ART-2022 (MATH 8852)
HOMEWORK 2

Due September 20, by 2pm — hard copy or email accepted
After the homework is graded, you will have 1 week to resubmit a corrected and revised version

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1.

Let X be a compact Polish space (complete, separable, metric), and G be a group acting on it by continuous maps. Let $\mathcal{I}$ be space of G-invariant probability measures on X.

(1) Show that $\mathcal{I}$ is a convex compact set.
(2) Show that if $\mu \in \mathcal{I}$ is not ergodic, then μ is not an extreme point of $\mathcal{I}$.
(3) Show that if a measure is ergodic, then it must be extreme. (Hint: use Radon-Nikodym)

Therefore, extreme points of $\mathcal{I}$ are precisely the ergodic measures.

2.

$\dim \lambda$ is the dimension of the irreducible representation of $S(|\lambda|)$ indexed by the partition λ. Using the following formula for $\dim \lambda$,

$$\dim \lambda = |\lambda|! \, \frac{\prod_{1 \le i < j \le N} (\lambda_i - i - \lambda_j + j)}{\prod_{i=1}^{N} (\lambda_i + N - i)!}, \tag{2.1}$$

where N is any number greater than $\ell(\lambda)$, prove directly that

$$\dim \lambda = \sum_{\mu = \lambda - \square} \dim \mu, \qquad \dim \lambda = \frac{1}{|\lambda| + 1} \sum_{\nu = \lambda + \square} \dim \nu. \tag{2.2}$$

Here "$\mu = \lambda - \square$" and "$\nu = \lambda + \square$" mean that μ is obtained from λ by deleting one box, and ν is obtained from λ by adding one box.

3.

(a) Using the previous problem, show that (2.1) holds for the dimension $\dim \lambda$ defined as the number of standard Young tableaux of shape λ.
(b) Using (a), prove the *hook formula*:

$$\dim \lambda = \frac{|\lambda|!}{\prod_{\square \in \lambda} h(\square)},$$

where $h(\square)$ is the hook length of $\square$, that is, the number of boxes in λ which are weakly to the right in the same row, or weakly below of $\square$ in the same column.

4.

Using only the recursions (2.2), show by induction that

$$\sum_{\lambda} (\dim \lambda)^2 = n!, \tag{4.1}$$

where the sum is over all partitions λ with n boxes.

Remark 4.1. Identity (4.1) is the Burnside's formula for the symmetric group $S(n)$, and we proved it in the lectures for an arbitrary finite group.

5.

Prove that for $n \geq 5$ the dimension of any irreducible representation of $S(n)$, except for the two one-dimensional representations, is at least $n-1$. For which λ do we have $\dim \lambda = n - 1$ exactly?

Remark 5.1. One can use this fact to derive that $S(\infty)$ has no finite–dimensional representations which are not direct sums of the one–dimensional ones.

6.

Let $\rho = (\rho_1 \geq \rho_2 \geq \ldots \geq \rho_k > 0)$ be a partition with $|\rho| = n$. Find the size of the conjugacy class in $S(n)$ indexed by ρ.

7.

Compute the dimension $\dim(n, n)$, where (n, n) is the Young diagram with two equal long rows. Do not use any of the formulas for the dimension, but rather identify standard tableaux of shape (n, n) with some familiar object.