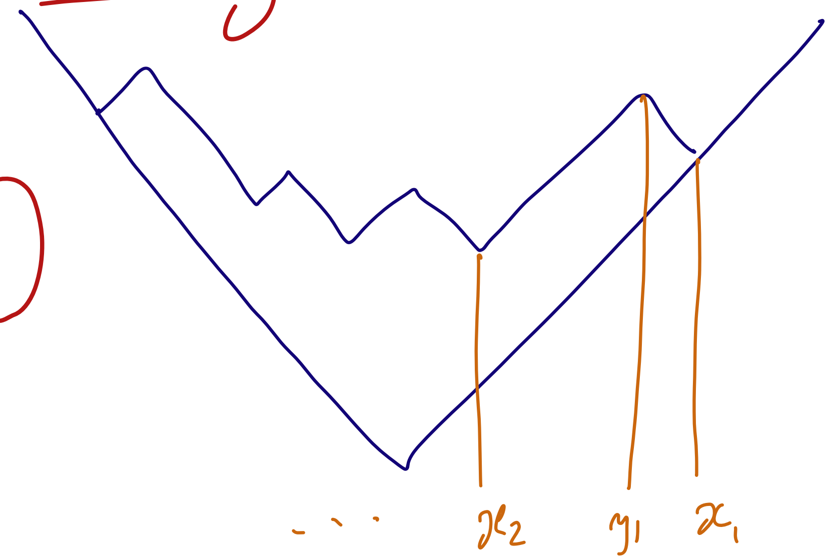
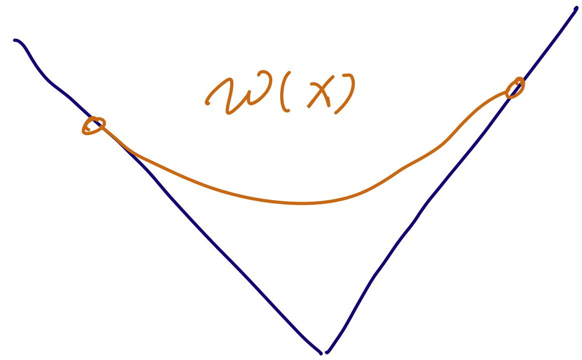
Summary.

1





$$p^{\uparrow}(\lambda,\nu) = \frac{\dim \nu}{(|\lambda|+1)\dim \lambda} = \pi_i^{\uparrow},$$

$$\frac{\prod_{i=1}^{d-1}(z-y_i)}{\prod_{i=1}^{d}(z-x_i)} = \sum_{i=1}^{d} \frac{\pi_i^{\uparrow}}{z-x_i}$$

$$\sigma(x) = \frac{1}{2}\left(\omega(x) - |x|\right)$$

$$S(z) = \int_{\mathbb{R}} \frac{\sigma'(x)\,dx}{z-x}$$

Def. $\pi^{\uparrow}(x)$: $|z|$ large

$$e^{S(z)} = \int_{\mathbb{R}} \frac{d\pi^{\uparrow}(x)}{1-x/z}$$

2

Moments: $$\tilde{p}_k = \int_{\mathbb{R}} x^k \, d\,\sigma'(x)$$

$$= -k \int_{\mathbb{R}} x^{k-1} \sigma'(x)\, dx$$

$$= - \int_{\mathbb{R}} \sigma'(x)\, d(x^k)$$

$$(\text{for rect.}) = \sum x_i^k - \sum y_i^k$$

$$S(z) = \sum_{n=1}^{\infty} \frac{\tilde{p}_n}{n} z^{-n}$$

$$\exp\left(S(z)\right) = \sum_{n=0}^{\infty} \tilde{h}_n z^{-n} = \int_{\mathbb{R}} \frac{d\pi^T(x)}{1 - x/z}$$

$\Rightarrow$ $\tilde{h}_n$ are moments of π^T:

$$\tilde{h}_n = \int_{\mathbb{R}} x^n \, d\pi^T(x)$$

③ Symm. f.

$p_k = \sum x_i^k$

h_n = complete homog.

$$\boxed{e^{\sum_1^\infty p_n/n\, t^n} = \sum_0^\infty h_n t^n}$$

Also $\exists$ s_λ (continual y.d.)

Ex. $s_\lambda(\Omega) = \det\left[\underbrace{h_{\lambda_i + j - i}(\Omega)}_{1,\ 0 \text{ or Catalan}}\right]$

$C_n = \frac{1}{n+1}\binom{2n}{n}$

$$\det(C_{j+i}) = \det\begin{bmatrix} C_1 & C_2 & C_3 & \cdots \\ C_2 & C_3 & \cdots & \\ C_3 & \cdots & & \\ \cdots & & & \end{bmatrix} = 1$$

④ VKLS $\quad \Omega$

$$\tilde{p}_{2m-1}(\Omega) = 0, \qquad \tilde{p}_{2m}(\Omega) = \binom{2m}{m}$$

$$S(z) = \sum_{n=1}^{\infty} \frac{\tilde{p}_n}{n} z^{-n}$$

$\arcsin(2z)$

$$= \log\frac{z}{2} + \boxed{\log\left(z - \sqrt{z^2-4}\right)}$$

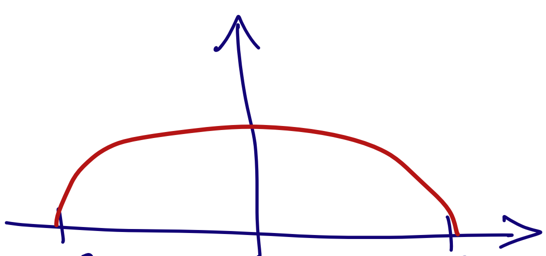
(Taylor series for arcsine)

$$\tilde{h}_{2m-1} = 0, \qquad \tilde{h}_{2m} = \underbrace{\frac{1}{m+1}\binom{2m}{m}}$$

Catalan

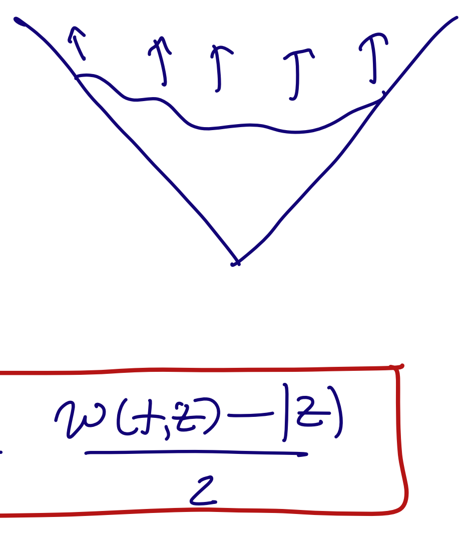
$$\Rightarrow d\pi^{\uparrow}(x) = \frac{1}{2\pi}\sqrt{4-x^2}\,dx$$

Semicircle density



13.4. Growth model

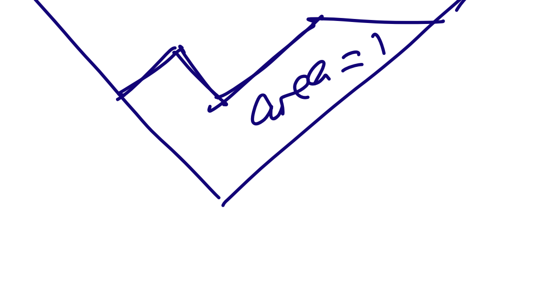
Large scale behavior of Plancherel growth.



Area $= t$ (time)

$$\| \qquad \int_{\mathbb{R}} \sigma(t,z)\, dz, \qquad \boxed{\sigma(t,z) = \frac{w(t,z) - |z|}{2}}$$

Start at $t=1$



π^0

Let $T(t,z) = \frac{\partial}{\partial t} \rho(t,z)$,

So $\int_{\mathbb{R}} T(t,z)\, dz = 1$

probab. distribution

Def. Planckerel growth ω_t :

$$\frac{\partial}{\partial t} \rho(t,x)\, dx = d\,\pi^{T}(\omega(t,\cdot))(x)$$

equality of 2 probab. densities

(∞-dim ODE in space of cont. y.d.

$$\boxed{\frac{\partial}{\partial t} \rho = F(\rho)}$$)

Rewrite equation

(1) $$\int_{\mathbb{R}} x^n T(t,x)\,dx =$$

$$= \frac{\tilde{p}_{n+2}(t)}{(n+1)(n+2)}$$

because

$$\int_{\mathbb{R}} x^n T(t,x)\,dx = \frac{\partial}{\partial t}\int_{\mathbb{R}} x^n \sigma(t,x)\,dx$$

$$\int_{\mathbb{R}} x^n \sigma(t,x)\,dx =$$

= twice by parts

$$= \int \frac{x^{n+2}}{(n+1)(n+2)} \sigma''(t,x)\,dx$$

$$= \frac{p_{n+2}(t)}{(n+1)(n+2)}. \qquad \square$$

(2) $$T(t,x)\,dx = d\pi^{T}(\omega(t,\cdot))(x)$$

$\Rightarrow$ moments:

$$\frac{\tilde{p}'_{n+2}(t)}{(n+1)(n+2)} = \overbrace{\tilde{h}_n(t)}^{\text{moments of } \pi^{T}}$$

③ via $S(t,z)$:

know

$$\exp\left(S(t,z)\right) = \int \frac{\overbrace{d\pi^T(w(t,0))(x)}^{\frac{\partial}{\partial t} b}}{1 - z/x}$$

$$\Rightarrow \boxed{\exp \int \frac{b'_x(t,x)\,dx}{z-x} = \int \frac{b'_t(t,x)\,dx}{1 - z/x}}$$

④ Define

"R-transform"

$$R(t,z) = \sum_{n=0}^{\infty} \tilde{h}_n(t)\, z^{-n-1}$$

$$\left(\sum_{n\geq 1}^{\infty} \frac{\tilde{p}_n(t)}{n} z^{-n}\right) = S(t,z) = \log\left(z R(t,z)\right)$$

$$\frac{\partial}{\partial t} \implies \boxed{S'_t = \frac{R'_t}{R}}$$

& know

specific to Plancherel growth

$$\boxed{\frac{\tilde{p}'_{n+2}(t)}{(n+1)(n+2)} = \tilde{h}_n(t)}$$

$\Rightarrow$ R satisfies $\boxed{R'_t + R R'_z = 0}$

Indeed, $S'_t = \sum_{n=0}^{\infty} \frac{\tilde{\rho}'_{n+2}(t)}{n+2} z^{-n-2}$ $\quad (\tilde{\rho}_1 = 0)$

$$= \sum_{n=0}^{\infty} \tilde{h}_n(t) \cdot (n+1)\, z^{-n-2}$$

$$= \boxed{-R'_z}$$

$\Rightarrow$ $-R'_z = \frac{R'_t}{R}$, $\boxed{R'_t + R R'_x = 0}$

$\left(\text{Burgers}: \quad \rho_t + (\rho(1-\rho))_x = 0 \right)$

$\boxed{\text{Appl. to VKLS}}$

For Ω at $t=1$

Let $r(x) = \frac{1}{2}\left(x - \sqrt{x^2-4}\right)$

Check :

$R(t,x) = \dfrac{r(x/\sqrt{t})}{\sqrt{t}}$ satisfies

$$\boxed{R_t' + R R_z' = 0}$$

Facts. $r(x/\sqrt{t})/\sqrt{t}$ is

(1) the unique automodel solution to

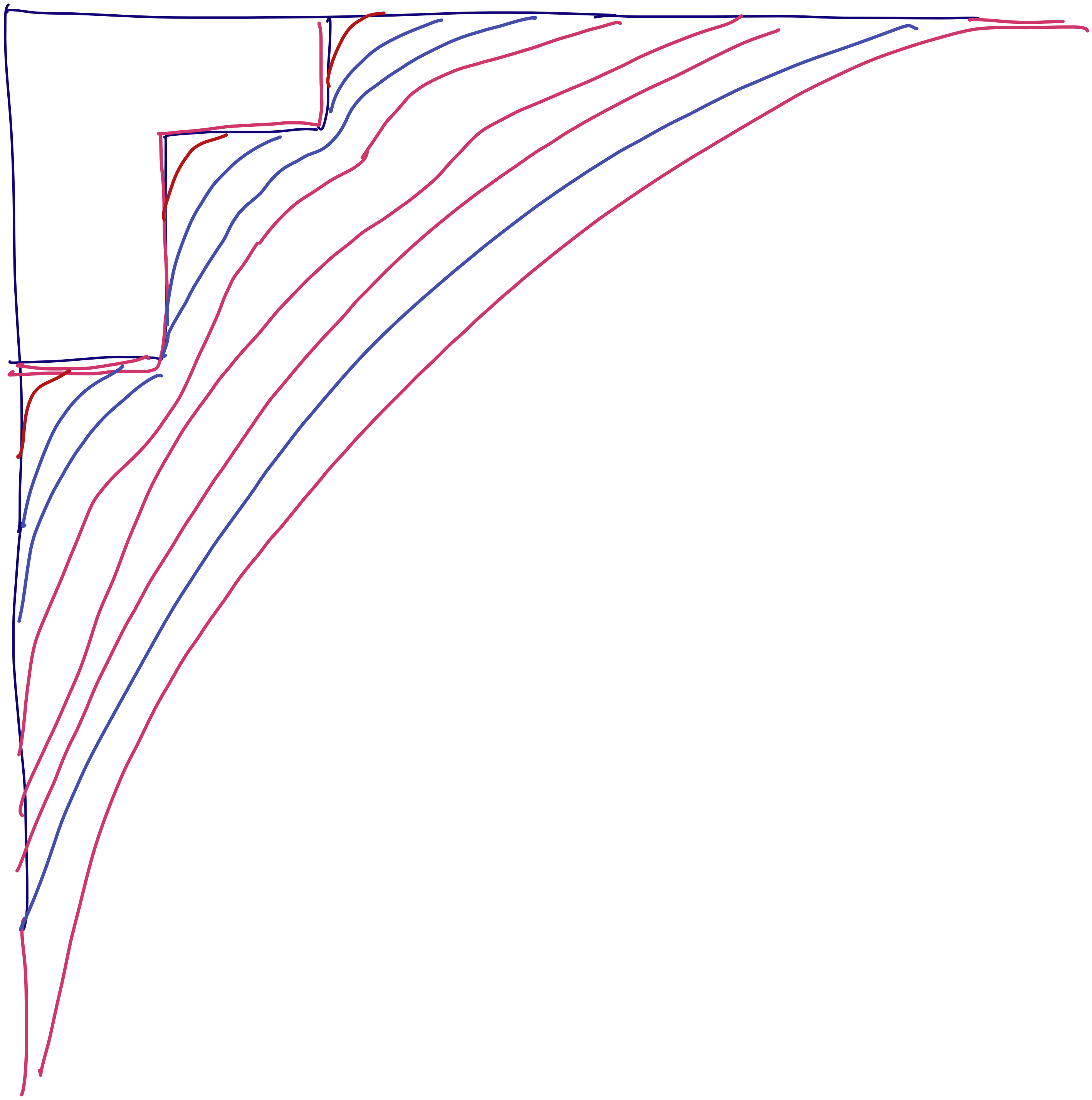
$$\boxed{R'_t + R R'_z = 0}$$

(2) Started from any continual Young diagram

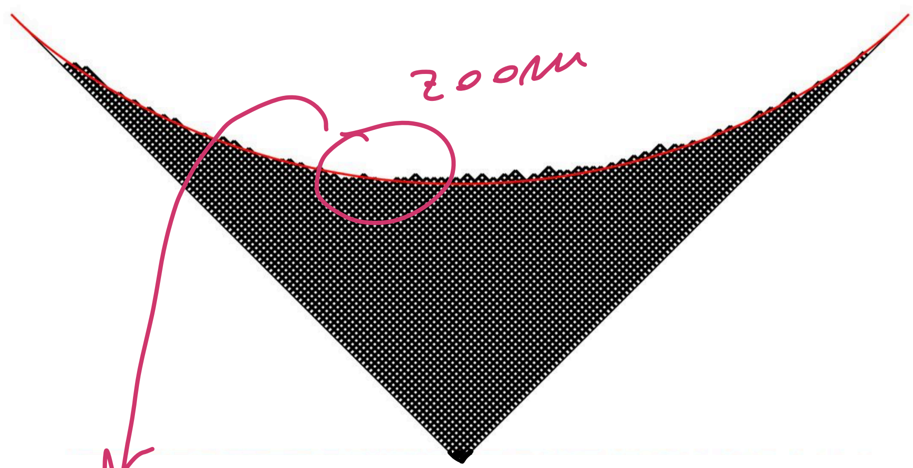
$R(t=1, x) = R_1(x)$,

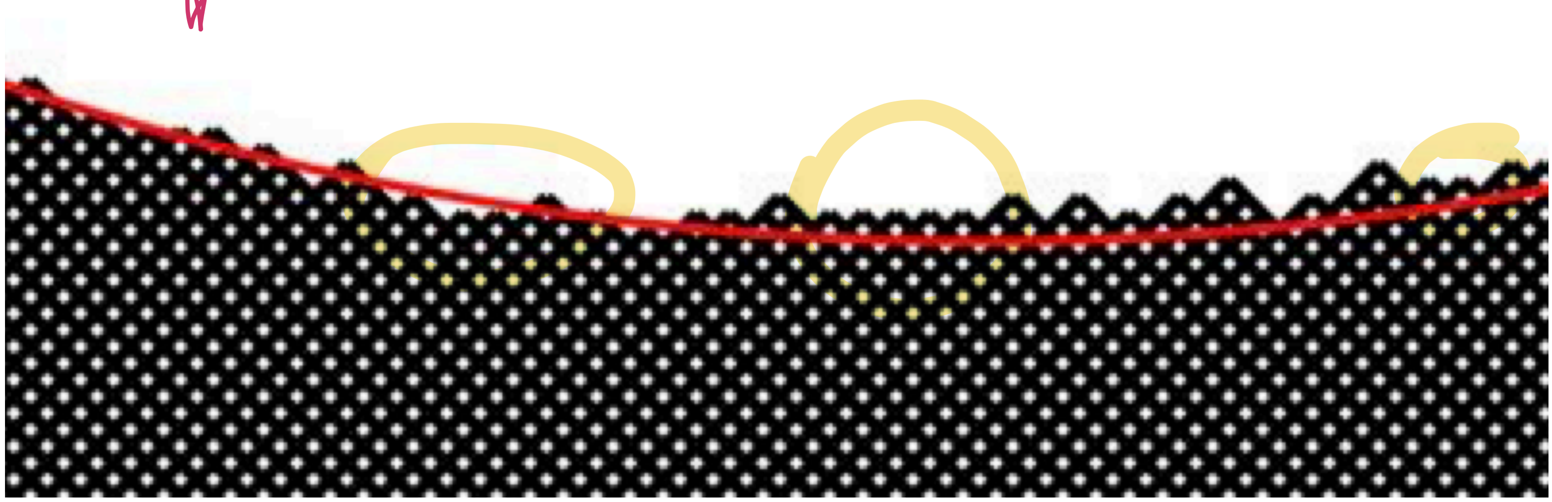
The equation's solution converges to the VKLS solution.

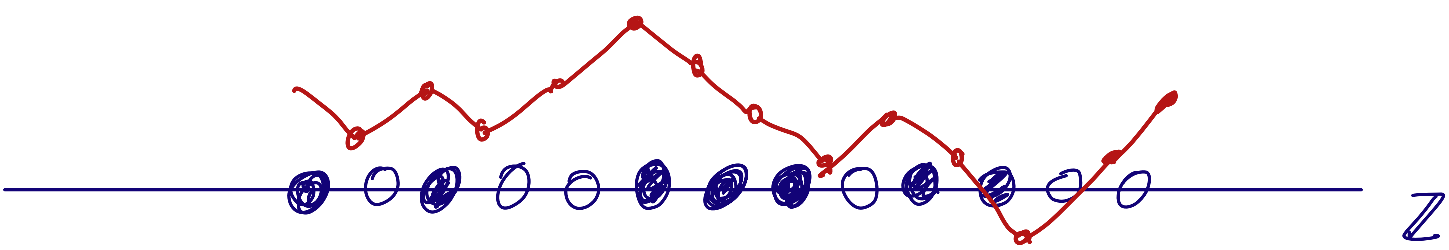
(So, $\forall$ initial y.d., the Plancherel growth produces VKLS)



14. Local correlations of Plancherel

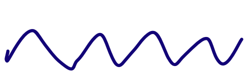
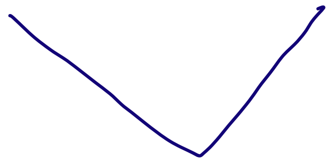






Locally Bernoulli? — $\boxed{\text{No}}$

Some other law?

$\mathbb{P}($  $)$ vs $\mathbb{P}($  $)$

more likely

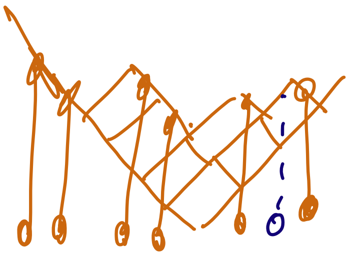
14.1 Infinite wedge space (Fock space)

[Kac ∞ dim Lie alg.]

[Okounkov 1999], ...

we only take a particular case

① $\lambda \in \mathbb{Y} \longrightarrow \{\lambda_i - i + \frac{1}{2}\}_{i \geq 1}$



$$v_\lambda = v_{\lambda_1 - 1 + \frac{1}{2}} \wedge v_{\lambda_2 - 2 + \frac{1}{2}} \wedge \cdots$$

$$\lambda = (4,3,1) \longrightarrow \boxed{v_\lambda = v_{3+\frac{1}{2}} \wedge v_{1+\frac{1}{2}} \wedge v_{-2+\frac{1}{2}} \wedge v_{-4+\frac{1}{2}} \wedge v_{-5+\frac{1}{2}} \cdots}$$

$$v_\emptyset = v_{-\frac{1}{2}} \wedge v_{-\frac{3}{2}} \wedge v_{-\frac{5}{2}} \wedge \cdots$$

(„vacuum")

$$\langle v_\lambda, v_\mu \rangle = \delta_{\lambda,\mu}$$

(Hilbert space)

$\boxed{\ell^2(\mathbb{Y})}$

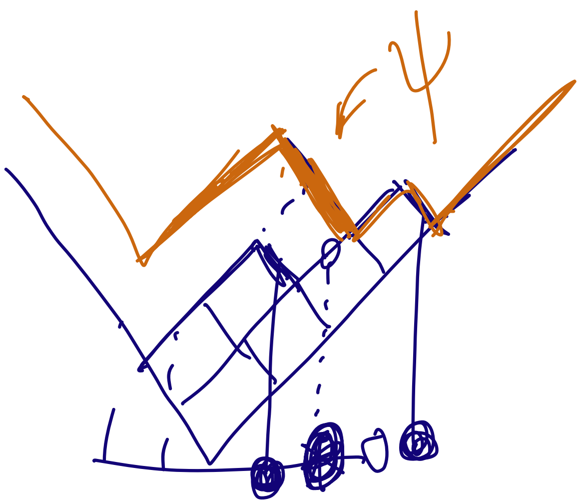
2. ψ_i, ψ_i^* $\boxed{i \in \mathbb{Z} + \frac{1}{2}}$

create

conjugate (annihilate)

$\psi_i v_\lambda = v_i \wedge v_\lambda$

anticommute to the place



$v_i \wedge v_i = 0$

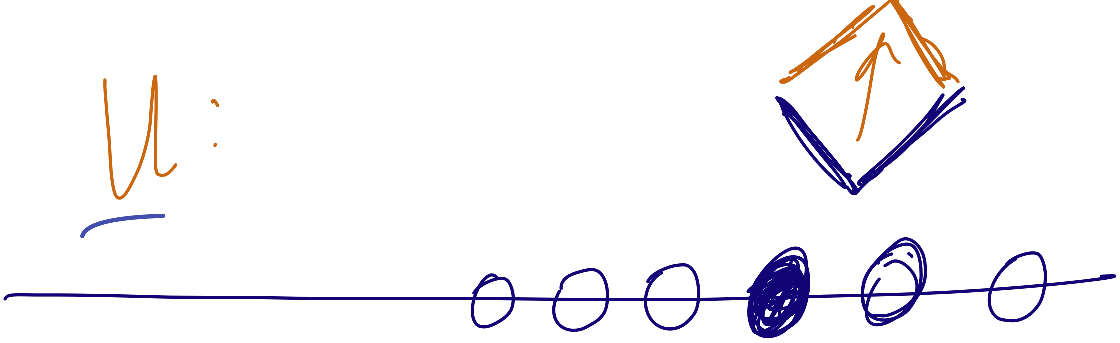
ψ_i^* - conjugate, removes v_i if it can

③ Let $\boxed{\begin{aligned} U v_\lambda &= \sum_{\mu=\lambda+\square} v_\mu \\ D v_\lambda &= \sum_{\mu=\lambda-\square} v_\mu \end{aligned}}$

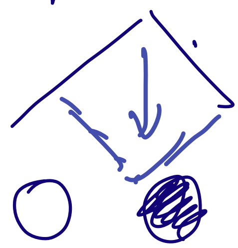
<u>Lemma</u>. $\boxed{\begin{aligned} U &= \sum_k \psi_k \psi^*_{k-1} \\ D &= \sum_k \psi_k \psi^*_{k+1} = U^* \end{aligned}}$

<u>Proof</u>.

$\mathcal{U}$:



$\mathcal{D}$:



(4) dim λ via $\boxed{\mathcal{U}, \mathcal{D}}$

$$\dim \lambda = \langle \mathcal{U}^n v_\emptyset, v_\lambda \rangle, \quad |\lambda| = n$$

$$= \langle \mathcal{D}^n v_\lambda, v_\emptyset \rangle$$

Plancherel measure via $\mathcal{U}, \mathcal{D}$

$$M_n(\lambda) = \frac{1}{n!} \underline{\langle \mathcal{U}^n v_\emptyset, v_\lambda \rangle} \underline{\langle \mathcal{D}^n v_\lambda, v_\emptyset \rangle}$$

vs $\langle B^? v_\emptyset, v_\emptyset \rangle$

Poissonized Plancherel (θ^2 - Parameter)

n - Poisson random $\sim \theta^2$

$$M_{\theta^2}(\lambda) = \operatorname{Prob}(\Pi_{\theta^2} = n) \cdot M_n(\lambda)$$

$$= e^{-\theta^2} \theta^{2n} \left(\frac{\dim \lambda}{n!}\right)^2 \quad \boxed{n = |\lambda|}$$

$$\# \text{boxes} \approx \theta^2 \pm c \cdot \theta$$

$$\langle e^{\theta U} v_\emptyset, v_\lambda \rangle = \sum_{n \geq 0} \frac{\theta^n}{n!} \langle U^n \emptyset, v_\lambda \rangle$$

$$M_{\theta^2}(\lambda) = e^{-\theta^2} \langle e^{\theta D} \mathbb{1}_\lambda e^{\theta U} v_\emptyset, v_\emptyset \rangle$$

$\mathbb{1}_\lambda$ operator, $\mathbb{1}_\lambda v_\mu = \begin{cases} v_\lambda, & \lambda = \mu \\ 0, & \lambda \neq \mu \end{cases}$

Ex. $\mathrm{Prob}\left(\exists i : \lambda_i - i + \frac{1}{2} = 5\right)$

$=$

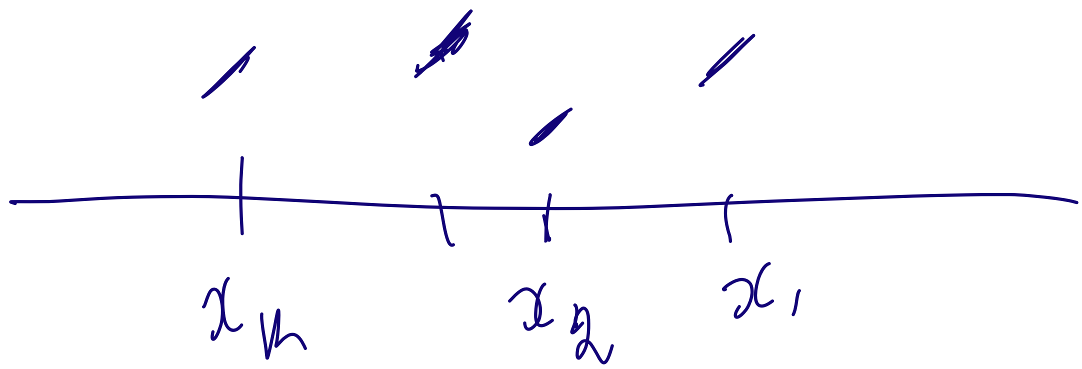
$$e^{-\theta^2} \left\langle e^{\theta D} \left(\text{indicator that } \bullet \text{ at } 5 \right) e^{\theta U} v_\emptyset , v_\emptyset \right\rangle$$

⑤ Correlation function of Poissonized Plancherel & its expression via ∞ wedge space

$\forall k$, $x_1 \ldots x_k \in \mathbb{Z} + \frac{1}{2}$ distinct

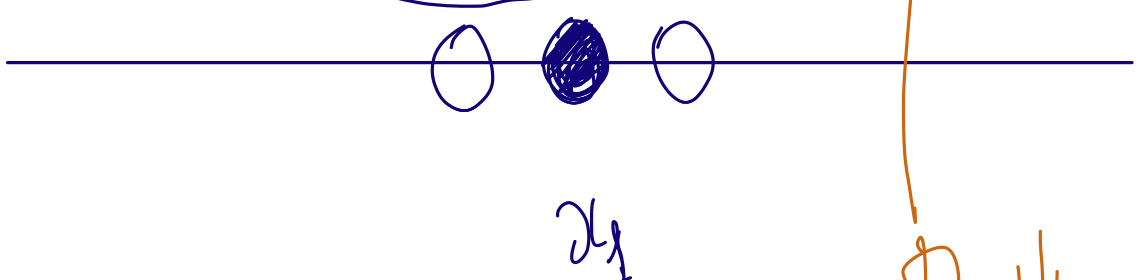
$X = \{x_1 \ldots x_k\}$

$$\rho_k(X), \quad \rho(X) = \text{Prob.}\left(\begin{array}{l}\text{config } \{\lambda_i - i + \tfrac{1}{2}\}_{i=1,2,\dots} \\ \text{contains each} \\ \text{of } x_1, x_2, \dots, x_k\end{array}\right)$$



$$\| \; e^{-\theta^2}\left\langle e^{\theta \mathcal{D}}\, \text{Indicator}(x_1 \dots x_k)\, e^{\theta \mathcal{U}} v_\emptyset, v_\emptyset \right\rangle$$

$$\boxed{\psi_{x_1}\psi^*_{x_1}}$$



$$\prod_i \psi_{x_i}\psi^*_{x_i}$$

$$e^{-\theta^2}\left\langle e^{\theta D}\prod_{i=1}^{k}\psi_{x_i}\psi^*_{x_i}\,e^{\theta U}\,v_\emptyset, v_\emptyset\right\rangle$$

$$\| $$

$$\rho_k(X)$$

Want to show:

$$\rho_k(X) = \det\left[K(x_i, x_j)\right]_{i,j=1}^{k}$$

Wick theorem

because

① $\psi_i\psi_j + \psi_j\psi_i = 0$

$$\psi^*_i \psi^*_j + \psi^*_j \psi^*_i = 0$$

$$\psi_i \psi^*_j + \psi^*_j \psi_i = 1_{i=j}$$

(2)

$$\langle e^{\theta D} \prod_i \psi_{x_i} \psi^*_{x_i} e^{\theta U} v_\emptyset, v_\emptyset \rangle$$

$$= \langle \prod_i \underbrace{\Psi_{x_i}} \underbrace{\Psi^{-*}_{x_i}} v_\emptyset, v_\emptyset \rangle$$

linear combin. of ψ_j, ψ^*_j resp.

$$\langle \prod_i \Psi_{x_i} \Psi^{-*}_{x_i} v_\emptyset, v_\emptyset \rangle$$

14.2 Correlations & density — formulas