ART-2022 (MATH 8852)
FINAL EXAM

Due December 13, by 2pm — send ONLY by email

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Course webpage: https://lpetrov.cc/art2022/

In solving the problems, you may use any results from the course, but make sure to mention them explicitly.

Full credit is given for any 3 correctly solved problems.

1.

Let $S(6)$ be the symmetric group permuting $\{1, 2, 3, 4, 5, 6\}$.

(1) Provide a list of all irreducible representations (irreps) of $S(6)$ and their dimensions.
(2) Check that the sum of squares of the dimensions of irreps equals the order of the group
(3) List all irreps of $S(6)$ which contain the sign representation of $S(3)$ (permuting $\{1, 2, 3\}$).

Some context for the last question. If $T: S(6) \to \mathrm{End}(V)$ is an irrep, then restricting it to $S(3)$ makes T a reducible representation of $S(3)$. List all irreps of $S(6)$ for which their restriction to $S(3)$ contains the sign representation. Recall that the sign representation $\mathrm{sgn}: S(3) \to \mathbb{R}$ maps each permutation into its sign, ± 1. It is a one-dimensional irreducible representation of $S(3)$.

2.

Let U, D be the up and down operators on the Young diagrams (recall HW3, problem 3). We know that $DU = UD + 1$, where 1 is the identity operator. Prove that for any $\alpha, \beta \in \mathbb{C}$,

$$e^{\alpha D} e^{\beta U} = e^{\alpha\beta} e^{\beta U} e^{\alpha D}.$$

Here $e^{\alpha D}, e^{\beta U}$ are Taylor series which are well-defined, and $e^{\alpha\beta}$ is a number.

A possible way to prove this (but you can do it in some other way):

(1) First, express $D^m U$ as a linear combination of UD^m and something else. In other words, compute the commutator $[D^m, U]$.
(2) Use this to compute $[e^{\alpha D}, U]$.
(3) Proceed to compute $\left[e^{\alpha D}, U^n\right]$.
(4) Finally, use this to prove the desired formula.

3.

If $s_\lambda(x_1, \ldots, x_n)$ is a Schur polynomial, then viewing it as a polynomial in $x_1, \ldots, x_k$, we may express it as a linear combination of $s_\mu(x_1, \ldots, x_k)$:

$$s_\lambda(x_1, \ldots, x_n) = \sum_\mu s_\mu(x_1, \ldots, x_k) C_{\lambda,\mu}.$$

The coefficients $C_{\lambda,\mu}$ depend on $x_{k+1}, x_{k+2}, \ldots, x_n$, and are called the *skew Schur polynomials.* Notation

$$C_{\lambda,\mu} = s_{\lambda/\mu}(x_{k+1}, x_{k+2}, \ldots, x_n).$$

Compute $s_{(3,2,1)/(2,1)}(x_3, x_4)$.

Hint: use the interpretation of the Schur polynomial as a generating function of semistandard Young tableaux.

4.

Let $\lambda^{(n)} = \left(\lfloor \frac{n}{3} \rfloor, \lfloor \frac{n}{3} \rfloor, \lfloor \frac{n}{6} \rfloor, \lfloor \frac{n}{6} \rfloor\right)$ be a sequence of Young diagrams. Find

$$\lim_{n \to +\infty} \frac{\dim(\mu, \lambda^{(n)})}{\dim \lambda^{(n)}},$$

where $\mu = (1, 1, 1)$.