Last time

dim λ, ...

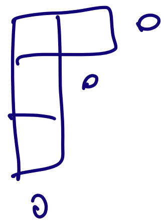
$S(\infty)$

harm. φ on $\mathbb{Y}$

$$\varphi(\lambda) = \sum_{\nu = \lambda + \square} \varphi(\nu)$$

$\varphi \geq 0$

$\varphi(\emptyset) = 1$



$$x^a y^{n-a}(x+y) = \dots$$

$$f_\lambda = \sum_{\nu \searrow \lambda} f_\nu$$

6. Symmetric functions

$f_n(x)$ $\qquad$ $\int f_n f_m \, d\mu = 1_{m=n}$

$1, x, x^2, x^3, \ldots$ $\qquad$ Gram - Schmidt

Symmetric functions

6.1. Algebra Λ

$$\Lambda_n = \mathbb{R}[x_1 \ldots x_n]^{S(n)} = \bigoplus_{k \geq 0} \Lambda_n^k$$

Symm pol in $x_1 \ldots x_n$

$\Lambda_n^k =$ homogeneous degree k poly.

Examples.

$$\left. \begin{array}{l} x_1^2 x_2 + x_2^2 x_1 + x_1^2 x_3 + x_3^2 x_1 \\ \quad + \ldots \end{array} \right) \leftarrow \text{degree 3.}$$

Def.

$$\Lambda_{n+1} \longrightarrow \Lambda_n$$

$$f(x_1 \ldots x_n, x_{n+1}) \longmapsto f(x_1 \ldots x_n, 0)$$

Inverse limit

$$\Lambda^k = \varprojlim_n \Lambda_n^k$$

$$\in$$

$$(f_1, f_2, f_3, \ldots)$$

$$\boxed{\Lambda = \bigoplus_{k=0}^{\infty} \Lambda^k}$$

(sometimes Sym)

$\deg f_i = k \quad \forall i, \qquad f_i$ - homogeneous

$$f_{n+1}\Big|_{x_{n+1}=0} = f_n \quad \forall n$$

Examples. $\quad p_1 = e_1 = h_1 = x_1 + x_2 + x_3 + \cdots$

deg = 1

$$\left(f_n = x_1 + \cdots + x_n \right)$$

Non-example $\quad \prod_{i=1}^{\infty} (1 + x_i) \notin \Lambda$

need bounded degree.

6.2 e_k, h_k & Generating functions

$\boxed{e_0 = h_0 = 1}$

$$e_k = \sum_{1 \le i_1 < \dots < i_k} x_{i_1} \cdot \ldots \cdot x_{i_k}$$

$\deg e_k = k$
$= \deg h_k$

elementary sym. poly

$e_k(x_1, \dots, x_n) = 0$ if $k > n$

complete homogeneous

$$h_k = \sum_{1 \le i_1 \le \dots \le i_k} x_{i_1} \cdot \ldots \cdot x_{i_k}$$

(every possible monomial of deg k)

$$\begin{cases} e_2 = x_1x_2 + x_1x_3 + x_2x_3 + \cdots \\ h_2 = e_2 + x_1^2 + x_2^2 + \cdots \end{cases}$$

$e_k = e_k(x_1, x_2, \dots)$

(Viet) $$\sum_{k=0}^{\infty} e_k t^k = \prod_{i=1}^{\infty} (1 + x_i t)$$
$$= \boxed{E(t)}$$

$$1 + (x_1 + x_2)t + x_1x_2t^2 = (1 + x_1t)(1 + x_2t)$$

$n = 2$ var's

$$e_k(\underbrace{1, \dots, 1}_{n}) = \binom{n}{k}$$

$$\sum_{k=0}^{\infty} h_k t^k = \prod_{i=1}^{\infty} \left(1 + tx_i + (tx_i)^2 + (tx_i)^3 + \dots\right)$$

$$= \prod_{i=1}^{\infty} \frac{1}{1 - tx_i} = H(t)$$

$$h_k(\overbrace{1 \dots 1}^{n}) = \binom{n+k-1}{n} \qquad \text{(exercise)}$$

$$E(t)\,H(-t) = 1.$$

$$\begin{array}{ccc} \| & & \| \\ \sum e_k t^k & & \sum h_k (-t)^k \end{array}$$

$1 + e_1 t + e_2 t^2 + \dots$

$1 - h_1 t + h_2 t^2 + \dots$

$(e_0 = h_0 = 1)$

$$e_0 h_0 = 1$$

$$e_1 - h_1 = 0$$

$$e_2 - e_1 h_1 + h_2 = 0$$

etc.

$\vdots$

$$\Rightarrow \quad e_k \in \mathbb{R}[h_1, \dots, h_k]$$

$$h_k \in \mathbb{R}[e_1, \dots, e_k]$$

6-3 p_k, m_λ, more relations

p_k & e_n, h_n | $p_k = x_1^k + x_2^k + x_3^k + \dots$

power sums

$$p(t) = \sum_{k=1}^{\infty} \frac{p_k}{k} t^k = \log \overbrace{\left(\prod_{i=1}^{\infty} \frac{1}{1 - x_i t} \right)}^{H(t)}$$

$|x_i t| < 1$

$$\sum_{k \geq 1} \frac{x^k t^k}{k} = \log (1 - xt)^{-1}$$

$$\boxed{H(t) = e^{p(t)} = \frac{1}{E(-t)}}$$

$\Downarrow$

relation for coeffs $\qquad h \leftrightarrow p$

$$1 + h_1 t + h_2 t^2 + \dots = \exp\left(p_1 t + p_2 t^2/2 + p_3 t^3/3 + \dots \right)$$

$x_1^2 + x_1 x_2 + \dots$

$$= 1 + p_1 t + p_2 t^2/2 + \dots$$
$$+ \frac{1}{2}\left(p_1 t + p_2 t^2/2 + \dots \right)^2$$
$$+ \dots$$

$\uparrow$

$$h_2 = \frac{p_2}{2} + p_1^2/2$$

Def. $m_\lambda = \sum x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_{\ell(\lambda)}^{\lambda_{\ell(\lambda)}}$

sum over all distinct monomials

$\lambda = \ydiagram{3,3,1}$

$$m_{\ydiagram{2,2}} = x_1^2 x_2^2 + x_1^2 x_3^2 + \cdots$$

$$m_{\ydiagram{1}} = p_1 = e_1 = h_1$$

$$m_{\underbrace{\ydiagram{3}}_{k}} = p_k \qquad m_{\left.\ydiagram{1,1,1,1}\right]k} = e_k$$

Prop. $(m_\lambda)_{\lambda \in \mathbb{Y}}$ is linear basis in Λ

$\square$

Orthogonality. (in Λ_n)

$$\langle f,g\rangle = \frac{1}{n!}\oint_{|z_1|=\dots=|z_n|=1} f(z)\,\overline{g(z)}\,\frac{dz_1\dots dz_n}{\prod_j (2\pi i z_j)}$$

$\bar z = \frac{1}{z}$ if $|z|=1$

$\langle m_\lambda, m_\mu\rangle =$

$$\oint z^k \frac{dz}{2\pi i z} = \begin{cases}1, & k=0\\ 0, & \text{else}\end{cases}$$

$k\in\mathbb{Z}$

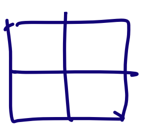
$$= \frac{1}{n!}\oint\dots\oint \sum_{i_1,\dots,i_n} z_{i_1}^{\lambda_1}\dots z_{i_n}^{\lambda_n}\, z_{i_1}^{-\mu_1}\dots z_{i_n}^{-\mu_n}\,\frac{d\vec z}{(2\pi i)^n\,\vec z}$$

$= 0$ if $\lambda\neq\mu$.

if $\lambda=\mu$

$$= \frac{1}{n!}\cdot\frac{n!}{(\text{combinatorial factor})}$$

Orthogonal basis in each Λ_n.

x_1, x_2 

$\frac{2!}{2} = 1$

$x_1^2x_2^2 + \cancel{x_1^2x_2^2}$

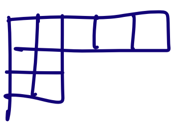
6.4 Fundamental theorem

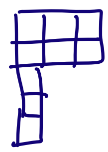
$$\Lambda_n = \mathbb{R}[e_1, \dots, e_n]$$

$$\Lambda = \underbrace{\mathbb{R}[e_1, e_2, \dots]}_{\text{finite linear comb.'s of monomials in } e_j\text{'s}}$$

(every symm. f. is a polynomial in e_i's)

Idea: $\lambda \rightsquigarrow \lambda' =$ transpose

$\lambda =$ 

$\lambda' =$ 

$$e_\lambda = e_{\lambda_1} e_{\lambda_2} \cdots e_{\lambda_k}$$

triangular change of var's.

$$e_{\lambda'} = m_\lambda + \sum_{\mu<\lambda} c_{\lambda\mu} m_\mu \qquad (*)$$

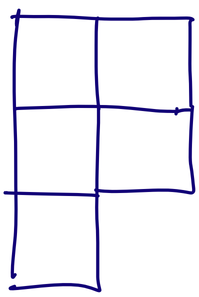
partial order

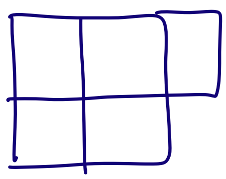
$$\boxed{\mu<\lambda : \quad \mu_1+\dots+\mu_k \le \lambda_1+\dots+\lambda_k \ \forall k \ \ \& \ \mu\neq\lambda}$$

λ is obtained from μ by moving some boxes up

Proof of $(*)$

$e_{\lambda'}$ = monomial expansion

λ 

$\lambda' =$ 

$e_3 \quad e_2$

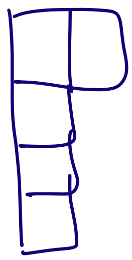
$x_1x_2x_3 \quad x_1x_2$

$m_\lambda \quad x_1^2x_2^2x_3$ leading coeff

Something else?

$x_1 x_2 x_3 \quad * \quad x_1 x_4$

$\Downarrow$

μ  $\longleftarrow \quad x_1^2 x_2 x_3 x_4$

$\mu < \lambda$ (move box down)

$\square$

<u>So;</u> $e_{\lambda'}$ also a linear basis of Λ

$e_{\lambda'} \longleftrightarrow m_\lambda$ related by unitriangular change of variables.

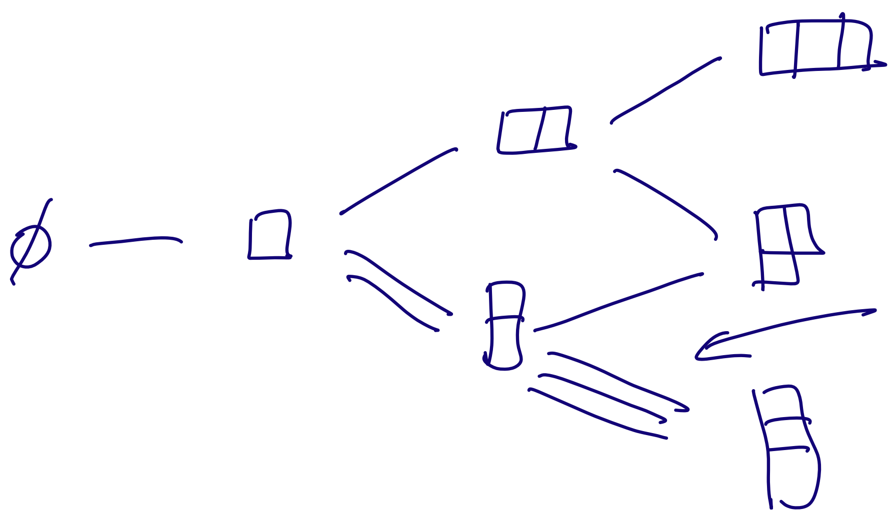
$\Downarrow$

m_λ = linear comb. of $e_{\mu'}$, so a polynomial in e_j's

$\Rightarrow$ Fundam theorem is done.

$$p_1 m_\lambda = \sum_{\mu = \lambda + \square} x_{\lambda\mu} m_\mu$$

$x_1 x_2 (x_1 + x_2 + x_3 + \cdots$



$x_{\lambda\mu}$ edge mult.

So, $\{m_\lambda\}_{\lambda \in \mathbb{Y}}$ is not the "right" basis for the Young graph

6.5 Antisymm. functions & Schur pol.

next lecture

6.6. Pieri rule

$$p_1 s_\lambda = \sum_{\nu = \lambda + \square} s_\nu$$

Proof In Λ_n

$$a_{\lambda+\delta}\, p_1 = \sum_{k=1}^{n} a_{\lambda+\delta+\underbrace{e_k}_{\text{basis vector}}}$$

$$\lambda+\delta+e_k = (\lambda_1+n-1,\ \lambda_2+n-2,\ \dots,\ \lambda_k+n-k+1,\ \dots,\ \lambda_{n-1}+1,\ \lambda_n)$$

$a_{\lambda+\delta+e_k}$ vanishes if ---