Chapter VIII
The Yang-Baxter Equation and (Co)Braided Bialgebras

Part II is centered around the now famous Yang-Baxter equation whose solutions are the so-called R-matrices. We introduce the concept of braided bialgebras due to Drinfeld. These are bialgebras with a universal R-matrix inducing a solution of the Yang-Baxter equation on any of their modules. This provides a systematic method to produce solutions of the Yang-Baxter equation. There is a dual notion of cobraided bialgebras. We show how to construct a cobraided bialgebra out of any solution of the Yang-Baxter equation by a method due to Faddeev, Reshetikhin and Takhtadjian [RTF89]. We conclude this chapter by proving that the quantum groups $GL_q(2)$ and $SL_q(2)$ of Chapter IV can be obtained by this method and that they are cobraided.

VIII.1 The Yang-Baxter Equation

Definition VIII.1.1. *Let V be a vector space over a field k. A linear automorphism c of $V \otimes V$ is said to be an R-matrix if it is a solution of the Yang-Baxter equation*

$$(c \otimes \mathrm{id}_V)(\mathrm{id}_V \otimes c)(c \otimes \mathrm{id}_V) = (\mathrm{id}_V \otimes c)(c \otimes \mathrm{id}_V)(\mathrm{id}_V \otimes c)$$

that holds in the automorphism group of $V \otimes V \otimes V$.

Finding all solutions of the Yang-Baxter equation is a difficult task, as will appear from the examples given below. Let $\{v_i\}_i$ be a basis of the vector space V. An automorphism c of $V \otimes V$ is defined by the family

$(c_{ij}^{k\ell})_{i,j,k,\ell}$ of scalars determined by

$$c(v_i \otimes v_j) = \sum_{k,\ell} c_{ij}^{k\ell}\, v_k \otimes v_\ell.$$

Then c is a solution of the Yang-Baxter equation if and only if for all i, j, k, ℓ, m, n, we have

$$\sum_{p,q,r,x,y,z} (c_{ij}^{pq}\delta_{kr})(\delta_{px}c_{qr}^{yz})(c_{xy}^{\ell m}\delta_{zn}) = \sum_{p,q,r,x,y,z} (\delta_{ip}c_{jk}^{qr})(c_{pq}^{xy}\delta_{rz})(\delta_{x\ell}c_{yz}^{mn}),$$

which is equivalent to

$$\sum_{p,q,y} c_{ij}^{pq} c_{qk}^{yn} c_{py}^{\ell m} = \sum_{y,q,r} c_{jk}^{qr} c_{iq}^{\ell y} c_{yr}^{mn}. \tag{1.1}$$

for all i, j, k, ℓ, m, n. Solving the non-linear equations (1.1) is a highly non-trivial problem. Nevertheless, numerous solutions of the Yang-Baxter equation have been discovered in the 1980's. Let us list a few examples.

Example 1. For any vector space V we denote by $\tau_{V,V} \in \mathrm{Aut}(V \otimes V)$ the *flip* switching the two copies of V. It is defined by

$$\tau_{V,V}(v_1 \otimes v_2) = v_2 \otimes v_1,$$

for any $v_1, v_2 \in V$. The flip satisfies the Yang-Baxter equation because of the Coxeter relation $(12)(23)(12) = (23)(12)(23)$ in the symmetry group S_3.

Here is a way to generate new R-matrices from old ones.

Lemma VIII.1.2. *If $c \in \mathrm{Aut}(V \otimes V)$ is an R-matrix, then so are λc, c^{-1} and $\tau_{V,V} \circ c \circ \tau_{V,V}$ where λ is any non-zero scalar.*

PROOF. This follows from the identities

$$(\lambda c \otimes \mathrm{id}_V) = \lambda(c \otimes \mathrm{id}_V), \quad (\mathrm{id}_V \otimes \lambda c) = \lambda(\mathrm{id}_V \otimes c),$$

$$(c^{-1} \otimes \mathrm{id}_V) = (c \otimes \mathrm{id}_V)^{-1}, \quad (\mathrm{id}_V \otimes c^{-1}) = (\mathrm{id}_V \otimes c)^{-1},$$

$$(c' \otimes \mathrm{id}_V) = \sigma(\mathrm{id}_V \otimes c)\sigma^{-1}, \quad (\mathrm{id}_V \otimes c') = \sigma(c \otimes \mathrm{id}_V)\sigma^{-1},$$

where $c' = \tau_{V,V} \circ c \circ \tau_{V,V}$ and σ is the automorphism of $V \otimes V \otimes V$ defined by $\sigma(v_1 \otimes v_2 \otimes v_3) = v_3 \otimes v_2 \otimes v_1$ for $v_1, v_2, v_3 \in V$. □

Example 2. Let us solve the Yang-Baxter equation when $V = V_1 = V_{1,1}$ is the 2-dimensional simple module over the Hopf algebra $U_q = U_q(\mathfrak{sl}(2))$ considered in Chapters VI–VII. More precisely, let us find all U_q-automorphisms of $V_1 \otimes V_1$ that are R-matrices. We freely use the notation of the above-mentioned chapters. Recall that if v_0 is a highest weight vector of V_1, then the set $\{v_0, v_1 = Fv\}$ is a basis of V_1. By the Clebsch-Gordan Theorem VII.7.1 we have $V_1 \otimes V_1 \cong V_2 \oplus V_0$. Lemma VII.7.2 implies that the vectors

$$w_0 = v_0 \otimes v_0 \quad \text{and} \quad t = v_0 \otimes v_1 - q^{-1} v_1 \otimes v_0$$

are highest weight vectors of respective weights q^2 and 1. We complete the set of linearly independent vectors $\{w_0, t\}$ into a basis for $V \otimes V$ by setting

$$w_1 = Fw_0 = q^{-1} v_0 \otimes v_1 + v_1 \otimes v_0 \quad \text{and} \quad w_2 = \frac{1}{[2]} F^2 w_0 = v_1 \otimes v_1$$

where $[2] = q + q^{-1}$.

Proposition VIII.1.3. *Any U_q-linear automorphism φ of $V_1 \otimes V_1$ is diagonalizable and of the form $\varphi(w_i) = \lambda w_i$ $(i = 0, 1, 2)$ and $\varphi(t) = \mu t$ where λ and μ are non-zero scalars. The automorphism φ is an R-matrix if and only if*

$$(\lambda - \mu)(q\lambda + q^{-1}\mu)(q^{-1}\lambda + q\mu) = 0.$$

PROOF. Since φ is U_q-linear, the image under φ of a highest weight vector is a highest weight vector of the same weight. Now, w_0 and t have different weights (we still assume that $q^2 \neq 1$); therefore, there exist λ and μ such that $\varphi(w_0) = \lambda w_0$ and $\varphi(t) = \mu t$.

As for the remaining basis vectors, we have

$$\varphi(w_i) = \frac{1}{[i]} \varphi(F^i w_0) = \frac{1}{[i]} F^i \varphi(w_0) = \lambda w_i$$

for $i = 1, 2$. This completes the proof of the first assertion in Proposition 1.3.

The second assertion results from tedious computation. Let us give some details. We first observe that the matrix Φ of φ with respect to the basis $\{v_0 \otimes v_0, v_0 \otimes v_1, v_1 \otimes v_0, v_1 \otimes v_1\}$ is given by

$$\Phi = \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \alpha & \gamma & 0 \\ 0 & \gamma & \beta & 0 \\ 0 & 0 & 0 & \lambda \end{pmatrix}$$

where

$$\alpha = \frac{q^{-1}\lambda + q\mu}{[2]}, \quad \beta = \frac{q\lambda + q^{-1}\mu}{[2]}, \quad \gamma = \frac{\lambda - \mu}{[2]}.$$

The automorphisms $\varphi \otimes \mathrm{id}$ and $\mathrm{id} \otimes \varphi$ can be expressed, respectively, by the 8×8-matrices Φ_{12} and Φ_{23} in the basis consisting of the elements $v_0 \otimes v_0 \otimes v_0$, $v_0 \otimes v_0 \otimes v_1$, $v_0 \otimes v_1 \otimes v_0$, $v_0 \otimes v_1 \otimes v_1$, $v_1 \otimes v_0 \otimes v_0$, $v_1 \otimes v_0 \otimes v_1$, $v_1 \otimes v_1 \otimes v_0$, and $v_1 \otimes v_1 \otimes v_1$ of $V \otimes V \otimes V$ where

$$\Phi_{12} = \begin{pmatrix} \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha & 0 & \gamma & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha & 0 & \gamma & 0 & 0 \\ 0 & 0 & \gamma & 0 & \beta & 0 & 0 & 0 \\ 0 & 0 & 0 & \gamma & 0 & \beta & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda \end{pmatrix}$$

and

$$\Phi_{23} = \begin{pmatrix} \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha & \gamma & 0 & 0 & 0 & 0 & 0 \\ 0 & \gamma & \beta & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha & \gamma & 0 \\ 0 & 0 & 0 & 0 & 0 & \gamma & \beta & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda \end{pmatrix}.$$

Now, $\Phi_{12}\Phi_{23}\Phi_{12} - \Phi_{23}\Phi_{12}\Phi_{23}$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & K & -\alpha\beta\gamma & 0 & 0 & 0 & 0 & 0 \\ 0 & -\alpha\beta\gamma & L & 0 & \alpha\beta\gamma & 0 & 0 & 0 \\ 0 & 0 & 0 & -K & 0 & \alpha\beta\gamma & 0 & 0 \\ 0 & 0 & \alpha\beta\gamma & 0 & M & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha\beta\gamma & 0 & -L & \alpha\beta\gamma & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha\beta\gamma & -M & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

where $K = \alpha((\lambda - \alpha)\lambda - \gamma^2)$, $L = \alpha\beta(\alpha - \beta)$ and $M = \beta(\gamma^2 + \lambda(\beta - \lambda))$. Suppose that we have proved that K, L and M are multiples of $\alpha\beta\gamma$. Then

$$\Phi_{12}\Phi_{23}\Phi_{12} - \Phi_{23}\Phi_{12}\Phi_{23} = \alpha\beta\gamma \times \Psi$$

where Ψ is a non-zero matrix. It follows that Φ is an R-matrix if and only if $\alpha\beta\gamma = 0$, which would complete the proof of Proposition 1.3.

It remains to show that K, L and M are multiples of $\alpha\beta\gamma$. An easy computation proves that

$$\lambda - \alpha = q\gamma, \quad \lambda - \beta = q^{-1}\gamma, \quad q^{-1}\lambda - \gamma = q^{-1}\alpha, \quad q\lambda - \gamma = q\beta$$

and $\beta - \alpha = (q - q^{-1})\gamma$. Therefore,

$$K = \alpha\gamma(q\lambda - \gamma) = q\,\alpha\beta\gamma, \quad L = -(q - q^{-1})\,\alpha\beta\gamma$$

and $M = \beta\gamma(\gamma - q^{-1}\lambda) = -q^{-1}\alpha\beta\gamma$. □

To sum up, the R-matrices of the U_q-module $V_1 \otimes V_1$ belong to the following three types depending on a parameter $\lambda \neq 0$:

1. If $\lambda = \mu$, φ is a homothety.

2. If $q\lambda + q^{-1}\mu = 0$, then

$$\Phi = q\lambda \begin{pmatrix} q^{-1} & 0 & 0 & 0 \\ 0 & q^{-1} - q & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & q^{-1} \end{pmatrix}.$$

3. If $q^{-1}\lambda + q\mu = 0$, then

$$\Phi = q^{-1}\lambda \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & q - q^{-1} & 0 \\ 0 & 0 & 0 & q \end{pmatrix}.$$

It is clear that Cases 2 and 3 are equivalent within a change of basis after exchanging q and q^{-1}. As we shall see in the next example, the minimal polynomial of Φ is of degree ≤ 2.

Example 3. We now give an important class of R-matrices with *quadratic minimal polynomial.* Such R-matrices will be used in Chapter XII to construct isotopy invariants of links in $\mathbf{R}^3$.

Let V be a finite-dimensional vector space with a basis $\{e_1, \ldots, e_N\}$. For two invertible scalars p, q and for any family $\{r_{ij}\}_{1 \leq i,j \leq N}$ of scalars in k such that $r_{ii} = q$ and $r_{ij}r_{ji} = p$ when $i \neq j$, we define an automorphism c of $V \otimes V$ by

$$\begin{aligned} c(e_i \otimes e_i) &= q\, e_i \otimes e_i \\ c(e_i \otimes e_j) &= \begin{cases} r_{ji}\, e_j \otimes e_i & \text{if } i < j \\ r_{ji}\, e_j \otimes e_i + (q - pq^{-1})\, e_i \otimes e_j & \text{if } i > j. \end{cases} \end{aligned}$$

Proposition VIII.1.4. *The automorphism c is a solution of the Yang-Baxter equation. Moreover, we have*

$$(c - q\,\mathrm{id}_{V\otimes V})(c + pq^{-1}\,\mathrm{id}_{V\otimes V}) = 0,$$

or, equivalently, $c^2 - (q - pq^{-1})c - p\,\mathrm{id}_{V\otimes V} = 0$.

PROOF. (a) We first show that c is an R-matrix. In order to simplify the proof, let us introduce the following notation. The symbol (ijk) will stand for the vector $e_i \otimes e_j \otimes e_k$, and $[i > j]$ for the integer 1 if $i > j$ and for 0 otherwise. Then c can be redefined as

$$c(e_i \otimes e_j) = r_{ji} e_j \otimes e_i + [i > j]\beta e_i \otimes e_j$$

where $\beta = q - pq^{-1}$.

An immediate computation yields

$$
\begin{aligned}
&(c\otimes \mathrm{id})(\mathrm{id}\otimes c)(c\otimes \mathrm{id})((ijk))\\
&\quad = r_{ji}r_{ki}r_{kj}(kji) + r_{ji}r_{ki}[j>k]\beta(jki)\\
&\qquad + r_{kj}r_{ki}[i>j]\beta(kij) + r_{kj}[i>j][j>k]\beta^2(ikj)\\
&\qquad + r_{ji}\Big([j>i][i>k] + [i>j][j>k]\Big)\beta^2(jik)\\
&\qquad + \Big(r_{ji}r_{ij}[i>k]\beta + [i>j][j>k]\beta^3\Big)(ijk)
\end{aligned}
$$

and

$$
\begin{aligned}
&(\mathrm{id}\otimes c)(c\otimes \mathrm{id})(\mathrm{id}\otimes c)((ijk))\\
&\quad = r_{ji}r_{ki}r_{kj}(kji) + r_{ji}r_{ki}[j>k]\beta(jki)\\
&\qquad + r_{kj}r_{ki}[i>j]\beta(kij) + r_{ji}[i>k][j>k]\beta^2(jik)\\
&\qquad + r_{kj}\Big([i>k][k>j] + [i>j][j>k]\Big)\beta^2(ikj)\\
&\qquad + \Big(r_{jk}r_{kj}[i>k]\beta + [i>j][j>k]\beta^3\Big)(ijk).
\end{aligned}
$$

We have to prove that these expressions are equal for all i, j, k. This is clearly the case if $i = j = k$. If i, j, k are distinct indices, they are equal in view of relations of the type

$$[i>j][i>k] = [i>j][j>k] + [i>k][k>j]$$

which express the fact that for distinct indices, we have $i > j$ and $i > k$ if and only if $i > j > k$ or $i > k > j$. If exactly two indices are equal, say $i = j \neq k$, then the desired equality is equivalent to $r_{ii}^2 = \beta r_{ii} + p$, which holds since $r_{ii} = q$ and $\beta = q - pq^{-1}$.

(b) One computes $c^2 - \beta c - p\,\mathrm{id}_{V\otimes V}$ on any vector of the form $e_i \otimes e_j$. If $i \neq j$, one immediately obtains 0. If $i = j$, one gets $(q^2 - \beta q - p)(e_i \otimes e_i)$, which is zero because of the value given to β. □

Consider the following two special cases:

(i) If $p = q^2$ and $r_{ij} = q$ for all i, j, then c is a homothety.

(ii) Take $p = 1$ and $r_{ij} = 1$ for $i \neq j$. Then c takes the form shown in Case 3 of Example 2 when V is two-dimensional. Thus, Example 2 turns out to be a special case of Example 3.

VIII.2 Braided Bialgebras

The aim of this section is to define the concept of a braided bialgebra. The importance of this concept comes from the fact proved in Section 3 that braided bialgebras generate solutions of the Yang-Baxter equation.