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LAGRANGE'S IDENTITY AND THE HOOK FORMULA

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A simple combinatorial derivation of the hook formula for the dimensions of irreducible representations of the symmetric group is given.

In [1] Vershik used some identities to prove the hook formula. We recall that the hook formula gives a "multiplicative" version of the formula for the dimension of an irreducible representation S^λ of the symmetric group S_n, corresponding to a Young diagram

$$\dim S^\lambda = \frac{n!}{\prod_{x\in\lambda} h(x)} . \tag{1}$$

Here $h(x)$ is the length of the hook of the diagram λ at the point $x=(i,j)\in\lambda$:

$$h(x) = \lambda_i + \lambda'_j - i - j + 1 . \tag{2}$$

There is a formula of Frobenius for the dimension of the representation $S^{\lambda/\mu}$ of the symmetric group S_n, corresponding to the skew Young diagram $\lambda\setminus\mu$:

$$\dim S^{\lambda/\mu} = n!\,\det\left|\frac{1}{(\lambda_i-\mu_j-i+j)!}\right| , \tag{3}$$

in the determinant of (3) it is assumed that $\frac{1}{k!}=0$ if $k<0$, $1\leqslant i,j\leqslant \ell(\lambda\setminus\mu)$. It is known that $\dim S^{\lambda/\mu}$ is also equal to the number $f^{\lambda/\mu}$ of standard tableaux (without repetitions) of the form $\lambda\setminus\mu$. Consequently,

$$f^\lambda = \frac{n!}{\prod_{x\in\lambda} h(x)} . \tag{4}$$

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The hook formula (1) was first proved in [6] using elementary transformations over matrices which occur in the Frobenius formula (3) for $\mu=0$ (cf. also [3], Sec. 20). A direct algebraic proof of the hook formula (4) using the theory of symmetric functions is given in [2], Sec. 5, Example 2. There is a bijective proof of the hook formula (4), cf. [7]; references are also given there to other recent proofs of this result.

The proof of (4) given by A. M. Vershik is by induction. We denote by f_λ the right side of (4). It is well-known that f_λ satisfies the recurrence relation (Young ramification rule $S^\lambda \downarrow S_{n-1}$)

$$f^\lambda = \sum_{\mu\subset\lambda, |\lambda-\mu|=1} f^\mu. \tag{5}$$

The recurrence relation (5) and initial data $f^{(n)}=1$ define the numbers f^λ uniquely. It is clear that $f_{(n)}=1$ and to prove the equality $f_\lambda=f^\lambda$ it suffices to show that the numbers f_λ also satisfy (5). To this and the following identity is used in [1]:

Let $a_0, a_1, \ldots, a_{2s+1}$ be a collection of variables. Then

$$\sum_{j=0}^{s}(a_0+a_1+\ldots+a_{2j})(a_{2j+1}+\ldots+a_{2s+1})\prod_{i=1}^{j}\frac{a_{2i}+\ldots+a_{2j}}{a_{2i-1}+\ldots+a_{2j}}\prod_{i=j+1}^{s}\frac{a_{2j+1}+\ldots+a_{2i-1}}{a_{2j+1}+\ldots+a_{2i}} = \sum_{j=0}^{s} a_{2j}\left(\sum_{k=j}^{s} a_{2k+1}\right). \tag{6}$$

We show that it follows from (6) that the numbers f_λ satisfy (5). Indeed we consider a partition of λ. It has the form $(l_1^{a_0}, l_2^{a_2}, \ldots, l_s^{a_{2s}})$, where $\lambda_1 = l_1 > l_2 > \ldots > l_s > 0$. Let $a_{2i+1} = l_i - l_{i+1}$. It is clear that the collection of positive integers $(a_0, a_1, a_2, \ldots, a_{2s+1})$ defines the diagram λ and conversely. The number of cells in the diagram λ can be established from the collection $(a_0, a_1, \ldots, a_{2s+1})$ as follows:

$$|\lambda| = \sum_{j=0}^{s} a_{2j}\left(\sum_{k=j}^{s} a_{2k+1}\right). \tag{7}$$

Now we note that the diagrams over which one sums in (5) are in one-to-one correspondence with collections $(a_0, \ldots, a_{2j-1}, a_{2j}-1, 1, 1, a_{2j+1}-1, a_{2j+2}, \ldots, a_{2s+1}), j=0,\ldots,s$. We denote the corresponding diagram by μ_j. It is easy to verify that

$$|\lambda|\cdot\frac{f_{\mu_j}}{f_\lambda} = (a_0+a_1+\ldots+a_{2j})(a_{2j+1}+\ldots+a_{2s+1})\prod_{i=1}^{j}\frac{a_{2i}+a_{2i+1}+\ldots+a_{2j}}{a_{2i-1}+a_{2i}+\ldots+a_{2j}}\prod_{i=j+1}^{s}\frac{a_{2j+1}+\ldots+a_{2i-1}}{a_{2j+1}+\ldots+a_{2i}}. \tag{8}$$

Thus, the recurrence relation (5) for the numbers f_λ follows from the identity (6) and the equalities (7) and (8).

We note that the proof given which is due to A. M. Vershik is completely analogous to Goode's proof [2] of the following conjecture of Dyson:

If $a_1, \ldots, a_n$ are arbitrary nonnegative integers, then the free term in the product

$$\prod_{1\le i\ne j\le n}\left(1-\frac{x_j}{x_i}\right)^{a_j} \tag{9}$$

is equal to

$$\frac{(a_1+\ldots+a_n)!}{a_1!\ldots a_n!}. \tag{10}$$

Indeed, we denote the free term sought by $C(a_1, \ldots, a_n)$. Now we use the following identity [2], p. 147

$$\sum_{i=1}^{n}\prod_{j\ne i}\frac{x_j-tx_i}{x_j-x_i} = \frac{1-t^n}{1-t}. \tag{11}$$

We multiply both sides of (11) for $t=0$ by the product (9). We get the recurrence relation

$$c(a_1,\dots,a_n)=\sum_{i=1}^{n} c(a_1,\dots,a_i-1,\dots,a_n). \tag{12}$$

Moreover it is clear that

$$c(a_1)=1,\ c(a_1,\dots,a_{i-1},0,a_{i+1},\dots,a_n)=c(a_1,\dots,a_{i-1},a_{i+1},\dots,a_n), \tag{13}$$

if all the a_i are nonnegative. Obviously (12) and (13) uniquely determine the coefficients $c(a_1,\dots,a_n)$ for all nonnegative a_i. To prove Dyson's conjecture it remains to note that the polynomial coefficients (10) also satisfy (12) and (13). Equation (11) is an analog of the identity (6) (for $t=0$).

To finish the proof of the hook formula (4) it remains to verify (6). The proof of this identity in [1] is rather long and involved. It takes place in the framework of probability-theoretic considerations. The goal of the present note is to give a simple proof of (6) on the basis of Lagrange's identity. We recall Lagrange's identity:

let $x_1,\dots,x_n$ be a collection of independent variables, $\varphi\in\mathbb{C}(z)$, $f(z)=(z-x_1)\dots(z-x_n)$. Then

$$L_f(\varphi):=\sum_{k=1}^{n}\frac{\varphi(x_k)}{f'(x_k)}=\det\begin{vmatrix}1 & \dots & 1\\ x_1 & \dots & x_n\\ \vdots & & \vdots\\ x_1^{n-2} & \dots & x_n^{n-2}\\ \varphi(x_1) & \dots & \varphi(x_n)\end{vmatrix}\Big/\Delta, \tag{14}$$

where Δ is the Vandermonde determinant in the variables $x_1,\dots,x_n$. The proof of Lagrange's formula (14) follows easily from the expansion of the determinant in the numerator of the right side of (14) along the last row.

Before passing to the derivation of (6) from (14), we write down several other useful identities:

$$\sum_{j=0}^{s}\prod_{i=1}^{j}\frac{a_{2i}+\dots+a_{2j}}{a_{2i-1}+\dots+a_{2j}}\prod_{i=j+1}\frac{a_{2j+1}+\dots+a_{2i-1}}{a_{2j+1}+\dots+a_{2i}}=1, \tag{15}$$

$$\sum_{j=0}^{s}(a_0+a_1+\dots+a_{2s})\prod_{i=1}^{j}\frac{a_{2i}+\dots+a_{2j}}{a_{2i-1}+\dots+a_{2j}}\prod_{i=j+1}\frac{a_{2j+1}+\dots+a_{2i-1}}{a_{2j+1}+\dots+a_{2i}}=a_0+a_2+\dots+a_{2s}, \tag{16}$$

$$\sum_{j=0}^{s}(a_{2j+1}+\dots+a_{2s+1})\prod_{i=1}^{j}\frac{a_{2i}+\dots+a_{2j}}{a_{2i-1}+\dots+a_{2j}}\prod_{i=j+1}^{s}\frac{a_{2j+1}+\dots+a_{2i-1}}{a_{2j+1}+\dots+a_{2i}}=a_1+a_3+\dots+a_{2s+1}. \tag{17}$$

<u>Proof of the Identities (6), (15)-(17).</u> Let

$$x_k=\sum_{j=0}^{2k}a_j,\quad \alpha_k=\sum_{j=0}^{2k+1}a_j, \tag{18}$$

$k=0,1,\dots,s$, and consider the polynomials $f(z)=\prod_{k=0}^{s}(z-x_k)$, $\varphi(z)=\prod_{k=0}^{s-1}(z-\alpha_k)$. Then the left side of (15) assumes the following form

$$L_f(\varphi):=\sum_{k=0}^{s}\frac{\varphi(x_k)}{f'(x_k)}, \tag{15'}$$

which is equal to 1 by virtue of the following elementary lemma:

if $f(z)=(z-x_1)\dots(z-x_n)$, $k\in\mathbb{Z}_+$, then

$$L_f(z^k)=h_{k-n+1}(x_1,\dots,x_n), \tag{19}$$

where $h_m(x)$ is a complete symmetric function of the variables $x_1,\dots,x_n$. The identity (15) is proved. Analogously, the left side of (16) becomes the sum

$$\sum_{k=0}^{s} x_k \frac{\varphi(x_k)}{f'(x_k)} = L_f(z^{s+1}) - \sigma_1(\alpha) L_f(z^s) = \sum_{k=0}^{s} x_k - \sum_{k=0}^{s-1} \alpha_k = \sum_{j=0}^{s} a_{2j},$$

which proves (16). Analogously, the left side of (17) becomes the sum

$$\sum_{k=0}^{s} (\alpha_s - x_k) \frac{\varphi(x_k)}{f'(x_k)} = \alpha_s - \sum_{j=0}^{s} a_{2j} = \sum_{j=0}^{s} a_{2j+1}.$$

Now we pass to the proof of (6). After the change of variables (18) the left side of (6) becomes the sum

$$\sum_{k=0}^{s} x_k(\alpha_s - x_k) \frac{\varphi(x_k)}{f'(x_k)} = \alpha_s\Big(\sum_{j=0}^{s} a_{2j}\Big) - \Big\{L_f(z^{s+2}) - \sigma_1(\alpha) L_f(z^{s+1}) +$$

$$+\sigma_2(\alpha)\Big\} = (\alpha_s - \sigma_1(x))\Big(\sum_{j=0}^{s} a_{2j}\Big) + \sigma_2(x) - \sigma_2(\alpha).$$

Finally, we prove the equality

$$(\alpha_s - \sigma_1(x))\Big(\sum_{j=0}^{s} a_{2j}\Big) + \sigma_2(x) - \sigma_2(\alpha) = \sum_{j=0}^{s} a_{2j}\Big(\sum_{k=j}^{s} a_{2k+1}\Big). \tag{20}$$

We recall that $x = (x_0, \ldots, x_s)$, $\alpha = (\alpha_0, \ldots, \alpha_{s-1})$. We denote the difference between the left and right sides of (20) by θ_s. An elementary calculation shows that $\theta_{s+1} - \theta_s = (x_{s+1} - a_{2s+2})(\sigma_1(x) - \sum_{j=0}^{s} a_{2j}) - \alpha_s \sigma_1(\alpha) = 0$ by virtue of the obvious equalities $\alpha_s = a_{s+1} - a_{2s+2}$, $\sigma_1(x) = \sigma_1(\alpha) + \sum_{j=0}^{s} a_{2j}$. Consequently, $\theta_{s+1} = \theta_s$ and hence $\theta_s = 0$ since $\theta_1 = 0$. Thus, (6) and hence also the hook formula, is compeltely proved.

<u>COROLLARY.</u> Let $a_1, \ldots, a_{n-1}$ be parameters. The following identity holds:

$$\sum_{k=1}^{n} \prod_{j=1}^{n-1} (1 - x_k a_j) \prod_{\substack{j=1 \\ j \neq k}}^{n} \Big(1 - \frac{x_k}{x_j}\Big)^{-1} = 1. \tag{21}$$

Indeed it was shown in the derivation of (15) (cf. 15) that

$$\sum_{k=1}^{n} \prod_{j=1}^{n-1} (x_k - a_j) \prod_{\substack{j=1 \\ j \neq k}}^{n} (x_k - x_j)^{-1} = 1. \tag{21'}$$

Equation (21') is obtained from (21) after the substitution $x_j \longrightarrow x_j^{-1}$.

The q-analog of (21') and (15) follows from (21'):

Let $[m] := \frac{1-q^m}{1-q}$. Then

$$\sum_{k=1}^{n} \prod_{j=1}^{n-1} [x_k - a_j] \prod_{\substack{j=1 \\ j \neq k}}^{n} [x_k - x_j]^{-1} = 1, \tag{22}$$

and consequently,

$$\sum_{j=0}^{s} \prod_{i=1}^{j} \frac{[a_{2i} + \ldots + a_{2j}]}{[a_{2i-1} + \ldots + a_{2j}]} \prod_{i=j+1}^{s} \frac{[a_{2j+1} + \ldots + a_{2i-1}]}{[a_{2j+1} + \ldots + a_{2i}]} = 1. \tag{23}$$

The natural q-analog of the hook formula (4) is the equality (cf. [2], Chap. III, Sec. 6, Example 2):

$$K_{\lambda,(1^n)}(q) = q^{n(\lambda')} \frac{\prod_{j=1}^{n} (1 - q^j)}{H_\lambda(q)}, \tag{24}$$

where $H_\lambda(q)=\prod_{x\in\lambda}(1-q^{h(x)})$ is a polynomial in the length of hooks. Here $K_{\lambda,\mu}(q)$ is the Kostka-Greene-Fulkes polynomial (cf. [2, 5]). It is easy to see that the polynomial $K_{\lambda,(1^n)}(q)$ does not satisfy the recurrence relation (5). In order to clarify the reason why (5) does not hold for the Kostka polynomials $K_{\lambda,(1^n)}(q)$ we give an interpretation of (5) on the level of the sets $STY(\lambda)$ and $STY(\mu)$ and not only the numbers $f^\lambda=|STY(\lambda)|$ and $f^\mu=|STY(\mu)|$. Namely,

$$STY(\lambda)\cong\bigcup_\mu STY(\mu), \tag{15}$$

where the union in (25) is taken over all Young diagrams μ such that $\mu\subset\lambda, |\lambda\setminus\mu|=1$. Here $STY(\lambda)$ is the set of standard Young tableaux of the form λ without repetitions.

As above let the collection $(a_0,a_1,a_2,\ldots,a_{2s-1})$ correspond to the diagram λ. We set

$$1+r_j=\sum_{k=0}^{j}a_{2k},\ j=0,1,\ldots,s. \tag{26}$$

We denote by $STY^{(r)}(\lambda)$ the set of tableaux $T\in STY(\lambda)$ for which the number n is located in the $(r+1)$-st row and by μ_j the diagram corresponding to the collection $(a_0,a_1,\ldots,a_{2j}-1,1,1,$ $a_{2j+1}-1,\ldots,a_{2s+1})$, $0\leqslant j\leqslant s$. It is clear that $STY^{(r)}(\lambda)\neq\emptyset$ if and only if $r=r_j$ for some $0\leqslant j\leqslant s$ and all diagrams μ appearing on the right side of (15) have the form μ_j. Moreover, removing the cell with number n in the tableau $T\in STY^{(r_j)}(\lambda)$ gives a bijection

$$STY^{(r_j)}(\lambda)\xrightarrow{\pi_j}STY(\mu_j). \tag{27}$$

Thus, we have obtained a (well-known) bijective proof of the decomposition (25). Further, we recall [2] the description of the Kostka-Green-Fulkes polynomials $K_{\lambda,\mu}(q)$ due to A. Lask and M.-P. Schützenberger as generating polynomials of the charge functional C on the set $STY(\lambda,\mu)$ of standard Young tableaux of the form λ and weight μ:

$$K_{\lambda,\mu}(q)=\sum_{T\in STY(\lambda,\mu)}q^{c(T)}. \tag{28}$$

One can find the definition of c(T), the charge of the tableau T, for example, in [2] or [5]. It is easy to see how the charge functional behaves under the map (27). The answer is the following:

$$c(T)=c(\pi_j(T))+n-1-p(T), \tag{29}$$

where p(T) is the number of descents in the tableau $T\in STY(\lambda)$. Indeed, we denote by $DES(T)$ the set of descents in the tableau T, i.e., the set of numbers $k\in T$ such that the number $k+1$ is located in the tableau T strictly below the number k. We set $des(T)=\sum k, k\in DES(T)$, $T'=\pi(T)$, $p(T)=|DES(T)|$. Then it is easily verified that

$$des(T)-des(T')=(n-1)(p(T)-p(T')). \tag{30}$$

Equation (29) follows from (30) and the relation (cf. [5])

$$c(T)=\frac{n(n-1)}{2}-p(T)\cdot n+des(T). \tag{31}$$

Returning to the question of the validity of (5) for the polynomials $K_{\lambda,(1^n)}(q)$ we see that the main reason obstructing this is (29), the discrepancy of the charge functional under the map (27). Nevertheless, the Kostka polynomials $K_{\lambda,(1^n)}(q)$ satisfy the following recurrence agreement

$$K_{\lambda,(1^n)}(q)=\sum_\mu q^{|\mu|}\frac{(q)_n}{(q)_{|\mu|}}K_{\mu,(1^{|\mu|})}(q), \tag{32}$$

the summation in (32) is over all diagrams μ for which

$$\lambda_1 \geqslant \mu_1 \geqslant \lambda_2 \geqslant \mu_2 \geqslant \lambda_3 \geqslant \ldots \geqslant \lambda_{n-1} \geqslant \mu_{n-1} \geqslant \lambda_n . \tag{33}$$

In two words, (5) reflects the ramification rule for the symmetric, and (32) that for the general linear groups. We pass to the derivation of (32).

We consider the irreducible representation $V_\lambda^{(n)}$ of the general linear group $GL(n)$, corresponding to the Young diagram λ. We recall D. Littlewood's formula for the dimension of the representation $V_\lambda^{(n)}$:

$$\dim V_\lambda^{(n)} = \prod_{x \in \lambda} \frac{n + c(x)}{h(x)} \tag{34}$$

here $h(x)$ is the hook length at the point $x = (i,j) \in \lambda, c(x) = j - i$. There are formulas of Weyl for the dimension of the representation $V_\lambda^{(n)}$:

$$\dim V_{\lambda/\mu}^{(n)} = \det \left| \binom{n}{\lambda_i - \mu_j - i + j} \right|_{1 \leqslant i, j \leqslant n} , \tag{35}$$

$$\dim V_\lambda^{(n)} = \frac{1}{1!\,2!\ldots(n-1)!} \prod_{1 \leqslant i < j \leqslant n} (\lambda_i - \lambda_j - i + j). \tag{36}$$

Littlewood's formula (34) can be obtained from Weyl's formula (35) (for $\mu=0$) with the help of elementary transformations over the matrix which appears on the right side of (35). We note that $\dim V_{\lambda/\mu}^{(n)}$ is a polynomial in n for $n \geqslant \ell(\lambda \backslash \mu)$ of degree equal to $|\lambda - \mu|$ while

$$\dim V_{\lambda/\mu}^{(n)} = \dim S^{\lambda/\mu} \cdot \frac{n^{|\lambda-\mu|}}{(|\lambda-\mu|)!} + O(n^{|\lambda-\mu|-1}), \tag{37}$$

here $\dim S^{\lambda/\mu}$ is given by the Frobenius formula (3).

We recall the ramification rule for $GL(n)$: Under restriction to the subgroup $GL(n-1) \hookrightarrow GL(n)$ the representation $V_\lambda^{(n)}$ splits into the direct sum of all irreducible representations $V_\mu^{(n-1)}$ of the group $GL(n-1)$, whose diagrams satisfy (33). On the level of characters

$$S_\lambda(x_1, \ldots, x_{n-1}, 1) = \sum_\mu S_\mu(x_1, \ldots, x_{n-1}), \tag{38}$$

the sum is taken over all diagrams μ satisfying (33). After the specialization $x_i = q^i, 1 \leqslant i \leqslant n-1$, we arrive at the identity for generalized q-binomial coefficients

$$q^{n(\lambda)} \begin{bmatrix} n \\ \lambda' \end{bmatrix}_q = \sum_\mu q^{n(\mu)+|\mu|} \begin{bmatrix} n-1 \\ \mu' \end{bmatrix}_q , \tag{39}$$

the sum in (39) is taken over all diagrams μ satisfying (33). We recall [2] the definition of the generalized q-binomial coefficients:

$$\begin{bmatrix} n \\ \lambda \end{bmatrix}_q = \prod_{x \in \lambda} \frac{1 - q^{n - c(x)}}{1 - q^{h(x)}} . \tag{40}$$

The recurrence relation (5) can be obtained from (39) after the substitution $q=1$ and equating the coefficients of maximal degree on the left and right sides of (39). Equation (39) follows from (38) and the well-known fact (cf., e.g., [2]):

$$S_\lambda(1, q, \ldots, q^{n-1}) = q^{n(\lambda)} \begin{bmatrix} n \\ \lambda' \end{bmatrix} . \tag{41}$$

Equation (32) follows from (39) after multiplication of both sides of this equality by $(q)_{|\lambda|-1}$ and passage to the limit as $n \to \infty$.

We give a combinatorial interpretation of (39). For this we consider the set $STY(\lambda \backslash \mu, \leqslant n)$ of standard Young tableaux of the form $\lambda \backslash \mu$ filled, possibly with repetitions, by the numbers from 1 to n. It is well-known [2] that

$$d^{(n)}_{\lambda\setminus\mu} := |STY(\lambda\setminus\mu, \leqslant n)| = \dim V^{(n)}_{\lambda/\mu}. \tag{42}$$

Thus,

$$d^{(n)}_{\lambda} = \begin{bmatrix} n \\ \lambda' \end{bmatrix}_{q=1}. \tag{43}$$

Further, let $T \in STY(\lambda, \leqslant n)$. We denote by $\|T\|$ the sum of all numbers appearing in the tableau T minus the total number of cells in λ. Then

$$d^{(n)}_{\lambda}(q) := \sum_T q^{\|T\|} = q^{n(\lambda)} \begin{bmatrix} n \\ \lambda' \end{bmatrix}_q. \tag{44}$$

We note that after multiplication of the right side of (44) by $q^{|\lambda|}$ we get the generating function for flat partitions of the form λ (cf. [2], p. 63), all of whose parts are $\leqslant n$. The Schützenberger involution [4] establishes a bijection between these sets.

We turn to the verification of (39). Removing from the tableau $T \in STY(\lambda, \leqslant n)$ all cells containing the number n, we get as a result that the polynomials $d^{(n)}_{\lambda}(q)$ [cf. (44)] satisfy (39). By virtue of (44) the generalized q-binomial coefficients (40) satisfy the same relation.

We note that one can derive (43) and (44) from Lagrange's identity (and its q-analog) completely analogously to the way the hook formula (4) was obtained in the present paper from Lagrange's identity (14).

The present note arose after Vershik's report [1] to the seminar on the theory of representations at the Leningrad Section of the Steklov Mathematical Institute. The author thanks A. M. Vershik profoundly for explaining the results of [1] and many helpful comments.

CONCLUSIONS

It is shown in the present paper that the hook formula for the number of standard Young tableaux of the form λ without repetitions can be derived from Lagrange's identity (14). The natural q-generalization of this derivation leads to the proof of the theorem that the generating function $d^{(n)}_{\lambda}(q)$ for standard Young tableaux of the form λ filled, possibly with repetitions, with the numbers from 1 to n, coincides up to degree q with the generalized q-binomial coefficient $\begin{bmatrix} n \\ \lambda' \end{bmatrix}$ [cf. (44)]. The hook formula (4) is obtained from (44) after considering the coefficients of higher degrees in n.

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