Recall

1) Symm. functions Λ

$e_k, \; h_k, \; p_k, \; m_\lambda$

$$s_\lambda(x_1, \dots, x_N) = \frac{\det\left[x_i^{\lambda_j + N - j}\right]_1^N}{\prod_{1 \le i < j \le N} (x_i - x_j)}$$

N var.

$$\det\left[x_i^{N-j}\right]_1^N$$

Note: s_λ vs $s_\lambda(x_1, \dots, x_N)$

2) $\Lambda \longleftrightarrow \mathbb{Y}$ graph

$$s_\lambda p_1 = \sum_{\nu = \lambda + \square} s_\nu$$

multiplicative graphs

$\updownarrow$

Algebra + basis

3) Characters of $S(\infty)$ (extreme, normalized)

$$\{\chi\} \longleftrightarrow \left\{ \begin{array}{l} \text{algebra homomorphisms } \Lambda \to \mathbb{R} \\ F((p_1 - 1)\Lambda) = 0 \\ F(s_\lambda) \geq 0 \quad \forall \lambda \end{array} \right\}$$

Then χ(cycle structure $\rho_1 \geq \rho_2 \geq \dots \geq \rho_\ell \geq 2$)

$$= F(p_{\rho_1})\, F(p_{\rho_2}) \cdots F(p_{\rho_\ell}).$$

Followed from general Ring Theorem & indep. from the functional equation for characters

$$F(p_k) = \begin{cases} 1, & k = 1 \\ \dots, & k \geq 2 \end{cases}$$

Our goal: to classify $\{\chi\}$.

Via ergodic method, need to look at

$$\lambda^{(n)} \in \mathbb{Y}_n, \quad n \to \infty,$$

s.t. $\forall \nu$ – fixed,

$$\boxed{\frac{\dim(\nu, \lambda^{(n)})}{\dim \lambda^{(n)}}}$$ has a limit in n

Thoma (1964), Edrei (1953)

– classification of irred. χ of $S(\infty)$

Vershik – Kerov (1981)

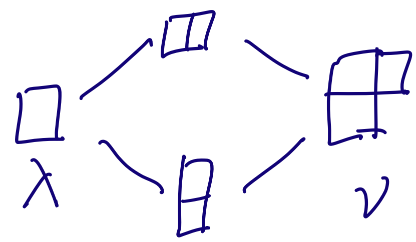
– asymptotic (ergodic) approach

Going along sec. 6 of [Bo] book.

7. Relative dimension & proofs

7.1. det formula for $\dim(\mu,\lambda)$

$$p_1 s_\lambda = \sum_{\nu=\lambda+\square} s_\nu$$



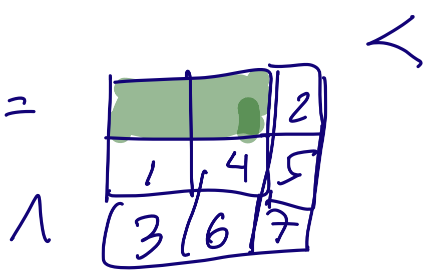
$p_1 = x_1 + x_2 + x_3 + \dots$

$$p_1^k s_\lambda = \sum_{\nu,\ |\nu|=n+k} \dim(\lambda,\nu)\, s_\nu$$

$|\lambda| = n$

$\lambda = \square\square$

$\nu =$



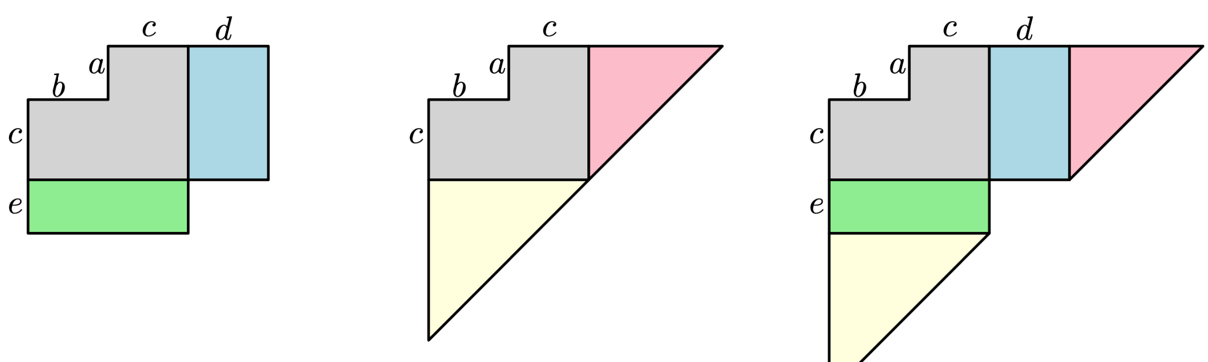
Note $\dim(\mu,\lambda) = f^{\lambda/\mu}$ in comb.

$= \#$ of SYT of skew shapes

(recent progress,
- Naruse hook length formula,
- special cases & asymptotics)

HOOK FORMULAS FOR SKEW SHAPES III. MULTIVARIATE AND PRODUCT FORMULAS

ALEJANDRO H. MORALES*, IGOR PAK*, AND GRETA PANOVA†



(i) (ii) (iii)

FIGURE 1. Skew shapes with product formulas for the number of SYT.

$$\sum_{\lambda} \frac{(\dim \lambda)^2}{n!} = 1$$

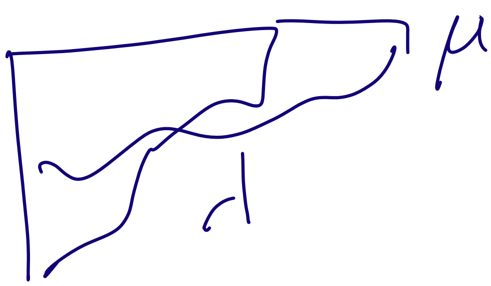
Prop. $N \geq \ell(\lambda)$, $|\lambda| = n$, $|\mu| = m$

$$\frac{\dim(\mu, \lambda)}{(n-m)!} = \det \left[\frac{1}{(\lambda_i - \mu_j + j - i)!} \right]_1^N$$

$\Gamma(n+1) = n!$, $\Gamma(-k) = \infty$

$$p_1^{n-m} s_\mu = \sum_{\lambda} \dim(\mu, \lambda) \, s_\lambda$$

Proof. 1) Vanishing $\mu \not\subset \lambda$



$\mu_i > \lambda_i$

2) $\lambda = \mu$

$$\lambda_i - \mu_j + j - i = 0$$

$$\det \begin{pmatrix} 1 & \ddots & * \\ 0 & & 1 \end{pmatrix} = 1$$

3) $\mu \subset \lambda$, $\mu \neq \lambda$, $\ell(\mu) \leq \ell(\lambda)$,

$m < n$

$\Downarrow$ $s_\mu = \dfrac{a_{\mu+\delta}}{a_\delta}$,

coeff in $\boxed{a_{\mu+\delta}(x_1 + \cdots + x_N)^{n-m}}$

by $\boxed{x^{\lambda+\delta}}$

$$\delta = (N-1, N-2, \ldots, 1, 0)$$

$$a_\alpha = \det [x_i^{\alpha_j}]_1^N$$

$$a_{\mu+\delta}(x_1+\dots+x_N)^{n-m} = \sum_{\lambda} \dim(\mu,\lambda) \circ a_{\lambda+\delta}$$

$\Rightarrow$ Follows from binomial theorem.

Let $\boxed{\begin{aligned} l_i &= \lambda_i + N - i \\ m_i &= \mu_i + N - i \end{aligned}}$

$$\sum_{\sigma \in S_N} (-1)^{\sigma} \cdot \prod x_i^{m_{\sigma i}} (x_1+\dots+x_N)^{n-m}$$

coeff. by $x_1^{l_1} \dots x_N^{l_N}$

Fixed $\sigma \Rightarrow$ coeff.

$$\binom{n-m}{l_1 - m_{\sigma 1} \; \dots \; l_N - m_{\sigma N}}$$

$\boxed{\binom{N}{k_1 \dots k_l} = \frac{N!}{k_1! \dots k_l!}}$ (multinomial)

$$\sum_{\sigma} (n-m)! \, (-1)^{\sigma} \prod_i \frac{1}{(l_i^0 - m_{\sigma_i})!}$$

$$\lambda_i + N - i - (\mu_{\sigma_i} + N - \sigma_i)$$

$\Rightarrow$ determinant $\square$.

7.2. Shifted Schur polynomials

$$\dim(\nu, \lambda^{(n)}) / \dim \lambda^{(n)} = \frac{f_\nu^*(\lambda^{(n)})}{n^{\downarrow m}}$$

$$|\lambda| = n, \quad |\nu| = m$$

$$\lambda = (a, n-a) \qquad \nu = (b, m-b)$$

$$f_\nu^*(x, y) = x^{\downarrow b} y^{\downarrow (m-b)}$$

$$x^{\downarrow k} = x(x-1)(x-2) \dots (x-k+1)$$

Recall Pascal : relative dim. belongs to the same algebra (not the case for $\mathbb{Y}$).

$$s_\lambda \leftrightarrow \frac{\det[x_i^{\lambda_j+N-j}]}{\det[x_i^{N-j}]} = V(\vec{x})$$

Okounkov – Olshanski

Sh. Sch. Pely

$$s_\mu^*(x_1, \dots, x_N) = \begin{cases} \dfrac{\det\left[(x_i+N-i)^{\downarrow \mu_j+N-j}\right]_1^N}{\det\left[(x_i+N-i)^{\downarrow N-j}\right]_1^N} \\ 0, \quad N < \ell(\mu) \end{cases}$$

- $s_\mu^*(x_1, \dots, x_N)$ not symm. in $x_1, \dots, x_N$
  is symm. in $x_1 - 1, \dots, x_N - N$

- Denominator

$$\det\left[x_i^{j-1}\right] = \text{Vandermonde}$$

$$= \underbrace{\det\left[p_{j-1}(x_i)\right]}$$

$p_j \leftarrow$ poly. of deg. j

$p_j(x) = x^j + \dots$

$$\begin{matrix} p_0(x_1) & \dots & p_0(x_N) & & 1 \\ p_1(x_1) & \dots & p_1(x_N) & \leftarrow & x + \cancel{c} \\ p_2(x_1) & \dots & p_2(x_N) & \leftarrow & x^2 + \cancel{b}x + \cancel{c} \\ \vdots & & & & \end{matrix}$$

$$\det\left[(\underbrace{x_i+N-i}_{\text{var's}})^{\overset{N-j}{\downarrow N-j}}\right]_1^N = \prod_{i<j}(x_i - i - x_j + j)$$

shifted Vandermonde

- Top degree term in $x_1, \dots x_N$:

$$s^*_\mu(x_1 \dots x_N) = s_\mu(x_1, \dots x_N) + \text{L.o.T.}$$

lower degree

- Stability: $x_{N+1} = 0$ (exercise)

$$s^*_\mu(x_1 \dots x_N, 0) = s^*_\mu(x_1, \dots, x_N)$$

(just as s_λ's)

- $s^*_\mu(\lambda)$ is well def $\forall\ \lambda$

$$\lambda = (\lambda_1, \dots, \lambda_n, 0, 0, \dots)$$

Theorem. $\forall\ \mu, \lambda$ $\qquad |\lambda| = n,\ |\mu| = m$

$$\boxed{\frac{\dim(\mu, \lambda)}{\dim \lambda} = \frac{s^*_\mu(\lambda)}{n^{\downarrow \mu}}}$$

$\dfrac{f^*_\mu(\lambda)}{n^{\downarrow m}}$

(Recall Pascal)

$$x^{\downarrow b} y^{\downarrow (m-b)} = x^b y^{m-b} + \dots$$

Proof.

$$\frac{\dim(\mu, \lambda)}{(n-m)!} = \det\left(\frac{1}{(\lambda_i - \mu_j + j - i)!}\right)$$

$$\frac{\dim \lambda}{n!} = (\text{HW}) = \frac{\prod_{i<j}(\lambda_i - \lambda_j + j - i)}{\prod_i (\lambda_i + N - i)!}$$

① ⊛ – shifted Vandermonde

② $n! / (n-m)! = n^{\downarrow m}$

$$\det\left(\frac{1}{(\lambda_i-\mu_j+j-i)!}\right)\prod_i(\lambda_i+N-i)!$$

$$=\det\left(\frac{(\lambda_i+N-i)!}{(\lambda_i-\mu_j+j-i)!}\right)$$

$$\frac{(\lambda_i+N-i)!}{(\lambda_i-\mu_j+j-i)!} = (\lambda_i+N-i)^{\downarrow \mu_j+N-j}$$

$\square$

7.3. Shifted symm. functions. Λ^*

(Not the same algebra)

$s_\mu^*(x_1, \dots, x_N) \in \Lambda_N^*$

Λ_N^* : polynomials symm. in $x_1 - 1, \dots, x_N - N$

Ex. $p_{k,c}^*(x_1, \dots, x_N) = \sum_{i=1}^{N} \left((x_i - i + c)^k - (-i + c)^k \right)$

$\in \Lambda_N^*$

$[p_{k,c}^*] = p_k$, top degree term

always symmetric

$\left(\text{so, } \Lambda_N^* \to \Lambda_N, \quad f \to [f] \right)$

filtered by degree

graded by degree

$\Lambda_N^{*,k} = \{ \text{all sh. symm. of deg} \le k \}$

$\Lambda^*_{N+1} \longrightarrow \Lambda^*_N$, $x_{N+1} = 0$

& $\Lambda^{*k} = \varprojlim_N \Lambda^{*k}_N$

$\Lambda^* = \bigcup_{k \geq 0} \Lambda^{*,k}$ | $\Lambda = \bigoplus_k \Lambda^k$

homog. symm. pol. of deg. k

Filtered / graded , $\Lambda^{*k} / \Lambda^{*k-1} = \Lambda^k$

& Shifted Schur functions $s^*_\mu \in \Lambda^*$

- basis in Λ^*
- $[s^*_\mu] = s_\mu$

$$p^*_{k,c}(\underbrace{x_1, x_2, \dots}) = \sum_{i=1}^{\infty} \left((x_i - i + c)^k - (-i+c)^k \right) \in \Lambda^*$$

finitely many nonzero

$$\frac{s^*_\mu(\lambda^{(n)})}{n^{\downarrow m}}$$

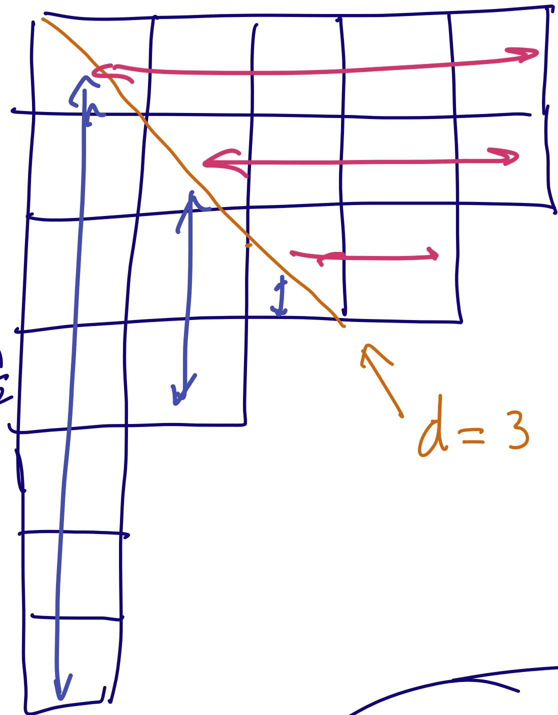
$p^*_{k,c}$ — algebraically indep in Λ^*

$$[p^*_{k,c}] = p_k$$

7.4. Modified Frobenius Coord.

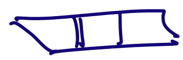
$a_i = \lambda_i - i + \frac{1}{2}$

$b_j = \lambda'_j - j + \frac{1}{2}$



λ

$\lambda = (a_1, \ldots, a_d \mid b_1, \ldots, b_d)$

lengths of  ,  $\in \mathbb{Z} + \frac{1}{2}$.

$$\boxed{|\lambda| = \sum a_i + b_i}$$

$$p^*_{k,\frac{1}{2}}(\lambda_1, \lambda_2, \ldots) = \sum_{i=1}^{\infty} \left(\left(\lambda_i - i + \tfrac{1}{2}\right)^k - \left(-i + \tfrac{1}{2}\right)^k \right)$$

Proposition.

$$= \sum_{i=1}^{d} \left(a_i^k - (-b_i)^k \right)$$

Lemma.

$$\prod_{i=1}^{\infty} \frac{u+i-1/2}{u+i-1/2-\lambda_i} = \prod_{i=1}^{d} \frac{u+b_i}{u-a_i}$$

Proof

$$\frac{u+i-1/2}{u+i-3/2} \cdot \frac{u+i-3/2}{u+i-5/2} \cdots \frac{u+i-\lambda_i+1/2}{u+i-\lambda_i-1/2}$$

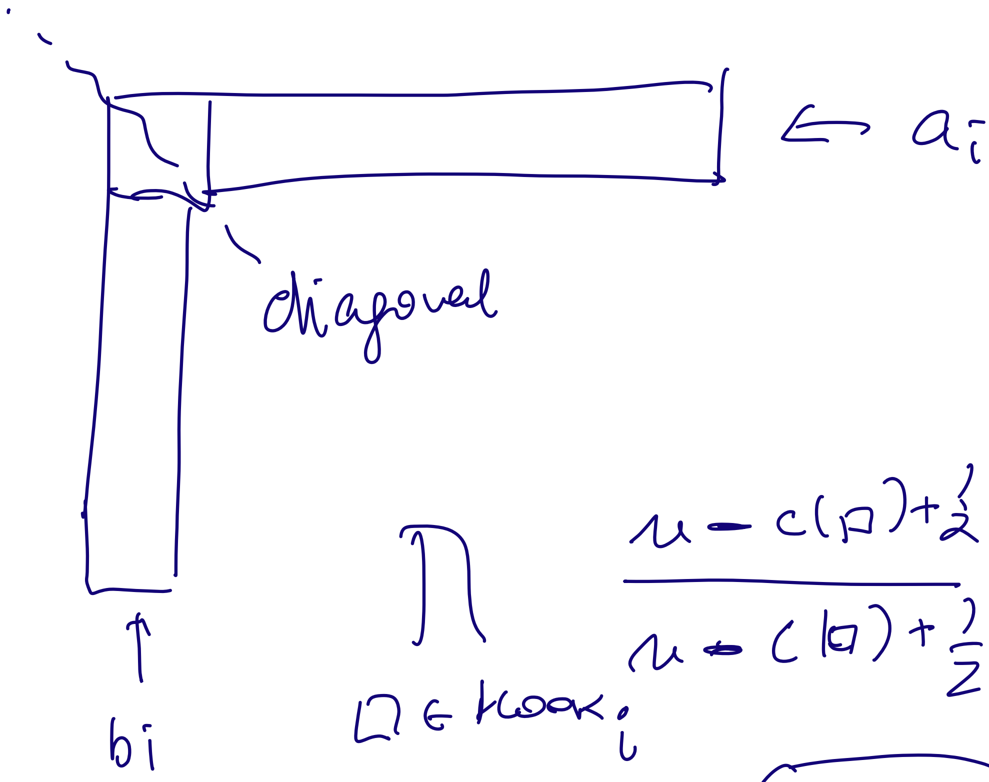
$i=2$

Content $\quad c(\square) = j-i$

	0	1	2	3	4
λ_2	-1	0	1	2	3
	-2	-1	0	1	
	-3	-2			
	-4				
	-5				
	-6				

$$\prod_{\square \in \lambda_i} \frac{u-c(\square)+\frac{1}{2}}{u-c(\square)-\frac{1}{2}}$$

$$\text{LHS} = \prod_{\square \in \lambda} \frac{u-c(\square)+\frac{1}{2}}{u-c(\square)-\frac{1}{2}}$$



$$\prod_{\square \in \mathrm{Hook}_i} \frac{u - c(\square) + \frac{1}{2}}{u + c(\square) + \frac{1}{2}} = \frac{u + b_i}{u - a_i}$$

$\square$

Next, p_n^* & Frobenius coord.