ART-2022 (MATH 8852)
HOMEWORK 1

Due September 13, by 2pm — hard copy or email accepted
After the homework is graded, you will have 1 week to resubmit a corrected and revised version

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1.

Let G be a finite group. Let $P = \dfrac{1}{|G|}\sum_{g\in G} g$ be an element of $\mathbb{C}[G]$. Show that

(1) $P^2 = P$
(2) In every representation V of G, P acts as a projection operator onto the subspace $V^G \subseteq V$ of invariant vectors, that is, vectors v for which $T(g)v = v$ for all $g \in G$

2.

Classify all groups whose irreducible representations have dimensions $1, 3$, and 4.

3.

Let G be a finite group. Characterize the center of the group algebra $\mathbb{C}[G]$.

4.

Let G be a finite group. Prove that if f is a nonzero function on the group satisfying

$$\frac{1}{|G|}\sum_{h\in G} f(g_1 h g_2 h^{-1}) = f(g_1)f(g_2) \tag{4.1}$$

for all $g_1, g_2 \in G$, then $f = \chi_\lambda / \dim V_\lambda$ for some irreducible representation λ of G.

Remark 4.1. Identity (4.1) is called the *functional equation for characters*, and it provides one more way of characterizing the irreducible characters without involving the trace in the representation space.

5.

(1) Let G be a finite group such that each element $g \in G$ is conjugate to its inverse g^{-1}. Show that any character of G is real-valued.
(2) Check that the symmetric group $S(n)$ satisfies the assumption in (a) and hence its characters are real-valued. (Because of this fact, we may deal with real valued functions on the groups $S(n)$.)

6.

(1) For a group G as in Section 5, show that the number of involutions in G (that is, g such that $g^2 = e$) is equal to the sum of dimensions of all irreducible representations of G.

(2) Show that in $S(n)$, the number of involutions is

$$\sum_{k=0}^{[n/2]} \frac{n!}{(n-2k)! \cdot 2^k k!}$$