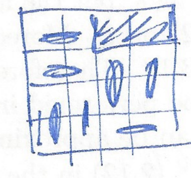
Integrable Seminar

Random tilings = beautiful subject,
uses combinatorics, probability, analysis, ...
Has physical sense, too.

Today: 2 example(s) of problems & enumeration.

Domino Tilings by 1×2 dominoes — of a region of the plane

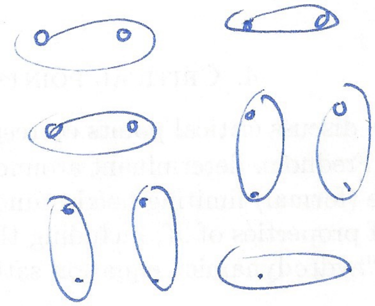
Rect.



$\Leftrightarrow$ Dimer coverings of a graph

$2\times n$

Fibonacci

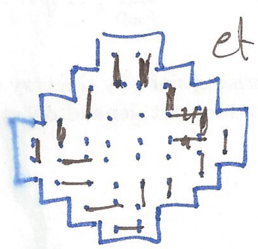


physical sense:
let us think of long molecules which pair with 1 another & they are on a lattice

$\Rightarrow$ even these pairings have some energy (free energy = log # config.)

$\Rightarrow$ want to study large systems

Aztec diamond



etc

(same question)

Random tilings of aztec (1990s)
— arctic circle
(show on comp)

or ASM

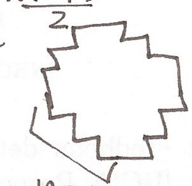
q: how many?

Originally: EKLP, Jun 1991 arXiv ess. by biject. w. some RT/algebraic objects ⊕ introduced Aztecs

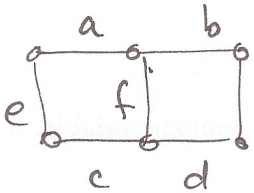
And there are many generaliz of the #. , by weighting edges of the graph.

$\boxed{\text{Thm.}}$ $2^{\frac{n(n+1)}{2}}$

(EKLP)

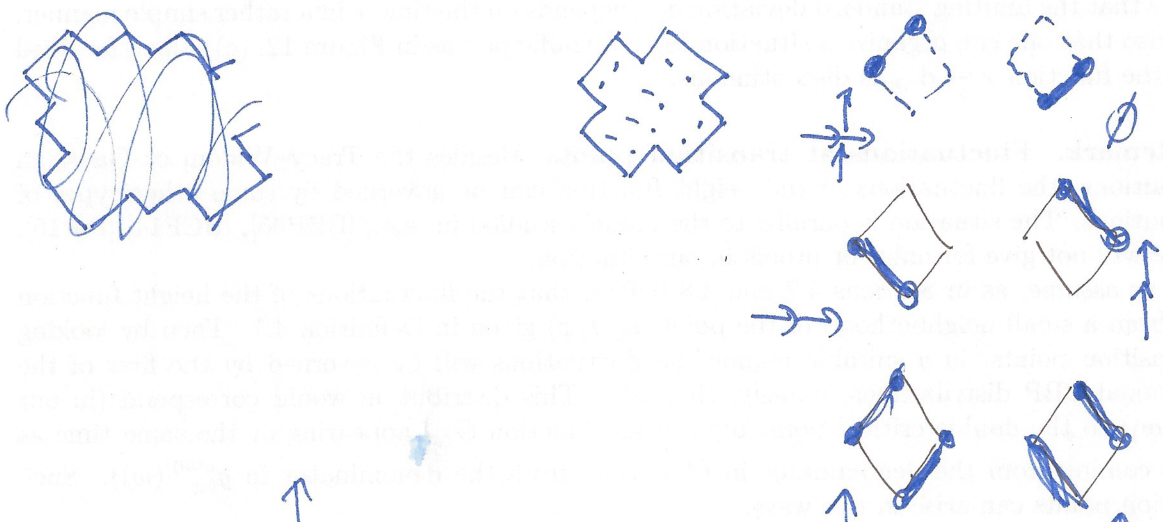


$n=3$

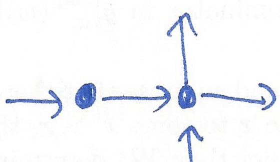


$\Rightarrow$ acg + bde + efg = partition f.

Another physics: bij of Aztec diamond tiling to square ice configurations

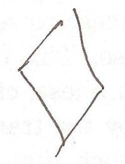


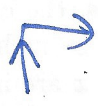
$\emptyset$



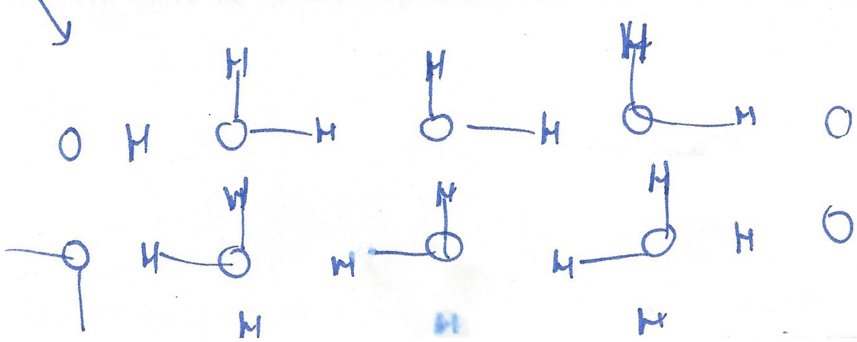
etc.

at each vertex : 2 holes & 2 arrows





at each vertex : 2 H's close and 2 are not



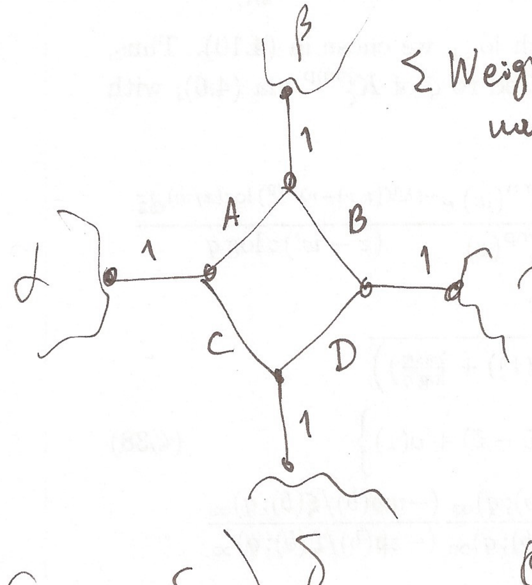
square ice

(2015: ice between 2 graphene layers)

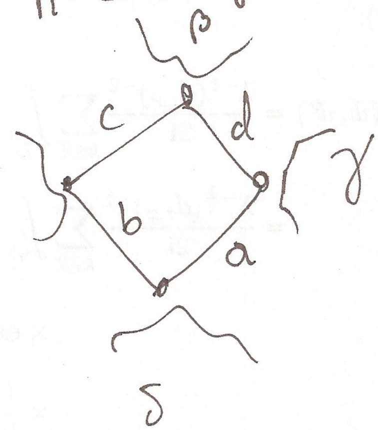
Proof. of aztec diamond theorem.

Lemma (Urban renewal) J. Propp, 1990s / Cluster alg. FZ 2005

$\sum$ Weights of matchings



$=$ $R \cdot \sum$ weights of

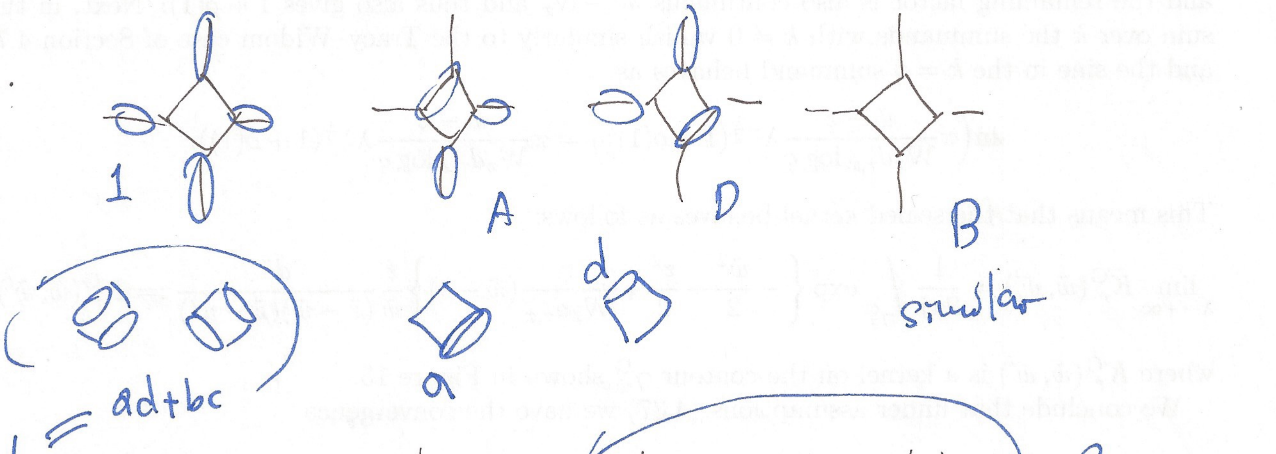


($\alpha\ \beta\ \gamma\ \delta$ — some graphs)

$$R = AD + BC$$

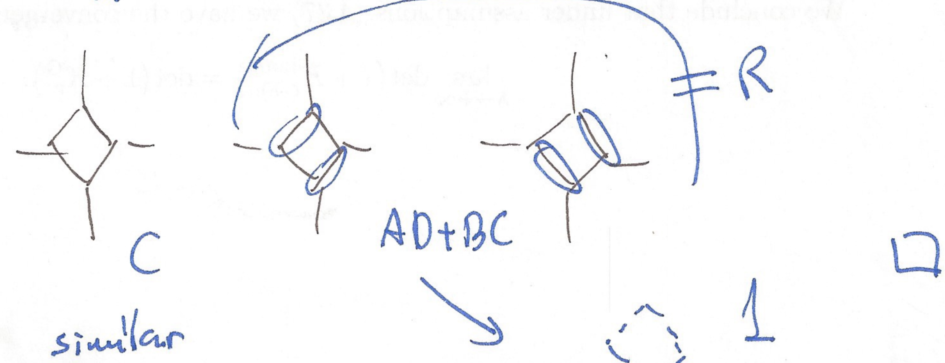
$$a = \frac{A}{R} \quad b = \frac{B}{R} \quad c = \frac{C}{R} \quad d = \frac{D}{R}$$

Proof.



$= \frac{1}{R}$

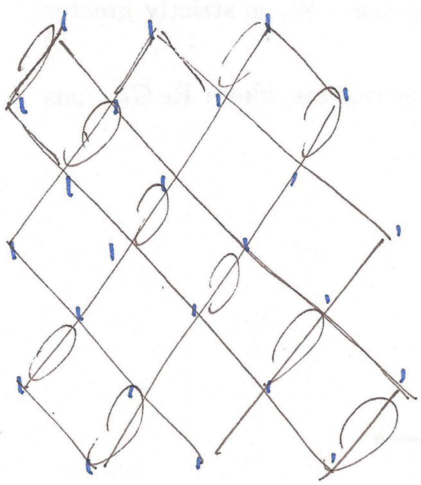
i.e. $(\sum \text{small}) * R = (\sum \text{large})$

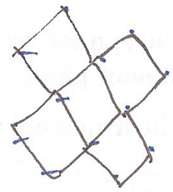


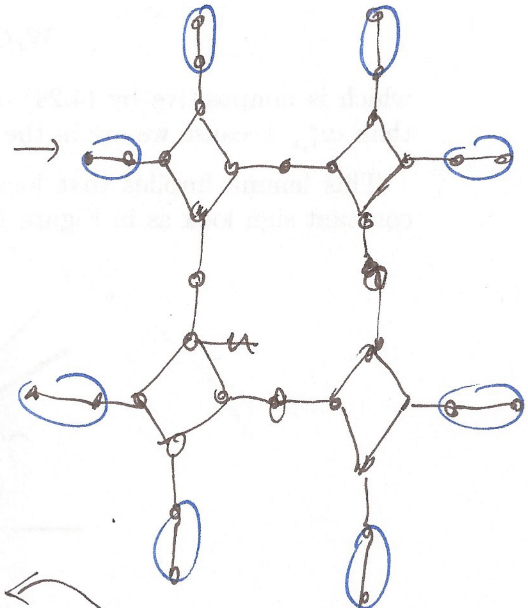
similar

Proof of the aztec diamond theorem

(dual graph)







n^2 "cells"

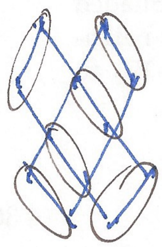
$\Rightarrow 4n^2$ edges total

~~$2n^2$ edges in tiling~~

# edges in tiling = # dominoes
Area = 4 × triang. #
# dom. = 2 × triang #

split each vertex into 3.
(does not change weight)

(n=2)



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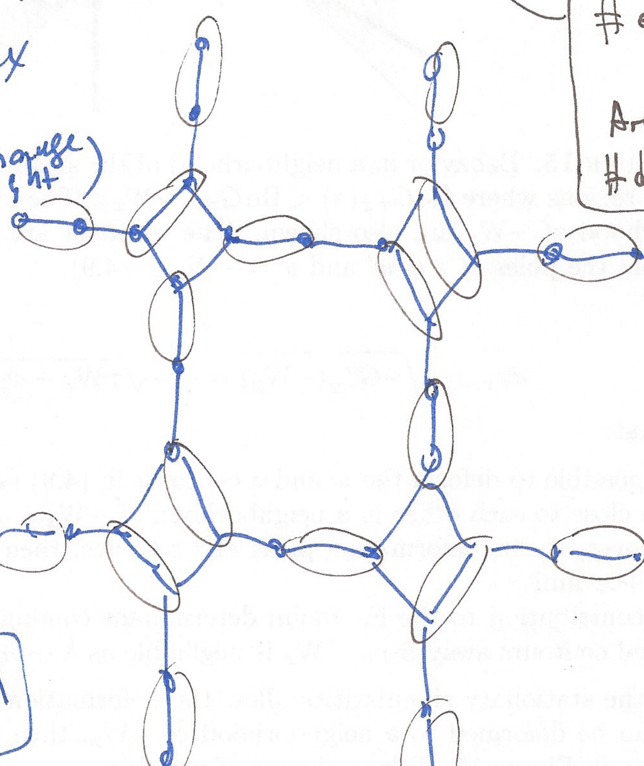
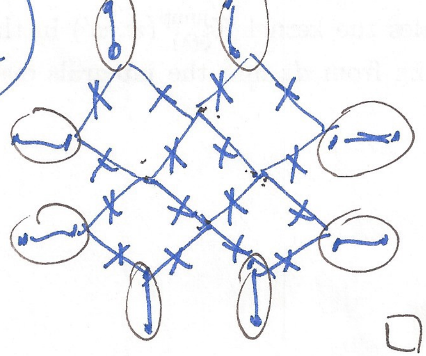


$\left(= 2^{n^2}\right.$ factor & by ind. gives factor $\boxed{2^n}$

apply L, 4 times

$\boxed{\times 2^4}$

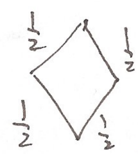
$x = \frac{1}{2} = \left(\frac{1}{R}\right)$



□

but we need to take boundary edges!

→



graph. of ord 1

$\Rightarrow \frac{1}{2^2}$

every tiling has 2 dominoes

$\left(= \frac{1}{2^{n(n-1)}}\right)$

□

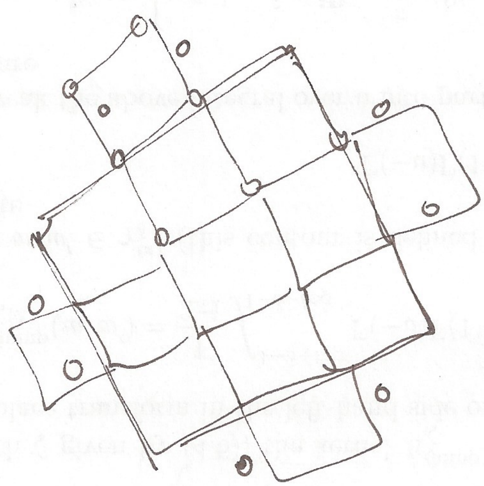
Generaliz. : $\mathbb{P}$(edge is in dom. tiling) 4.5

$\downarrow$

just put x (var.) on one of the edges & look at urban renewal process.

(in principle..)

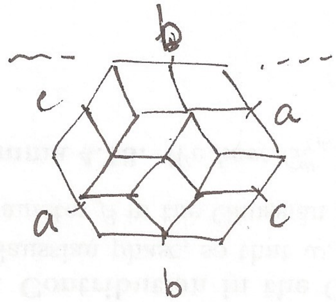
(+) for rect. domino tilings



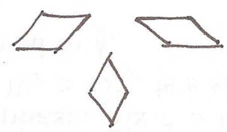
$\Rightarrow$ 3x3 rect.

5

Couple of words on lozenge tilings



a, b, c, a, b, c hexagon



$$\prod_{1}^{a}\prod_{1}^{b}\prod_{1}^{c}\frac{i+j+k-1}{i+j+k-2}$$

Aztec 1991

Hexagon – 1900s ! (MacMahon; also w. Hardy/Ramanujan in movie

$$\sum_{n=0}^{\infty} p(n)x^n = \prod_{i=1}^{\infty}\frac{1}{(1-x^i)} \text{ – Euler}$$

$$\sum_{n=0}^{\infty} \text{plane}(n)x^n = \prod_{i=1}^{\infty}\frac{1}{(1-x^i)^i}$$

But more recently connections to RT were revisited & understood

Tilings enumerate basis in repr. of $U(a+c)$ of hex.

(irrep corresp. to $\lambda = (\underbrace{b, b \ldots b}_{a}, \underbrace{0, 0 \ldots 0}_{c})$)

i.e. basis in a vector space w. action of unitary group by (unitary) transforms w. no proper inv. subspaces.

(decipher next time)

$U(N)$: $A^{*t} = A^{-1}$

example

$$s_\lambda(x_1 \ldots x_N) = \frac{\det\left[x_i^{\lambda_j+N-j}\right]_1^N}{\underbrace{\det\left[x_i^{N-j}\right]_1^N}_{\prod_{i<j}(x_i - x_j)}}$$

$\lambda = (\lambda_1 \geq \ldots \geq \lambda_N) \in \mathbb{Z}^N$

branching

$$s_\lambda(x_1 \ldots x_{N-1}, 1) = \sum_{\mu \prec \lambda} s_\mu(x_1 \ldots x_{N-1})$$