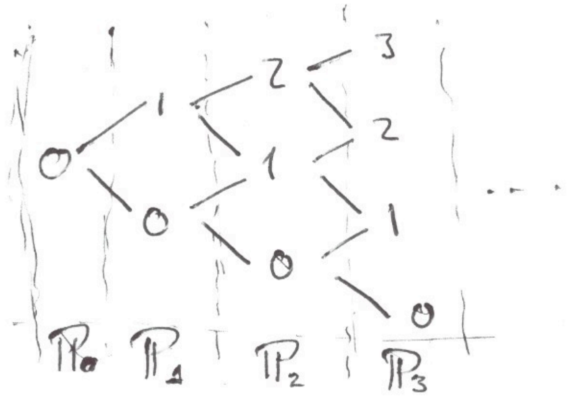
Pascal's Triangle & Branching Graphs

<u>Def</u>: The Pascal triangle is a graded posed given as follows:

$$\mathbb{P} = \coprod_{n=0}^{\infty} \mathbb{P}_n = \coprod_{n=0}^{\infty} \{ (k,l) \in \mathbb{Z}^2_{\geq 0} \mid k+l=n \}$$

$=$ 

<u>Def</u>: A normalized harmonic function, $\varphi : \mathbb{P} \to \mathbb{R}$, on the $\mathbb{P}$ is a function that satisfies the condition

$$\varphi(k,l) = \varphi(k+1,l) + \varphi(k,l+1)$$

and it is normalized so that $\varphi(0,0)=1$. We denote

$$\mathcal{H} = \{ \varphi, \text{ nor. har on } \mathbb{P} \}$$

<u>Claim</u>:

- $\mathcal{H}$ is a convex set
- Extreme elements of $\mathcal{H}$ are given by

$$\varphi_\alpha(k,l) = \alpha^k (1-\alpha)^l$$

for some $\alpha \in [0,1]$

- Any $\varphi \in \mathcal{U}$ may be written as

$$\varphi(k,l) = \int_0^1 \varphi_\alpha^{(k,l)} P(d\alpha) = \int_0^1 \alpha^k (1-\alpha)^l P(d\alpha)$$

for some prob measure P on $[0,1]$

<u>Def:</u> Let $F : \mathbb{R}[x,y] \to \mathbb{R}$ be a linear functional satisfying:

(i) $F(x^k y^l) \geq 0 \qquad k, l \geq 0$

(ii) F vanishes on the principal ideal generated by $(x+y)-1$.

(iii) $F(1) = 1$

We denote

$$\mathcal{F} = \{F, \text{ lin. fun w/ (i)-(iii)}\}.$$

<u>Def:</u> A measure ν on $\{0,1\}^\infty = \{(x_1, x_2, x_3, \dots) \mid x_i \in \{0,1\}\}$ is given by a coherent system of measures $\nu^{(n)}$ on $\{0,1\}^n = \{(x_1, \dots, x_n) \mid x_i \in \{0,1\}\}$ w.r.t. the proj. map

$$(x_1, \dots, x_n) \to (x_1, \dots, x_{n-1}).$$

In particular,

$$\nu^{(n-1)}(x_1, \dots, x_{n-1}) = \nu^{(n)}(x_1, \dots, x_{n-1}, 0) + \nu^{(n)}(x_1, \dots, x_{n-1}, 1)$$

Moreover, $\nu = \{\nu^{(n)}\}_{n=1}^\infty$ is called $S(\infty)$-invariant if

$$\nu^{(n)}(x_1, \dots, x_n) = \nu^{(n)}(x_{\sigma(1)}, \dots, x_{\sigma(n)}) \quad \forall\ \sigma \in S_n$$

We denote

$$\mathcal{M} = \{ \nu \text{ , } S(\infty)\text{-inv measure on } \{0,1\}^{\infty} \}.$$

<u>Lemma</u>: There is a three way bijection

$$\begin{tikzcd} & \mathcal{H} \arrow[dl, leftrightarrow] \arrow[dr, leftrightarrow] & \\ \mathcal{F} \arrow[rr, leftrightarrow] & & \mathcal{M} \end{tikzcd}$$

<u>Pf</u>:

$(\nu \longmapsto \varphi)$

Introduce the events (sets)

$$C_{k,\ell} = \{ \vec{x} \in \{0,1\}^{\infty} \mid x_1 = \dots = x_k = 1 \text{ , } x_{k+1} = \dots = x_{k+\ell} = 0 \}$$

$$\sigma \cdot C_{k,\ell} = \{ \sigma \cdot \vec{x} \mid \vec{x} \in C_{k,\ell} \}$$

$$\Rightarrow \quad \varphi(k,\ell) := \nu(C_{k,\ell}) = \nu(\sigma \cdot C_{k,\ell})$$

Note $\quad C_{k,\ell} = \sigma \cdot C_{k+1,\ell} \sqcup C_{k,\ell+1} \quad$ for $\sigma = (k+1, k+\ell+1)$

By coherence of ν $\Rightarrow$ $\nu(C_{k,\ell}) = \nu(\sigma \cdot C_{k+1,\ell}) + \nu(C_{k,\ell+1})$
$= \nu(C_{k+1,\ell}) + \nu(C_{k,\ell+1})$

$\Rightarrow \varphi(k,\ell) = \varphi(k+1,\ell) + \varphi(k,\ell+1)$

$\Rightarrow \varphi$ is har on $\mathbb{P}$.

$(\varphi \mapsto \nu)$

Set $\nu^{(n)}(x_1, \dots, x_n) := \varphi(k, l)$ if k ones & l zeros

$$\Rightarrow \nu^{(n+1)}(x_1, \dots, x_n, 0) + \nu^{(n+1)}(x_1, \dots, x_n, 1)$$
$$= \varphi(k, l+1) + \varphi(k+1, l)$$
$$= \varphi(k, l)$$
$$= \nu^{(n)}(x_1, \dots, x_n)$$

$\Rightarrow \{\nu^{(n)}\}_{n \geq 0}$ is ~~co~~ coherent.

$(\varphi \leftrightarrow F)$

Set $F(x^k y^l) := \varphi(k, l)$ and extend linearly

$$F([(x+y) - 1]\, x^k y^l)$$
$$= F(x^{k+1} y^l + x^k y^{l+1} - x^k y^l)$$
$$= F(x^{k+1} y^l) + F(x^k y^{l+1}) - F(x^k y^l)$$
$$= \varphi(k+1, l) + \varphi(k, l+1) - \varphi(k, l)$$
$$= 0$$

$\Rightarrow F$ vanishes on the ideal $(x+y) - 1$.

$\blacksquare$

Cor
- $\{\varphi_\alpha(k,l) = \alpha^k(1-\alpha)^l \mid \alpha \in [0,1]\}$ are the extremes of $\mathcal{H}$
- $\{\{\nu_\alpha^{(n)}(1,\dots,1,0,\dots,0) = \alpha^k(1-\alpha)^l\}_{n\geq 0} \mid \alpha \in [0,1]\}$ are the extremes for $\mathcal{M}$.
- $\{F_\alpha(x) = \alpha \mid \alpha \in [0,1]\}$ are the extremes on $\mathcal{F}$.

All of these are equivalent statements.

Claim: $F \in \mathcal{F}$ is extreme if it is multiplicative. That is,

$$F(x^k y^l) = F(x)^k F(y)^l,$$

meaning that F is an alg. hom on $\mathbb{R}[x,y]$ satisfying (i) - (iii).

Pf:

(F extreme $\Rightarrow$ F mult)

Case 1 $F(x) = 0$

$$\Rightarrow \quad 0 \leq F(x^{k+1}y^l) = F(x\, x^k y^l) \leq \frac{F(x(x+y)^{k+l})}{\binom{k+l}{k}} = \binom{k+l}{k}^{-1} F(x) = 0$$

Since $F\left((x+y)^{k+l} - \binom{k+l}{k} x^k y^l\right) \geq 0$.

So, $$F(x^{k+1}y^{\ell}) = 0 = F(x)F(x^k y^{\ell})$$

Case 2 $F(x) > 0$

$$\Rightarrow F(x^k y^{\ell}) = F((x+y)(x^k y^{\ell}))$$
$$= F(x^{k+1}y^{\ell}) + F(x^k y^{\ell+1})$$
$$= F(x)\frac{F(x^{k+1}y^{\ell})}{F(x)} + F(y)\frac{F(x^k y^{\ell+1})}{F(y)}$$
$$= F(x)F_x(x^k y^{\ell}) + F(y)F_y(x^k y^{\ell})$$

with $F_x(x^k y^{\ell}) := \dfrac{F(x^{k+1}y^{\ell})}{F(x)}$ & $F_y(x^k y^{\ell+1})\ \dfrac{}{F(y)}$.

Note that $F_x, F_y \in \mathfrak{F}$. Then, we must have

$$F_x(x^k y^{\ell}) = F(x^k y^{\ell})$$

since F is extreme

$$\Rightarrow F(x^k y^{\ell}) = \frac{F(x^{k+1}y^{\ell})}{F(x)} \Rightarrow F(x^{k+1}y^{\ell}) = F(x)F(x^k y^{\ell})$$

(F mult $\Rightarrow F$ extreme)

Let $E \subset \mathcal{F}$ be the set of extreme functionals in $\mathcal{F}$.

$$\Rightarrow \quad F(f) = \int_E G(f) P(dG)$$

for some prob measure on E.

$$\Rightarrow \quad \left(\int_E G(f) P(dG) \right)^2 = F(f)^2 = F(f^2) = \int G(f)^2 \cdot P(dG)$$

$$\Rightarrow \quad \int_E \left(G(f) - \int_E G(f) P(dG) \right)^2 P(dG) = 0$$

$$\Rightarrow \quad P(G) = \begin{cases} 1 & G = G' \\ 0 & G = G' \end{cases} \Rightarrow F = G' \Rightarrow F \text{ is extreme.}$$

<u>Rem</u>: We may construct

$$\varphi(k, l) = \int_0^1 \alpha^k (1-\alpha)^l P(d\alpha)$$

explicitely. We need the following observations

- $\varphi(k_0, l_0) = \sum_{k+l=n} \dim((k_0, l_0) \to (k, l)) \, \varphi(k, l)$

with
$$\dim((k_0, l_0) \to (k, l)) = \binom{k+l-k_0-l_0}{k-k_0}$$
$$= \# \text{ paths in } \mathbb{P} \text{ from } (k_0, l_0) \text{ to } (k, l).$$

- $M^{(n)}(k,l) = \dim((0,0)\to(k,l))\,\varphi(k,l)$
$= \binom{k+l}{k}\varphi(k,l)$
is a prob measure on $\mathbb{P}^{(n)}$

- $\dfrac{\dim((k_0,l_0)\to(k,l))}{\dim((0,0)\to(k,l))} = \left(\dfrac{k}{k+l}\right)^{k_0}\left(\dfrac{l}{k+l}\right)^{l_0} + O\left(\dfrac{1}{k+l}\right)$

Then,

$$\varphi(k_0,l_0) = \sum_{k+l=n}\left(\frac{\dim((k_0,l_0)\to(k,l))}{\dim((0,0)\to(k,l))}\right) \times \dim((0,0)\to(k,l))\,\varphi(k,l)$$

$$= \sum_{k+l=n}\left(\frac{k}{k+l}\right)^{k_0}\left(\frac{l}{k+l}\right)^{l_0} M^{(n)}(k,l) + O\left(\frac{1}{n}\right)$$

Set $P^{(n)}(\alpha) = \sum_{k+l=n} M^{(n)}(k,l)\,\mathbb{1}\left(\alpha = \frac{k}{k+l}\right)$

$\Rightarrow P^{(n_j)} \to P$ for some subseq
since the set of prob meas
are "compact"

$\therefore\ \varphi(k_0,l_0) = \int_0^1 \alpha^{k_0}(1-\alpha)^{l_0}\,P(d\alpha)$.