_1,\dots,z_k)}{s_\mu(1,\dots,1)}}$$

goal: express P_t via Schur (= solve g.f.)

(do not prove this fact; it uses skew Cauchy identities & nothing else. It interprets them as probab. statements)

Schur measures [Ok. 00]

$$P(\lambda) = \frac{1}{Z} s_\lambda(a_1,\dots,a_N)\, s_\lambda(b_1,\dots,b_M), \quad \lambda \text{ - all y.d.}$$

$\boxed{\text{Cauchy id.}}$ $$\sum_\lambda s_\lambda(a_1,\dots,a_N)\, s_\lambda(b_1,\dots,b_M) = \prod_{i,j} \frac{1}{1-a_i b_j}$$

prove via dets!

(only in Schur; repr. th. meaning too)

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Proof of Cauchy id, $N=M$ (can set zeros to compensate, $s_\lambda(\_\_\_, 0) = s_\lambda(\_\_\_\_)$)

$$\sum_\lambda \frac{\det\left[a_i^{\lambda_j+N-j}\right]\det\left[b_i^{\lambda_j+N-j}\right]}{V(a)V(b)}$$

$$= \underbrace{\sum_\lambda}_{} \sum_{\sigma,\tau} \frac{(-1)^{\sigma+\tau}}{V(a)V(b)} \prod_{i=1}^N a_{\sigma(i)}^{\lambda_i+N-j}\, b_{\tau(i)}^{\lambda_i+N-i}$$

goes inside, $*\, N!$ and sum over arbitrary collections of indices $\{\lambda_i+N-i \geq 0\}$

$$= N! \sum_{\sigma,\tau} \frac{(-1)^{\sigma+\tau}}{V(a)V(b)} \prod_i \frac{1}{1-a_{\sigma(i)}b_{\tau(j)}}$$

$$= \frac{1}{V(a)V(b)} \det\left[\frac{1}{1-a_ib_j}\right]_{i,j=1}^N$$

Cauchy det of $\left(\frac{1}{x_i+y_j}\right)$ – eval. by induction & row/col transformations

$$= \text{eventually} = \boxed{\prod_{i,j}\frac{1}{1-a_ib_j}}$$

$$\det\left[\frac{1}{x_i+y_j}\right] = \frac{V(x)V(y)}{\prod_{i,j}(x_i+y_j)}$$

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Limit: $b_i = \frac{t}{M}$, $M \to \infty$

$$\Rightarrow \prod_{i,j} \frac{1}{1-a_i b_j} = \prod_i \frac{1}{(1-a_i t/M)^M} \longrightarrow \boxed{e^{t\sum a_i}}$$

let us denote lim. of $S_\lambda(b_1, \dots b_M)$ (it exists!)

by $S_\lambda(\rho_t)$.

So,

$$\underset{e^{-tN}}{\nearrow} e^{t\sum a_i} = \sum_\lambda S_\lambda(a_1 \dots a_N) S_\lambda(\rho_t)$$

$$= \sum_\lambda \frac{S_\lambda(a_1 \dots a_N)}{S_\lambda(1 \dots 1)} S_\lambda(\rho_t) S_\lambda(1 \dots 1) \underset{e^{-tN}}{\nearrow}$$

$$\Rightarrow \boxed{\mathrm{Prob}_t(\mu_1 \dots \mu_K) = e^{-tK} S_\lambda(\overbrace{1 \dots 1}^{K}) S_\lambda(\rho_t)}$$

So, we connected TASEP to Schur measures.

How to solve Schur measures?

& In what sense solve?

$$P(\lambda) = \frac{1}{Z} S_\lambda(a_1 \dots a_N) S_\lambda(b_1 \dots b_M)$$

$\longrightarrow$ Okounkov: via math phys (infinite wedge Fock space)

$\longrightarrow$ Borodin: via biorthog. ensembles

$$P(x_1 \dots x_N) = \frac{1}{Z} \det[\varphi_i(x_j)] \det[\psi_i(x_j)]$$

$\longrightarrow$ etc.

Solve in a sense of <u>determinantal point processes</u>

$$\lambda \longrightarrow \text{configuration } \{x_i - i\}_{i\geq 1}$$

$$(4,3,1,0,0,\dots) \longrightarrow (3,1,-2,-4,-5,-6,\dots)$$

Fact: $\exists\ K(x,y)$; $x,y\in\mathbb{Z}$ s.t.

$$P(\exists \text{ points of } \{x_i - i\} \text{ at } y_1,\dots,y_k \in \mathbb{Z}) = \det\left[K(y_i,y_j)\right]_{i,j=1}^{k}$$

for any <u>pairwise distinct</u> $y_1 \dots y_k$.

$K(x,y)$ is explicit: (for $a_1 \dots a_N \mid b_1 \dots b_M$)

$$K(x,y) = \frac{1}{(2\pi i)^2}\oint\oint_{|w|<|v|} \frac{H(v)}{H(w)}\,\frac{\overline{H}(w^{-1})}{\overline{H}(v^{-1})}\,\frac{dv\,dw}{(v-w)\,v^{x+1}\,w^{-y}}$$

(+ certain contours – condit. to have Taylor exp. in denom.)

$$H(v) = \prod_{i=1}^{N}\frac{1}{1-va_i}$$

$$\overline{H}(v) = \prod_{i=1}^{M}\frac{1}{1-vb_i}$$

For TASEP: $a_i \equiv 1$ & $\boxed{\overline{H}(v) = e^{v}}$.

Back to TASEP:

$x_N \overset{\mathcal{D}}{=} \lambda_N - N$ where λ_N is the N-th coord. in λ distr. as Schur measure

$$\frac{1}{Z} s_\lambda(\underbrace{1,\dots,1}_{N})\, s_\lambda(\rho_t),$$

and for this measure we know the correlation kernel

This opens up a path to asymptotic analysis.