ART-2022 (MATH 8852)
HOMEWORK 10

Due December 9, by 2pm — hard copy or email accepted
After the homework is graded, you will have 1 week to resubmit a corrected and revised version

E-mail: leniapetrov+art2022@gmail.com
Course webpage: https://lpetrov.cc/art2022/

1. Area of a continual Young diagram

A *continual Young diagram* is a function $\omega(u)$, $u \in \mathbb{R}$, such that

(1) $|\omega(u_1) - \omega(u_2)| \le |u_1 - u_2|$ for all u_1, u_2, and
(2) $\omega(u) = |u - z|$ for some $z \in \mathbb{R}$ and all sufficiently large $|u|$

Show that for $z = 0$, the area under the continual diagram can be computed as the double integral

$$A(\omega) = \frac{1}{2} \iint\limits_{u<v} d\,(u + \omega(u))\, d\,(v - \omega(v))\,.$$

2. θ-contents

Let $\theta > 0$.

Recall that the boxes in the Young diagram λ are encoded by their coordinates (i, j), where $1 \le j \le \lambda_i$. Fix λ, and let (i_k, j_k) be the coordinates of the boxes which can be added to λ, while (i'_l, j'_l) be the coordinates of the boxes which can be removed from λ. Both sequences are in decreasing order. Denote

$$x_k = (j_k - 1) - \theta(i_k - 1), \qquad y_l = j'_l - \theta i'_l.$$

For example, for $\lambda = (3, 3, 1)$ we have

$$\vec{x} = (3, 1 - 2\theta, -3\theta), \qquad \vec{y} = (3 - 2\theta, 1 - 3\theta).$$

Show that

(1) The sequences $\vec{x}, \vec{y}$ interlace as $x_1 > y_1 > x_2 > y_2 > \ldots > x_{d-1} > y_{d-1} > x_d$.
(2) We have

$$\sum_{i=1}^{d} x_i - \sum_{j=1}^{d-1} y_j = 0.$$

3. θ-contents, down expansion

Let u be a complex variable and consider the following expansion in partial fractions

$$\frac{\prod_{i=1}^{d}(u-x_i)}{\prod_{j=1}^{d-1}(u-y_j)} = u - \sum_{j=1}^{d-1} \frac{\pi_j^{\downarrow}}{u-y_j}.$$

View this as a definition of $\pi_i^{\downarrow}$.

Show that

$$\sum_{j=1}^{d-1} \pi_j^{\downarrow} = \theta|\lambda|.$$

4. θ-contents, up expansion

Denote $\pi_i^{\uparrow}$ from the expansion

$$\frac{\prod_{j=1}^{d-1}(u-y_j)}{\prod_{i=1}^{d}(u-x_i)} = \sum_{i=1}^{d} \frac{\pi_i^{\uparrow}}{u-x_i}$$

Show that

$$\sum_{i=1}^{d} \pi_i^{\uparrow} = 1, \qquad \sum_{i=1}^{d} x_i \pi_i^{\uparrow} = 0, \qquad \sum_{i=1}^{d} x_i^2 \pi_i^{\uparrow} = \theta|\lambda|.$$

5. RSK

Recall what we proved about the RSK and the length of the longest increasing subsequence in a permutation.

If the RSK algorithm is used only to find the length of the longest increasing subsequence of $\pi \in \mathcal{S}_n$, find its computational complexity in the worst case.

6.

Use RSK to prove the Erdős-Szekeres theorem: Given any $\pi \in \mathcal{S}_{nm+1}$, then π contains either an increasing subsequence of length $n+1$ or a decreasing subsequence of length $m+1$